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Beneficial Changes in Random Variables Under Multiple Sources of Risk and Their Comparative Statics*

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Abstract

The consequences of a change in a random parameter are determined for a decision model with more than one source of randomness. The two cases of independent and stochastically dependent sources of risk are discussed. Four comparative static theorems are given. These state the effect of first degree stochastically dominant shifts or risk decreases for one random variable while the other random variable is held fixed. Deterministic transformations are used to represent random parameter changes. The results are presented in the context of the coinsurance demand model with a risky insurable asset and background risk.

Key words: Risk, Insurance Demand, Deterministic Transformations, Stochastic Dominance, Expected Utility.

1. Introduction

Determining the consequences of a change in a random parameter is an important and frequently studied comparative statics problem. Rothschild and Stiglitz [1970, 1971] analyzed the effect of an increase in risk and others have examined the impact of first or second degree stochastically dominant shifts. Refinements of this work have considered special types of risk increases or particular deterministic transformations as the means of altering the random parameter. These research efforts have been carried out in specific models, such as that of the competitive firm, and in general decision models. In a majority of the analysis, however, the decision model includes only one source of randomness.[1] Papers which do include multiple sources of risk often assume independent risks.

An important direction in which this comparative static analysis can and should be extended is to examine decision models with multiple sources of randomness including cases where the risks are not independent of one another. This work takes modest steps in that direction. As a preliminary matter, two conceptual questions which arise in models with multiple random parameters are addressed.

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The first asks which changes in a random parameter are most usefully analyzed. This question arises because the various definitions of risk increases or stochastically dominant shifts formulated for single random parameter models do not necessarily have the same meaning in the multiple random parameter case. The second question deals with which assumption to make concerning the other random parameters, as the parameter of interest is shifted. This is an especially important question when the random parameters are not stochastically independent.

The analysis and examples used in this paper are formulated within a specific rather than a general decision model. The insurance demand model with two sources of randomness is used. This model has the advantage of being linear in the random parameters. This linearity makes the analysis less cumbersome, but also implies that the insurance model is similar in some respects to the portfolio model with two risky assets. This allows recent analysis by Hadar and Seo [1990] and Meyer and Ormiston [1991] to be exploited. The results that are obtained replicate and extend findings in the insurance literature, but more importantly, illustrate the issues involved in comparative static analysis in decision models with multiple sources of risk.

The paper is organized as follows. First, the insurance demand problem is presented and two examples are given to illustrate the issues which can arise when more than one random parameter is included in a decision model. Next, in section 3, the questions of which changes in the random parameter are to be analyzed, and what is to be assumed concerning the other random parameters, are addressed. An appropriate *ceteris paribus* assumption is made and beneficial changes in random parameters are defined. Both the case of stochastic independence and dependence are handled. Deterministic transformations are an important facet of the modeling procedure. Finally, in section 4, these assumptions and definitions are applied, and their effects illustrated, in the insurance demand model. Several propositions concerning the demand for insurance are developed.

2. The insurance purchase decision model

Assume an individual decision maker possesses two random assets $\tilde{y}$ and $[M-\tilde{x}]$, where the support of $\tilde{y}$ is $[0,B]$ and the support of $[M-\tilde{x}]$ and $\tilde{x}$ is $[0,M]$. The random variable $\tilde{y}$ represents the value next period of all assets the decision maker possesses, excluding the asset of primary interest. This asset, whose value next period is $[M-\tilde{x}]$, has maximum value M which is subject to random loss of size $\tilde{x}$. The decision maker can choose to insure against the loss by acquiring δ units of insurance which provides a payment $[\tilde{x}-P]$, where $0 < P < M$ is the price paid for this insurance. δ is the coinsurance rate. The decision maker is assumed to choose δ to maximize the expected utility from $\tilde{z} = [M-\tilde{x}] + \delta[\tilde{x}-P] + \tilde{y}$. $R_A(z)$ and $R_R(z)$ are used to denote the decision maker's measures of absolute and relative risk aversion, respectively. $R_A(z) > 0$ is assumed.

