TWO- AND THREE-LAYER MODELS OF URANUS

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(Received 4 August, 1983)

Abstract. We present simple two-layer models of Uranus with rocky core and polytropic envelope satisfying exactly the observed mass, radius and the gravitational moments. The models show that the value of the fourth order zonal harmonic is $J_4 \lesssim -38 \times 10^{-6}$, while $J_6 \approx 10^{-6}$. More elaborate three-layer models fail to satisfy the observational constraints of the ice/rock ratio and/or of the rotation period. We conclude that three-layer models with uniform chemical composition in each layer may be too restrictive. More realistic models should account for variable chemical composition within each layer.

1. Equation of State

According to Podolak (1982) there are two basic hypotheses for the origin of the Jovian planets: (i) accretional hypotheses and (ii) giant protpolanet hypotheses. Accretional hypotheses start with a core composed of rocks (SiO_2, MgO, FeS, FeO, etc.) and of ice (H_2O, CH_4, NH_3, CO) that accrets an envelope of helium and hydrogen. On the other hand, giant protoplanet theories start with large gas balls of uniform composition that contract into the Jovian planets.

Fortunately, leaving aside the difficulties of giant protoplanet hypotheses as *ad hoc* assumptions regarding the formation of protoplanets and inability of Earth and Venus to lose amounts of gas larger than several percents of their mass (e.g., Horedt, 1982), both theories come up with a rocky core surrounded by variable amounts of H_2, He, H_2O, CH_4, NH_3, CO, etc. The rocky cores of the Jovian planets are formed by segregation of their rocky, heavy constituents towards the centre of the planet. The temperature-independent equation of state for the rocky core (38% SiO_2, 25% MgO, 25% FeS, 12% FeO) is given by (cf. Hubbard and MacFarlane, 1980)

$$p = \rho^{4.40613} \exp(-6.57876 - 0.176368\rho + 2.02239 \times 10^{-3}\rho^2). \qquad (1)$$

The pressure p is measured in megabars (Mb) and the density in g cm^{-3}. This version of the fitting formula is valid for the relatively low Uranian pressures.

For the two-layer models (Figures 1–3) we have adopted Equation (1) for the core and for the envelope a simple polytropic equation of state with index n:

$$p = k\rho^{1+1/n}, \quad k = \text{const.} \qquad (2)$$

The Moon and the Planets 29 (1983) 229–236. 0165–0807/83/0293–0229$01.20.

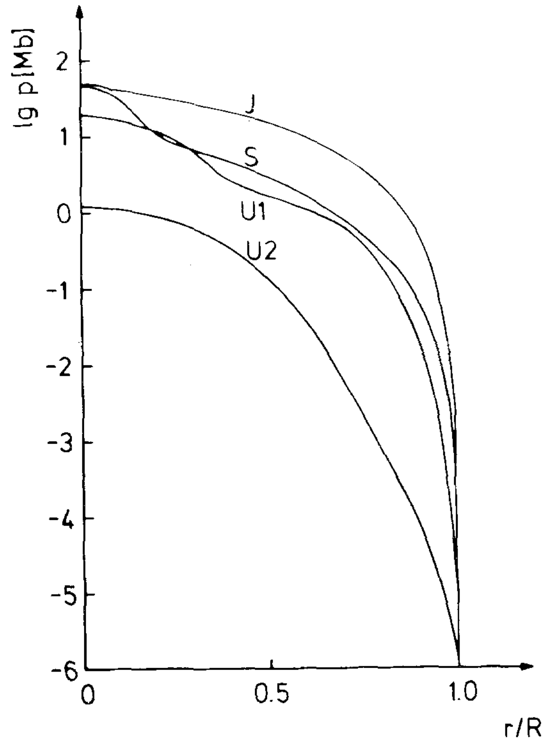

Fig. 1. Run of pressure for the two-layer polytropic models. J, Jupiter model; S, Saturn, Ul, Uranus with $J_4 = -38 \times 10^{-6}$; U2, Uranus with $J_4 = -50 \times 10^{-6}$ ($T = 16.31$ h).

For the three-layer models the core also obeys Equation (1), while the middle layer is composed of 56.5% H_2O, 32.5% CH_4, 11% NH_3 with the equation of state (cf. Hubbard and MacFarlane, 1980)

$$p = \rho^{3.71926} \exp(-2.75591 - 0.271321\rho + 7.00925 \times 10^{-3}\rho^2). \qquad (3)$$

This version of the fitting formula is also valid for Uranian pressures.

