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On Convex Vectorial Optimization in Linear Spaces

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Abstract. We observe that the results and proofs of Ref. 1 are valid only for finite convex functionals, and we give some other corrections to Ref. 1.

Key Words. Vectorial and scalar optimization, convex programs, best approximation, characterization of minimal elements, existence of minimal elements, uniqueness of minimal elements.

All the results and proofs in Ref. 1 are valid under the assumption that the convex functionals f_1, f_2 defined on the linear space E are finite, i.e.,

$$f_i(E) \subset R = (-\infty, +\infty) \qquad \text{for } i = 1, 2.$$

The weaker assumption, made in Ref. 1, that f_1, f_2 are proper, i.e.,

$$f_i(E) \subset (-\infty, +\infty] \qquad \text{and } f_i \not\equiv +\infty \qquad \text{for } i = 1, 2,$$

must be replaced by the assumption that f_1, f_2 are finite, as shown by the following example.

Example. Let $E = R = (-\infty, +\infty)$. Define proper convex functionals f_1, f_2 on E by

$$f_1(x) = \begin{cases} 1 & \text{for } x \in [-1, +1], \\ +\infty & \text{for } x \notin [-1, +1], \end{cases}$$

$$f_2(x) = x \qquad \text{for } x \in R,$$

and let $G = [-2, 2]$. Then, it is not true that the equality

$$\mathcal{V}_G(f_1, f_2) \cap \{y \in E | f_1(y) = c\} = \mathcal{S}_{G \cap \{y \in E | f_1(y) \leq c\}}(f_2), \tag{6}$$

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of Ref. 1, Theorem 2.1, holds for all $c \in R = (-\infty, +\infty)$ satisfying

$$\inf_{g \in G} f_1(g) \leq c \leq \inf_{g \in \mathcal{S}_G(f_2)} f_1(g). \tag{5}$$

Indeed,

$$\inf_{g \in G} f_1(g) = 1,$$

$$\mathcal{S}_G(f_2) = \{-2\}, \qquad \inf_{g \in \mathcal{S}_G(f_2)} f_1(g) = +\infty;$$

but, for any c with $1 < c < +\infty$, we have

$$\{y \in E \mid f_1(y) = c\} = \varnothing,$$

$$\mathcal{S}_{G \cap \{y \in E \mid f_1(y) \leq c\}}(f_2) = \mathcal{S}_{[-1, +1]}(f_2) = \{-1\} \neq \varnothing.$$

There are two incorrect claims in the proof of Ref. 1, Theorem 2.1 (for f_1, f_2 proper), which are also shown by this example, namely:

(a) On page 178, it is claimed that, for any $c \in (-\infty, +\infty)$ satisfying (5), any

$$g_0 \in \mathcal{S}_{G \cap \{y \in E \mid f_1(y) \leq c\}}(f_2) \setminus \{y \in E \mid f_1(y) = c\},$$

and any $g \in G$ such that

$$f_2(g) < f_2(g_0), \tag{9}$$

the convex function

$$\varphi(\lambda) = f_1(\lambda g_0 + (1 - \lambda) g), \qquad 0 \leq \lambda \leq 1,$$

is continuous on $[0, 1]$. However, for any c with $1 < c < +\infty$ and for $g_0 = -1$, $g = -2$, we have now

$$\varphi(\lambda) = f_1(-2 + \lambda) = \begin{cases} +\infty & \text{for } \lambda \in [0, 1), \\ 1 & \text{for } \lambda = 1. \end{cases}$$

(b) On page 180, it is claimed that, for any $g_0 \in \mathcal{V}_G(f_1, f_2)$, we have

$$c = f_1(g_0) \in R = (-\infty, +\infty).$$

However, in the above example,

$$\mathcal{V}_G(f_1, f_2) = \{-2\} \cup \{-1\};$$

and, for $g_0 = -2$, we have

$$f_1(-2) = +\infty.$$

If we assume that f_1, f_2 are finite, then the above claims become correct. Indeed, if f_1 is finite (and convex) on E, then the restriction of f_1 to the two-dimensional subspace spanned by g_0 and g is continuous; hence, $\varphi(\lambda)$ is continuous on $(-\infty, +\infty)$ so the claim (a) becomes correct.

We note here that the claim (a) was used in Ref. 1 only to prove the existence of a number λ_0, with $0 \leq \lambda_0 < 1$, such that

$$\varphi(\lambda_0) = f_1(\lambda_0 g_0 + (1-\lambda_0)g) \leq c,$$

but this can be also deduced directly from the convexity of φ, without using the continuity of φ. Indeed, if

$$\varphi(\lambda) \geq c$$

for all $0 \leq \lambda < 1$, then, since

$$\varphi(1) = f_1(g_0) < c,$$

we obtain, fixing any λ_0 with $0 \leq \lambda_0 < 1$ and taking $\alpha > 0$ sufficiently small,

$$\begin{aligned} c &\leq \varphi(\alpha\lambda_0 + (1-\alpha)1) \leq \alpha\varphi(\lambda_0) + (1-\alpha)\varphi(1) \\ &< \alpha\varphi(\lambda_0) + (1-\alpha)c < c, \end{aligned}$$

which is impossible.

On the other hand, it is obvious that the claim (b) becomes correct if f_1 is finite. We note here that, in the part following this claim on page 180 of Ref. 1, the relation

$$g_0 \in \mathscr{V}_G(f_1, f_2) \cap \{y \in E | f_1(y) \leq c\}$$

should be replaced by

$$g_0 \in \mathscr{V}_G(f_1, f_2) \cap \{y \in E | f_1(y) = c\},$$

and the order of the subsequent arguments should be interchanged as follows: first show, as in Ref. 1, that

$$c = f_1(g_0)$$

satisfies (5); then, finally, conclude by (6) that

$$g_0 \in \mathscr{S}_{G \cap \{y \in E | f_1(y) \leq c\}}(f_2).$$

Remark. If we assume that f_1, f_2 are finite, then the results of Ref. 1 remain also valid if we delete everywhere in Ref. 1 the relations $-\infty < c < +\infty$, since, for $c = \pm\infty$, (6) reduces to

$$\phi = \phi.$$

Indeed, for $c=+\infty$, by (5),

$$\inf_{g\in\mathcal{S}_G(f_2)} f_1(g)=+\infty,$$

whence, since f_1 is finite,

$$\mathcal{S}_G(f_2)=\phi;$$

thus,

$$\mathcal{S}_{G\cap\{y\in E|f_1(y)\leq c=+\infty\}}(f_2)=\mathcal{S}_G(f_2)=\phi,$$

while the left-hand side of (6) is again ϕ, by the finiteness of f_1.

We also note that, throughout Ref. 1, $\mathcal{U}_G(f_1, f_2)$ should read: $\mathcal{V}_G(f_1, f_2)$, where the letter $\mathcal{V}$ stands for "vectorial." Furthermore, on page 176 of Ref. 1, in the definition of the partial order relation

$$(\alpha_1, \alpha_2)\leq(\beta_1, \beta_2) \qquad \text{in } R^2,$$

the inequalities

$$\alpha_1\leq\alpha_2 \qquad \text{and } \beta_1\leq\beta_2$$

should be replaced by

$$\alpha_1\leq\beta_1 \qquad \text{and } \alpha_2\leq\beta_2,$$

respectively.

Finally, we mention that some particular cases of Ref. 1, Theorem 2.1, have been also obtained, independently, by Gearhart (Ref. 2).

References

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