The first order condition for an interior maximum for this problem requires that $E\{u'(\tilde{z})[\tilde{x}-P]\} = 0$. If this equation has a solution, it yields a maximum when $E\{u''(\tilde{z})[\tilde{x}-P]^2\} < 0$. This condition holds under strict risk aversion. As usual, comparative static analysis in this model proceeds by determining the impact of whatever change is of interest on the first order condition expression, $E\{u'(\tilde{z})[\tilde{x}-P]\}$. Signing the change in this expression is equivalent to determining the direction in which the decision maker would adjust δ, as long as an interior solution prevails initially.

Because there are two random parameters in this decision model, results quite different from the one parameter version, where $\tilde{y}$ is nonrandom, are possible. Two examples are used to illustrate these differences, and to indicate which types of changes in a random parameter can lead to unusual results.

Consider the following initial situation. Let $\tilde{y}$ take on values 20, 10 and 10 and $[M-\tilde{x}]$ take on values 20, 50, and 50 in states S_1, S_2 and S_3, respectively. The probability of S_1 is 1/2 and for S_2 and S_3 it is 1/4. Also, $M = 50$. Let the price of the insurance policy, P, equal 15, the actuarially fair value. For all risk averse decision makers, the optimal δ is 2/3. This value yields a risk free wealth next period of 50. Even though this insurance is priced in an actuarially fair manner, the optimal δ is less than one because the decision maker wishes to preserve some of the riskiness of $[M-\tilde{x}]$ in order to offset the riskiness of $\tilde{y}$. $[M-\tilde{x}]$ and $\tilde{y}$ are negatively correlated in this case.

For the first example, which displays the effect of altering a random parameter, change $[M-\tilde{x}]$ so that it takes on values 30, 30 and 50 in states S_1, S_2 and S_3. This change is a mean preserving contraction of the initial distribution of $[M-\tilde{x}]$ or $\tilde{x}$; that is, it is a Rothschild and Stiglitz decrease in risk according to their mean preserving spread definition. This change in the insurable asset, however, has eliminated the possibility of reducing the decision maker's risk to zero. For all decision makers, their next period wealth has a mean value of 50, but risk cannot be reduced to zero by an appropriate level of insurance. Thus, all risk averse decision makers are made worse off by this mean preserving contraction. The change is not beneficial to risk averse agents because it alters the diversification possibilities in this two asset portfolio.

Consider now a second example where the random parameter $[M-\tilde{x}]$ is altered so that it takes on values 50, 20 and 20 in the states S_1, S_2 and S_3, respectively. The distribution function for asset $[M-\tilde{x}]$, or the loss $\tilde{x}$ itself, is *unchanged* from that initially given. The insurable asset, now and in the initial setting, are equal in distribution to one another. For this new $[M-\tilde{x}]$ it is still possible to eliminate risk completely. Choosing $\delta = 4/3$ does this, and this is the coinsurance rate all risk averse decision makers would select. The change in $[M-\tilde{x}]$ has caused the correlation between $\tilde{y}$ and $[M-\tilde{x}]$ to become positive rather than negative. The decision maker now chooses to over insure $[M-\tilde{x}]$ in order to also provide protection against losses in $\tilde{y}$. The important feature of this example is that even though no change in the marginal distribution function for either random parameter has occurred, the optimal δ has changed.

Two main points can be extracted from these examples. The first is that changes in a random parameter which increase expected utility and are considered beneficial in a one random parameter setting need not increase expected utility if other random parameters are present.[2] Second, in discussing changes in random parameters, it is necessary that one do more than restrict the distribution functions describing the random alternatives on an individual basis; the relationship between the random parameters must also be restricted. Next, ways of dealing with these points are suggested.