The outer (H_2 + He)-envelope of three-layer models obeys an equation of state of the form (cf. Hubbard *et al.*, 1980)

$$p = \rho^{X_1} \exp(X_2 + X_3\rho + X_4\rho^2) + \rho^{X_5} \exp(X_6 + X_7/\rho), \qquad (4)$$

where the coefficients X_i are given by

$$X_i = B_{i1} + B_{i2}y + B_{i3}t + B_{i4}y^2 + B_{i5}yt + B_{i6}t^2, \quad i = 1\text{–}7, \qquad (5)$$

and

$$y = Y - 0.2, \quad t = (T_{1\,\mathrm{bar}} - 140\,\mathrm{K})/100\,\mathrm{K}; \qquad (6)$$

Y denotes the fraction of helium by mass and $T_{1\,\mathrm{bar}}$ the temperature of the outer atmosphere at the 1 bar pressure level. The coefficients B_{ij} ($i = 1$–$7, j = 1$–6) are given by the matrix

$$B_{ij} = \begin{pmatrix} 1.40552 & 8.35340-2 & 2.19554-2 & 8.36655-2 & 4.85471-2 & 5.24978-3 \\ 1.20435+1 & -4.00838-2 & 8.52064-1 & 4.58141-1 & 4.51209-1 & -3.47402-1 \\ 6.99899 & -3.73628 & -2.57516 & -4.03739-1 & 1.17191 & 1.09677 \\ -7.04121 & 3.33766 & 2.62985 & -7.87440 & 1.21139 & -1.46941 \\ 1.31988 & -5.99219-1 & 2.85968-2 & 2.34703 & 4.26509 & 4.43703-2 \\ 1.63426+1 & 2.34944-1 & -1.29238-2 & -5.14266 & -1.00308-1 & -9.69252-2 \\ 2.42842 & 2.36287 & 2.44870-1 & 8.08992 & 4.29976-1 & 9.33803-2 \end{pmatrix} \tag{7}$$

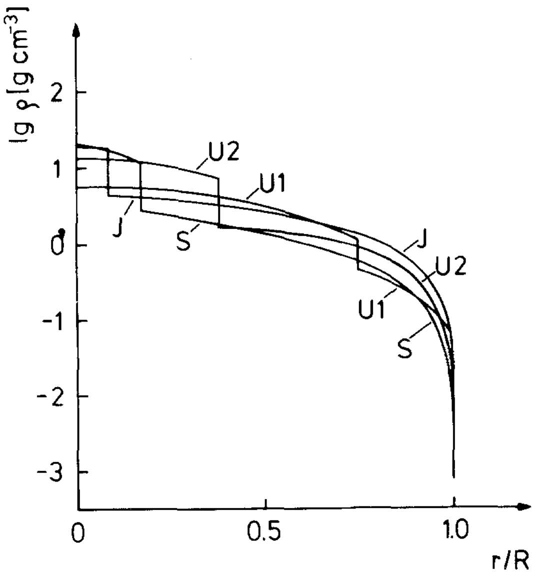

Fig. 2. Same as Figure 1 for the run of density.

2. Method

To determine the internal structure we have adopted the third-order theory of rotating figures in the form given by Zharkov and Trubitsyn (1978). We have solved their equations (29.1)–(29.3), (30.6)–(30.11) by successive approximations, starting with a linear density profile and zero flattening. Generally, over 50 iterations are needed to complete one model with 100–400 mesh-points. After completion of one model we have used a simple Newton–Raphson iteration procedure to adjust model parameters in order to satisfy up to a relative precision of 10^{-6} the following observational constraints (cf. Hubbard and Horedt, 1983): The equatorial radius a at the 1 bar pressure level and the gravitational moments J_2, J_4 of order 2 and 4, respectively. The mass of the planet was kept constant and equal to its observational value. Another important parmeter was the observed rotation period T, and for the three-layer models a mean observed temperature $T_{1\,\mathrm{bar}}$ of the atmosphere at the 1 bar pressure level (Table I).

The three observational constraints a, J_2, J_4 allow for the determination of three parameters of the internal structure. These parameters have been chosen to be: (i) for the two-layer models the polytropic index n, the polytropic constant k and the mass of the rocky core M_r; (ii) for the three-layer models the mass of rock and ice $M_r + M_i$, the ratio f = ice/rock $= M_r/M_i$, (M_r, M_i = mass of rock and ice, respectively); and (iii) the helium content Y of the outer (H_2 + He)-envelope.