3. Beneficial changes and *ceteris paribus*

Decision models with random parameters allow many comparative static questions to be formulated. In deciding which of these to address, a researcher must determine which are of interest, and which can be answered in an interesting way. Questions must be both interesting and solvable. This section deals with posing interesting and solvable comparative static questions concerning the effects of shifting one random parameter in a multiple random parameter decision model.

In one random parameter models, the effects of a change in that random parameter are of interest if that change is known to increase expected utility for all agents in a well defined group. Presumably this is because those agents can and would take actions which result in that type of change. Rothschild and Stiglitz decreases in risk and the various stochastically dominant shifts are examples of changes which increase expected utility for specific groups of decision makers.[3]

For other changes in random parameters, such as simple or strong decreases in risk, or option like changes, researchers argue they are worthy of study by pointing to situations where they do occur. The effects of these changes are interesting to analyze because they can and do occur in the real world. Each of these reasons is used here in determining which types of random parameter changes to analyze in multiple random parameter models.

Recall the first example given earlier which shows that a distribution function change can cause all risk averse decision makers to be worse off even though for that single random parameter, it is mean preserving contraction, and in a single random parameter setting would be a Rothschild and Stiglitz decrease in risk. The point of the example is that in decision models with multiple random parameters, it is not immediately obvious which distribution function changes increase expected utility. It is important, however, that this be determined. When a change in a random parameter alters expected utility in a known way, this fact can be, and usually is, exploited in determining its impact on the level of the choice variable selected by the decision maker.

In summary, if a change in a random parameter increases expected utility, the change is of interest. In addition, this also gives information which can lead to a solution to the comparative statics question. For these two reasons, this analysis focuses on changes in random parameters which alter expected utility in a known way. In some instances it is possible to demonstrate that a particular change increases or decreases expected utility for the decision maker or a group of decision makers. In other cases, that it does so is one of the assumptions characterizing the change. Changes which increase expected utility are referred to as beneficial changes in the random parameter.

A second methodological issue which arises in decision models with two random parameters concerns which *ceteris paribus* assumption to make. It is particularly important to decide what to assume concerning the other random parameters as the parameter of interest is shifted. Obviously, to focus on the impact of the change of interest, the other random parameters should be kept fixed. It is not clear, however, what this means, or if this is even possible when the random parameters are not independently distributed. Thus, the cases of independence and dependence are discussed separately.

When the random parameters are independently distributed, the *ceteris paribus* assumption used here is that of Hadar and Seo, and requires that the distribution functions for the other random parameters be fixed and remain independent as changes are made in the random parameter of interest. In the insurance demand model, this assumption implies that the distribution of $\tilde{y}$ is not changed as that for $[M-\tilde{x}]$ is shifted and vice versa. Notice that the relationship and correlation between these random parameters is not allowed to change under this assumption since independence is maintained.

When the random parameters, such as $\tilde{y}$ and $[M-\tilde{x}]$, are stochastically dependent, one cannot alter the distribution function for the one without changing the marginal and/or conditional distribution functions for the other.[4] Stochastic dependence, like deterministic dependence, requires that when one of the two variables is changed, the other must change also. Unlike deterministic dependence, however, there are a large number of possible changes which can occur. Hence, a researcher still can impose any one of a variety of *ceteris paribus* assumptions concerning the other random parameters.

The second example presented earlier gives a clue as to how to proceed. The example shows that it is not sufficient to only require that the marginal distribution of the other random parameter be held fixed. The relationship and correlation properties must be held constant to the extent that this is possible. Deterministic transformations are a means of doing this.

A deterministic transformation of a random parameter replaces every realization of that parameter by a new value determined according to a function defined over the support of the random parameter. This function is often required to be nondecreasing. When random parameter $\tilde{x}$ is transformed by a nondecreasing $t(x)$, the resulting random variable has marginal cumulative distribution function (CDF) $F^1(\cdot)$ which is related to the original $F^0(\cdot)$ by the equation $F^1(t(x)) = \sup\{F^0(w)$: $t(w) \leq t(x)\}$. For the case of strictly increasing transformations this reduces to $F^1(t(x)) = F^0(x)$.

When $\tilde{x}$ is transformed deterministically the marginal CDF for $\tilde{y}$ is not changed. Because $t(x)$ is nondecreasing, the correlation between $\tilde{x}$ or $[M-\tilde{x}]$ and $\tilde{y}$ cannot be completely reversed as was the case in the second example. Even stronger statements can be made. For instance, if $\tilde{x}$ is transformed linearly, the correlation between $\tilde{x}$ and $\tilde{y}$ does not change at all as a result of the transformation. For nonlinear nondecreasing transformations the correlation can change, but the change is limited. Thus, deterministic transformations are a way to change one random parameter while keeping the marginal distribution of the other fixed, and also maintaining the relationship and correlation properties between the parameters to the extent that this is possible.

Requiring that a change in a random parameter be able to be accomplished by means of a deterministic transformation does not severely restrict the types of changes which can be analyzed. A nondecreasing t(x) can be used to accomplish any change in $\tilde{x}$ as long as $\tilde{x}$ is continuously distributed. A shift from any initial marginal CDF $F^0(\cdot)$ to any final $F^1(\cdot)$ is carried out by $t(x) = \inf\{w: F^1(w) \geq F^0(x)\}$. Thus, virtually all of the analysis carried out assuming independence and using CDF shifts can also be accomplished using deterministic transformations.[5]

In summary, when $[M-\tilde{x}]$ or $\tilde{y}$ are transformed deterministically, the marginal distribution of the other is kept unchanged and the relationship between the two random parameters is also maintained. This makes analysis of the effects of deterministic transformations more likely to yield determinate results. Since deterministic transformations are also of interest because they represent methods of changing a random parameter which are available in the marketplace, this is the manner in which the *ceteris paribus* assumption in models with dependent random parameters is specified in the analysis. Specifically, when the random parameters $[M-\tilde{x}]$ and $\tilde{y}$ are not independently distributed, the impact of a change in one of them on the demand for insurance will be determined by calculating the effect of transforming it in a deterministic fashion.

4. Analysis of the insurance demand model with two risky assets

The series of questions which are addressed in the following theorems all concern the effect of a change in the loss distribution on the agent's demand for insurance. The demand for insurance is measured by the optimal coinsurance rate. In addressing these questions, it is assumed that the price of insurance, P, and the maximum value of the asset, M, are fixed. Assuming P to be fixed is an important simplifying assumption. It can be justified in two ways. First, it is a reasonable assumption when the price of insurance is not tied to a specific decision maker's loss distribution, but is instead based on a population loss distribution which is not assumed to change. Alternatively, when the price for insurance is based only on the mean of $\tilde{x}$ and a mean preserving change in $\tilde{x}$ occurs, then P would not change when $\tilde{x}$ is shifted.

When $\tilde{x}$ and $\tilde{y}$ are independently distributed, the analysis follows a standard methodology replacing one cumulative distribution function (CDF) by another in the expression to be signed and then determining the impact of this change. Let $F^0(x)$ and $F^1(x)$ denote the initial and final CDFs for $\tilde{x}$, respectively, and G(y) the CDF for $\tilde{y}$.

The impact of the change from $F^0(x)$ to $F^1(x)$ on expected utility in this insurance demand model is given by: $\int_0^B \int_0^M u(z)d[F^1(x) - F^0(x)]dG(y)$. Integrating by parts this becomes $-(\delta - 1)\int_0^B \int_0^M u'(z)[F^1(x) - F^0(x)]dxdG(y)$. If the shift from $F^0(x)$ to $F^1(x)$ does not change the mean of $\tilde{x}$, the expression can also be written as: $(\delta - 1)^2 \int_0^B \int_0^M u''(z) \int_0^x [F^1(s) - F^0(s)]dsdxdG(y)$.