The gravitational moments $J_2^{(0)}$ and $J_4^{(0)}$ are determined for a certain reference radius a_0 and a reference mass M_0; these differ generally from the values of a and M adopted for our 1 bar pressure level surface. The connection between the published reference values

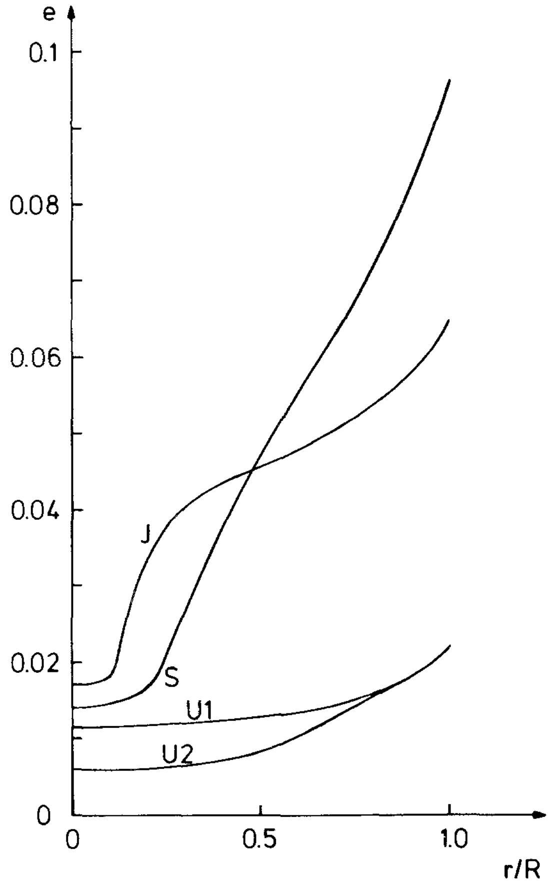

Fig. 3. Same as Figure 1 for the run of the oblateness of level surfaces.

TABLE I

Adopted values for the mass M, the equatorial radius a at pressure 1 bar, the gravitational moments J_2 and J_4, the rotation period T, and the temperature $T_{1\,\mathrm{bar}}$ at the 1 bar pressure level ($M_\oplus = M(\mathrm{Earth}) = 5.976 \times 10^{27}$ g; Allen, (1973)).

Planet	M ($M_\oplus$)	a (km)	$10^{-6} \times J_2$	$10^{-6} \times J_4$	T	$T_{1\,\mathrm{bar}}$
Jupiter	317.735[a]	71 492[b]	14 694 ± 4[a]	− 584 ± 7[a]	$9^{\mathrm{h}}55^{\mathrm{m}}29\overset{\mathrm{s}}{.}7$[c]	165[c]
Saturn	95.156[d]	60 200[c]	16 370 ± 18[d]	− 925 ± 38[d]	$10^{\mathrm{h}}39\overset{\mathrm{m}}{.}9 \pm 0\overset{\mathrm{m}}{.}3$[c]	140[c]
Uranus	14.511[e]	25 900[f]	3425 ± 8[e]	− 38 ± 12[e]	$16\overset{\mathrm{h}}{.}31 \pm 0\overset{\mathrm{h}}{.}27$[g]	78[f]

[a] Null (1976). [b] Lindal *et al.* (1981). [c] Hubbard *et al.* (1980). [d] Null *et al.* (1981). [e] Nicholson *et al.* (1982). [f] Hubbard and MacFarlane (1980). [g] Goody (1982).

and the gravitational moments at the 1 bar pressure level is given by (e.g., Zharkov and Trubitsyn, 1978; Equation (26.2)): $Ma^nJ_n = M_0a_0^nJ_n^{(0)}$, ($n = 1, 2, \ldots$). To make the Newton–Raphson procedure convergent for the three-layer models each iteration implements only 0.01–0.1 parts of the calculated corrections.

3. Results

Since the most uncertain observational constraint for Uranus is the gravitational moment of order 4, we made calculations with the most probable J_4 and its upper and lower observational limit: $J_4 = -38 \times 10^{-6}$, -26×10^{-6} and -50×10^{-6}, respectively. An uncertainty that seems to be largely removed by the discussion of Goody (1982) concerns the rotation period of Uranus. Using observed values for the zonal harmonic J_2 and for the flattening of Uranus (cf. Elliott, 1982), Goody finds for the rotation period of Uranus 16.7 h + 0.5 h from the theoretical relationship between these three quantities. Independent spectrographic measurements (T = 16.31 h + 0.27 h) agree well with the latter value, the sole discrepancy found by Goody (1982) being with the results of Hayes and Belton (1977). However in view of the recent downward revision of the rotation period of Neptune from 22 h (Hayes and Belton, 1977) to 18.2 h (Belton *et al.*, 1981), we assign a low weight to the 24 h rotation period of Uranus found with similar techniques by Hayes and Belton (1977). We conclude that a rotation period for Uranus of about 16.3 h is probably close to the real one.