To determine whether a change in $\tilde{x}$ would cause the decision maker to increase or decrease δ, one must determine its effect on $E\{u'(\tilde{z})[\tilde{x} - P]\}$. If $E\{u'(\tilde{z})[\tilde{x} - P]\}$ becomes positive (negative), then the optimal δ is larger (smaller) than its initial value assuming an initial interior solution. One begins with the first order condition: $\int_0^B \int_0^M \{u'(z)[x - P]\}dF^0(x)dG(y) = 0$. This is subtracted from the similar expression with $F^1(x)$ replacing $F^0(x)$. This yields: $\int_0^B \int_0^M \{u'(z)[x - P]\}d[F^1(x) - F^0(x)]dG(y)$ as the expression to be signed.

If one can show that $\{u'(z)[x - P]\}$ is increasing (decreasing) or concave (convex) in x *for all y*, then well known results for the one random parameter case can be used to sign the inner integral and the expression. This proof method is used in theorems 1 and 3.

The first theorem which is presented examines the effect of unambigiously reducing the loss distribution by means of a first degree stochastically dominant shift assuming independence between the random parameters. That is, the CDF for $\tilde{x}$ is changed from $F^0(x)$ to $F^1(x)$ satisfying $[F^1(x) - F^0(x)] \geq 0$. Under independence, the effect of such a shift on expected utility is easily determined. If $\delta < 1$, decreases in the loss distribution increase expected utility, but if $\delta > 1$ the reverse is true. It is well known that under independence all risk averse decision makers choose $\delta < 1$ whenever P is greater than the mean of $\tilde{x}$. This is assumed here. Thus, these FSD shifts are beneficial for this group, although this fact is not used in the proof of the theorem.

Theorem 1: *If $\tilde{x}$ and $\tilde{y}$ are independent and $\tilde{x}$ undergoes a first degree stochastically dominant decrease and remains independent of $\tilde{y}$, then decision makers with $R_A(z) > 0$ and $R_R(z) \leq 1$ will decrease* δ.

Proof: It is sufficient to show that $\{u'(z)[x - P]\}$ is increasing in x for all values for y. The derivative of this expression with respect to x is equal to:

$$
\begin{aligned}
&\{u'(z) + u''(z)[x - P](\delta - 1)\} = \\
&\{u'(z) + u''(z)(\delta x - x - \delta P + P)\} = \\
&\{u'(z) + u''(z)(z - M - y + P)\} = \\
&u'(z)[1 - R_A(z)(z - M - y + P)] = \\
&u'(z)[1 - R_R(z) + R_A(z)(M + y - P)] > 0
\end{aligned}
$$

under the hypotheses of the theorem.

QED

This result says that, under independence, adverse selection will occur. That is, if losses for one decision maker are lower than those of another in the sense of FSD, then for a fixed price of insurance and all else equal, the decision maker with smaller losses will also purchase less insurance. If a fixed price of insurance is offered to a pool of otherwise identical agents, those with higher loss distributions would select more insurance.

Theorem 2 considers the same question but removes the assumption of independence. Examples can easily be constructed to show that if $\tilde{x}$ is made smaller in a first degree sense, and the correlation between $\tilde{x}$ and $\tilde{y}$ is also increased, then the decision maker could choose more insurance because doing so would now provide insurance against the risk in $\tilde{y}$ as well. In theorem 2, the first degree change in $\tilde{x}$ is represented as resulting from a deterministic transformation. Thus, large changes in correlation are ruled out. This is sufficient to allow Theorem 1 to be extended to the case of stochastic dependence. The stringent independence restriction is replaced by a less restrictive one involving deterministic transformations.

When $\tilde{x}$ and $\tilde{y}$ are not independently distributed, let H(x,y) be the joint CDF for $\tilde{x}$ and $\tilde{y}$ and $d^2H(x,y)$ denote $[\partial^2 H(x,y)/\partial x \partial y] dx \cdot dy$. To determine the effect on δ of transforming random parameter $\tilde{x}$ using t(x) one must sign the first order condition expression with x replaced by t(x). That is, one must show that $\int_0^B \int_0^M u'(z)[x-P]d^2H(x,y)$ is positive or negative when x is replaced by t(x). One can do this directly, or alternatively, another way in which this can be accomplished is as follows. Define $k(x) = t(x) - x$ and $\tilde{z}(\theta)$ by $\tilde{z}(\theta) = M - (\tilde{x} + \theta \cdot k(\tilde{x})) + \delta[\tilde{x} + \theta \cdot k(\tilde{x}) - P] + \tilde{y}$. Determining the sign of the derivative of $\int_0^B \int_0^M u'(z(\theta))[x + \theta \cdot k(x) - P]d^2H(x,y)$ for all θ in [0,1] gives the desired sign. This procedure has been used for many years beginning at least with Sandmo [1970, 1971], and is followed here in the next theorem.

Theorem 2: *If $\tilde{x}$ is transformed by t(x) satisfying $t(x) - x = k(x) \leq 0$, then decision makers with $R_A(z) > 0$ and $R_R(z) \leq 1$ will decrease* δ.

Proof: The derivative of $\int_0^B \int_0^M u'(z(\theta))[x + \theta \cdot k(x) - P]d^2H(x,y)$ with respect to θ is: $\int_0^B \int_0^M \{u'(z(\theta)) + u''(z(\theta))[x + \theta \cdot k(x) - P](\delta - 1)\}k(x)d^2H(x,y)$. Using the same substitutions as in the proof of theorem 1, this can be rewritten as: $\int_0^B \int_0^M \{u'(z(\theta))[1 - R_R(z(\theta)) + R_A(z(\theta))(M + y - P)]\} \cdot k(x)d^2H(x,y)$. This is negative for all θ in [0,1] and hence the first order condition expression must be negative at $\theta = 1$, i.e. when x is replaced by t(x).

QED

Meyer [1989] shows that t(x) leads to a first degree stochastically dominant decrease in $\tilde{x}$ if and only if $k(x) \leq 0$ for all x. Thus, this theorem indicates that when $\tilde{x}$ and $\tilde{y}$ are stochastically dependent, the effect of a FSD decrease, accomplished by means of a deterministic transformation so the relationship between $\tilde{x}$ and $\tilde{y}$ is held fixed, is still to reduce δ. Adverse selection still occurs as was the case under independence.

First degree stochastically decreasing transformations of the loss parameter $\tilde{x}$ do increase expected utility for all decisions makers who prefer more to less, as long as $\delta < 1$. To see this note that $dEU/d\theta = \int_0^B \int_0^M u'(z(\theta))(\delta - 1)k(x)d^2H(x,y)$. This is positive for all θ if $k(x) \leq 0$, $u'(z) \geq 0$ and $\delta < 1$. Thus, EU evaluated at $\theta = 1$ is larger than EU evaluated at $\theta = 0$.

The next two theorems examine the effect of a change in the riskiness of the loss distribution. With independence, this is characterized by a change in the CDF for $\tilde{x}$ satisfying $\int_0^x [F^1(s) - F^0(s)]ds \leq 0$. With independence, these decreases in the riskiness of the loss distribution do increase expected utility for all risk averse decision makers. As the example presented earlier shows, without independence this is not necessarily the case. No restriction on δ is required in this statement.

Theorem 3: *Suppose $\tilde{x}$ and $\tilde{y}$ are independent and $\tilde{x}$ undergoes a Rothschild and Stiglitz decrease in risk that maintains independence between $\tilde{x}$ and $\tilde{y}$. Then decision makers who are increasingly relative and decreasing absolute risk averse with $R_R \leq 1$ will decrease δ.*

Proof: It is sufficient to show that $\{u'(z)[x - P]\}$ is convex in x for all values for y. The first derivative of this expression with respect to x as derived in the proof of theorem 1 is: $u'(z)[1 - R_R(z) + R_A(z)(M + y - P)]$. Hence the second derivative with respect to x is: $(\delta - 1)\{u'(z)[-R_R'(z) + R_A'(z)(M + y - P)] + u''(z)[1 - R_R(z) + R_A(z)(M + y - P)]\}$. This expression is positive under the hypotheses of the theorem as long as $\delta < 1$, which is the case.