Although the two-layer models are rudimentary they confirm previous findings (Hubbard and MacFarlane, 1980; Hubbard *et al.*, 1980) concerning the mass of the rocky core M_r. The ratio between M_r and the total planetary mass M is about 0.1 for Jupiter and Saturn and between 0.43 and 0.95 for Uranus, depending on the value of J_4 and T (Table I). For the two-layer models (Table II), the polytropic index n of the envelope is about 1 for Jupiter and Saturn, and about 0.4 for Uranus. For the upper limit of $J_4 = -26 \times 10^{-6}$ no Uranus models were obtained because M_r tends to be larger than M. Since the three-layer models were not able to fit the -26×10^{-6} value either, we conclude that for Uranus $J_4 \lesssim -38 \times 10^{-6}$. The third order level surface theory also automatically yields a value for J_6, shown in Table II.

Within the more restrictive conditions imposed on the three-layer models, there exists only a narrow range of rotation periods of Uranus, or even none, within which observational constraints can be fitted.

The results are shown in Table III and discussed below. The most probable value of $J_4 = -38 \times 10^{-6}$ can be satisfied only for a rotation period between about 14.5–15.5 h, which marginally fits the value quoted by Goody (1982). However, the value of $f = M_i/M_r$, which should be about 2–3 (Hubbard and MacFarlane, 1980; solar value, $f \simeq 3$) is unrealistic: $f \simeq 10$.

The $J_4 = -50 \times 10^{-6}$ figure is fitted for realistic values of f, but the rotation periods are 12–14 h, not in accordance with the previously quoted most probable value of T = 16.3 h. The most probable rotation period is fitted only for a narrow range of

TABLE II

Two-layer models of Jupiter, Saturn and Uranus showing fractional rocky core mass M_r/M, polytropic index n, polytropic constant k of the envelope and gravitational moment of order six J_6

Planet	M_r/M	n	k	$10^{-6} \times J_6$	Observations
Jupiter	0.088	0.934	1.972	33	J
Saturn	0.117	1.016	1.613	75	S
Uranus	0.892	0.376	0.029	0.74	U1, ($T = 16.31$ h, $J_4 = -38 \times 10^{-6}$)
Uranus	0.430	0.450	0.518	1.1	U2, ($T = 16.31$ h, $J_4 = -50 \times 10^{-6}$)
Uranus	0.945	0.356	0.584		($T = 14$ h, $J_4 = -38 \times 10^{-6}$)
Uranus	0.834	0.375	0.032		($T = 14$ h, $J_4 = -50 \times 10^{-6}$)

TABLE III

Results for three-layer Uranus models

$10^{-6} J_4$	f	Y	T (h)	$(M_r + M_i)/M$
−38	9.8	0.27	14.72	0.903
−38	10.8	0.40	15.00	0.890
−50	2.8	0.42	13.00	0.853
−50	2.2	0.75	14.00	0.738
−35	38.9	0.61	16.31	0.849

J_4-values ($-36 \times 10^{-6} \lesssim J_4 < -34 \times 10^{-6}$) but with a completely unrealistic value of $f = 38.9$.

The helium abundance Y in the outer envelope turns out to be for all calculated models considerably larger than its solar value of about 0.2, indicating the presence of elements heavier than helium in the outer parts of Uranus.

4. Conclusions

We have calculated for the first time Uranus models satisfying exactly the observational constraints of mass, radius, surface temperature, rotation period and gravitational moments. Our principal conclusion is that three-layer models encounter serious difficulties for Uranus, and because of its similarity probably also for Neptune. Since no J_4-value is known for Neptune, this planet is not yet suitable for a more accurate study.

From a cosmogonic viewpoint layered planetary models should be best applicable to the outermost planets. Due to the low temperatures the rocky planetesimals can accumulate ices like H_2O, CH_4, NH_3, CO, and form cores of rock and ice that accrete afterwards the remaining H_2 and He as an outer envelope. Despite these arguments, and even with the comfortable observational uncertainties, three-layer models of Uranus fail to fit the observations within the assumed constraints.

More suitable models for Uranus and Neptune may be: (i) two-layer models with a core of rock and ice and an evelope of chemical composition changing with depth, composed mainly of H, He, H_2O, CH_4, NH_3, CO, etc.; (ii) models without core but variable chemical composition. Such models can be contrived by fitting the equation of state of the proposed constituents, with appropriate composition gradients, to the polytropic models of Table II.

Acknowledgements

This research was supported in part by NASA Grants NAGW-192 and NSG-7045.

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