QED

This theorem indicates that with independent sources of risk, a risk averse decision maker will choose to insure less risky assets at lower coinsurance levels. For this case the assumption that P is fixed is based on the notion that insurance pricing depends only on the mean of the loss distribution. Thus, the interpretation here can be for a reduction in risk for a particular agent rather than a comparison across agents within a fixed pool.

As yet no general theorem concerning arbitrary Rothschild and Stiglitz decreases in risk has been demonstrated for the dependent case. Even the use of deterministic transformations does not seem to make such a result possible. In part at least, this is because these changes can reduce rather than increase expected utility even when accomplished by means of a deterministic transformation. Theorem 4 illustrates an attempt to make a general statement concerning the effect of particular risk changes on insurance demand when the risks are not independent, but it is not a very general result.

Assume that the decision maker's loss distribution is changed so that the relative ordering of losses is maintained, but small losses become larger and large losses become smaller, where small and large are defined relative to the price of the insurance policy. That is, a deterministic transformation t(x) which is increasing, with $t(x) \geq x$ for $x \leq P$ and $t(x) \leq x$ for $x \geq P$ is used to alter the riskiness of $\tilde{x}$. Because $t(P) = P$ and t(x) is nondecreasing the changes in the losses are not sufficiently large to alter their size relative to P. Small losses get larger, but not larger than P and large losses get smaller, but remain larger than P.

The above restriction on the change in risk is not enough. Moreover, even if one requires that such a transformation also maintain the mean of $\tilde{x}$, it is not sufficient for a comparative static result. In fact, this is not even enough to imply that the decision maker's expected utility is increased. Instead of adding a mean preserving assumption, an assumption which explicitly requires that expected utility be preserved or increased is made. The following theorem shows that *if* expected utility is increased by the special change in $\tilde{x}$ defined above, then such a transformation also leads to a reduction in the demand for insurance.

Theorem 4: *Let t(x) be increasing with $t(x) \geq x$ for $x \leq P$ and $t(x) \leq x$ for $x \geq P$. If expected utility does not decrease when $\tilde{x}$ is transformed by t(x), then the transformation causes the risk averse decision maker to reduce δ when $\delta < 1$.*

Proof: Recall from the proof of theorem 2 that the derivative to be signed is: $\int_0^B\int_0^M\{u'(z(\theta)) + u''(z(\theta))[x+\theta\cdot k(x)-P](\delta-1)\}k(x)d^2H(x,y)$. This equals: $\int_0^B\int_0^M u'(z(\theta))k(x)d^2H(x,y) + \int_0^B\int_0^M u''(z(\theta))[x+\theta\cdot k(x)-P](\delta-1)k(x)d^2H(x,y)$. The second portion of this expression is negative for all θ in [0,1], since k(x) and $[x+\theta\cdot k(x)-P]$ are opposite in sign for all θ in [0,1] and both u'' and $(\delta-1)$ are negative. The first portion of the expression is opposite in sign to $dEU/d\theta$ at $\theta = 0$. Because EU is concave in θ and t(x) does not decrease EU by assumption, it must be that $dEU/d\theta > 0$ at $\theta = 0$. Hence, the first portion is negative also, and one can conclude that the optimal δ is decreased.

QED

The conclusion reached in theorem 4 is consistent with the general finding for independent risks presented earlier as theorem 3. A particular reduction in risk which increases expected utility leads to a lower level of insurance purchase even with dependent risks. Notice that in theorem 4 only risk aversion is assumed although very strong conditions are placed on the change in $\tilde{x}$ which occurs. Certain strong and simple risk changes can be represented by transformations satisfying the conditions in theorem 4.

5. Conclusions

Those familiar with the Hadar and Seo portfolio paper will recognize the results and the method of proof in theorems 1 and 3. They are very similar to Hadar and Seo's theorems and proofs concerning the effect of a first degree improvement or mean preserving contraction in $\tilde{x}$ on the optimal amount of $\tilde{x}$ to hold in a portfolio. Hadar and Seo are concerned with the selection of an optimal α to maximize $Eu(\tilde{z})$ where $\tilde{z} = \alpha\tilde{x} + (1-\alpha)\tilde{y}$, for independently distributed assets. They show that FSD improvements or Rothschild and Stiglitz decreases in risk cause the decision maker to include more of the asset in the optimal portfolio.

Theorems 1 and 3 can be interpreted in a portfolio choice context since choosing δ is equivalent to choosing how much of the insurance "asset" to include in the optimal portfolio. The insurance asset, however, is not independent of $[M-\tilde{x}]$, but is in fact perfectly negatively correlated with it. This means that the results here need not be consistent with those of Hadar and Seo. In fact, one is and the other is directly opposite.

When losses get smaller in the FSD sense, insurance becomes a lower valued asset in the FSD sense, and hence according to Hadar and Seo, if independently distributed, less insurance would be included in the optimal portfolio. The conclusion here is the same even though insurance is not independently distributed from $[M-\tilde{x}]$. For Rothschild and Stiglitz decreases in risk, however, the situation is quite different. When the loss parameter becomes less variable, insurance becomes less risky and if independently distributed more would be included in the optimal portfolio. Because of the perfect negative correlation, the finding here is just the opposite. When insurance becomes less risky it is also less necessary for diversification reasons and δ is decreased.[6]

One can easily combine the results in theorems 1 and 3 to obtain a theorem concerning SSD changes in the loss distribution. Also, theorems concerning strong, relatively strong, or the most general restriction of this form, relatively weak increases in risk, defined by Dionne, Eeckhoudt and Gollier [1991], can be demonstrated under independence. Finally, the same methodology can be used to address the subject of background risk and the effect of changing it. This is carried out by changing the CDF for $\tilde{y}$ when independence is assumed or by transforming $\tilde{y}$ in a deterministic fashion for the dependent random parameter case. This extension is left to "future research."

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Notes

1. Exceptions include the analysis of incomplete insurance markets in Doherty and Schlesinger [1983], Schlesinger and Doherty [1985] and Doherty and Schlesinger [1986]. More recently, Hadar and Seo [1990] and Meyer and Ormiston [1991] deal with multiple random assets in the portfolio models, Kimball [1991] includes multiple risks in a general decision model, and Eeckhoudt and Kimball [1991] consider insurance demand with background risk.
2. This same point can be made in decision models with only one random variable. For instance, if the outcome variable $\tilde{z}$ is a convex function of the random variable $\tilde{x}$, then risk increases in $\tilde{x}$ can lead to increases in expected utility for the decision maker who maximizes $Eu(\tilde{z})$. Walter Oi [1961] exploited this fact in analyzing the competitive firm many years ago. If the decision model

is positive linear in a single random variable, as is the most basic insurance model, then increases in risk for the random variable lead to increases in risk for the outcome variable.
3. Sometimes it is more convenient analytically to consider the opposite changes, such as increases in risk, which decrease expected utility.
4. To see this observe that $G(y) = \int_0^B G(y|x)dF(x)$ where G(y) and F(x) are the marginal cumulative distribution functions for $\tilde{y}$ and $\tilde{x}$. Hence if F(x) is changed, either G(y) or G(y|x) for some x must change.
5. Deterministic transformations cannot be used to break up mass points of probability, only to relocate it. Transformations which have a stochastic component can be used to break up mass points. The specific forms of deterministic transformations which lead to decreases in risk or stochastically dominant shifts have been determined elsewhere and will be mentioned as the need arises in the next section. See Meyer [1989] and Meyer and Ormiston [1989].
6. By constructing clever counterexamples, Hadar and Seo show that their conditions are necessary as well as sufficient. This may be possible here as well but has not been attempted as yet.

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