THE EVOLUTION OF THE LUNAR ORBIT REVISITED. I

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Abstract. After recalling the contribution of Halley, J. Kepler, and G. Darwin to our understanding of the secular acceleration of the Moon, we establish a set of differential equations for the variation of the semi-major axis, and the inclination of the Moon on the maximum area plane. These equations are obtained without expanding the disturbing function, due to the tidal bulge, in term of the elliptic elements. The equations thus obtained are simple enough to allow us a qualitative discussion of the solution, followed by a numerical integration.

The results obtained show the Moon was in the distant past in a retrograde orbit, approaching the Earth, its inclination increasing towards 90°; once after a closer approach to the Earth, the Moon receeded and it will finally reach an equilibrium point, the orbital and the equatorial planes being blended.

The solution of the equations appears as a fascicle of curves, becoming extremely dense as we come nearer to the present. Owing to the high sensitivity of the solution to the initial conditions, a weak disturbance added to our modeled forces may lead to a past situation very different from the conclusion drawn by Goldreich (1966) and MacDonald (1964); the minimal approach distance could be greater than 10 Earth's radii.

1. Introduction

Numerous papers have been (and will be) published which are concerned with the Moon's orbit and, more precisely, its evolution under the effects of the tides.

As a matter of fact, E. Kant, followed by Lalande in 1774, was the first to point out, in 1754, that the tides had to be the cause of the lengthening of the day.

However, these ideas were expressed with the aim of explaining the apparent secular acceleration of the Moon's longitude, namely the variation of the Moon's mean motion.

Finally, P. S. Laplace announced in 1787 (exactly one century after the first edition of Newton's *Principia Mathematica*) he had come to an explanation which was in perfect agreement with the observations. Unfortunately for Kant and Lalande and some others, the calculations of Laplace were only in terms of Newtonian mechanics without any tidal hypothesis.

Some decades later, Airy and Hansen increased the observed values while C. Delaunay and Adams decreased the theoretical value found by P. S. Laplace.

We must wait until the end of the last century, and for the work of G. H. Darwin to get a global treatment of the effects of the tides on the Moon's orbit and on the Earth's rotation. Darwin concluded that in the distant past, the Moon was closer to the Earth than at present and that the duration of the day was shorter. But these conclusions depended upon the rheological behaviour of the Earth. In Darwin's theory, the Earth is assumed to be viscous. Modern investigators, such as Gerstenkorn (1955), Goldreich (1966), Kaula (1964), MacDonald (1964), reexamined the problem and their conclusions

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are in accordance with Darwin's theory concerning the present, but are different concerning the past; that is to say, the time when the Moon was closer to the Earth. Moreover, their conclusions differ one from another. This is due to the different patterns used to get the tidal torque between the Earth and the Moon. MacDonald used a constant geometric lag angle, whereas Goldreich and Kaula kept a constant phase lag for all frequencies.

In our opinion, these methods are not free from defects. If we consider the simplest case (i.e., the equatorial circular orbit of the Moon), only the M2 tide acts upon the Moon in the spherical harmonic of the second order. The tidal torque must vanish if the Moon's orbital mean motion (n) and the Earth's angular velocity (ω) become equal. It appears to be true in Darwin's work but not in MacDonald's and Kaula's papers.

These facts led us to keep to Darwin's formulation of a time delay, namely: an interval of time Δt between the stress in the Earth due to the Moon and the moment when the Earth gets its equilibrium figure. The span of tidal frequencies (2×10^{-5}–10^{-6} Hz) allows us to assume no dependence of Δt with the frequency.

Darwin derived a relationship between this time delay and the viscosity coefficient, in accordance with his model of dissipation. We are here only concerned with a mathematical model, while the time delay represents the whole dissipation in the Earth and oceans. In this case, Δt is related to the factor Q (Goldreich and Soter, 1966).

Apart from the introduction of Δt, the tidal torque is computed from the tidal interaction potential, using Love numbers formalism, as in Kaula's (1964) and Lambeck's (1975) papers. It is obtained in arbitrary coordinates of an inertial frame of reference. A derivation of the Fourier expansion of this potential can be found in Mignard (1978).

Though there is no doubt that the tidal dissipation is responsible for the evolution of the Moon's orbit, we are not able today to take precise account of it. Of course, the location of the dissipation in the Earth or (and) in the oceans has an effect upon the tidal torque between the Earth and the Moon. For the ocean tides, it is now well known that most of the energy exchange takes place in the shallow seas, and no information is available about their past location. Furthermore, the generally accepted early history of the Earth supposes that the atmosphere and the hydrosphere originated by outgassing from the inner Earth during the first aeon of its life. The dissipation, then, was not caused by the ocean; so one can imagine that the present state is not the mean one. Besides, the strong value of the secular acceleration of the Moon ($\dot{n} \simeq 30'' \text{ cy}^{-2}$) involves too short a time of evolution. For these reasons, it seems to us interesting and reasonable to investigate the effect of a solid Earth by means of the Love numbers and a time delay.

This paper is composed of four sections. Section 1 contains the computation of the part of the tidal torque from which secular terms in the Moon's motion are arising. Section 2 follows, with the derivation of dynamical equations, restricted to the circular case, using an average process on the precessional period. The constancy of the angular momentum reduces the problem to a second-order one and the trajectory can be visualized in a two-dimensional phase space. In Section 3, we attempt to offer a qualitative discussion of the set of the trajectories and, in addition, some properties are established for the cases of the Moon and Triton. In this section, we shall discuss the problem set by the initial conditions. Section 4 is devoted to a numerical integration of this second-order system of equations.

2. Tidal Torque

The action of the Moon upon a fluid element on the Earth can be studied with the help of a second-order limited expansion of the potential

$$U_0 = Gm^* \frac{r^2}{r^{*3}} P_2(\cos s) = \frac{Gm^*}{2r^{*5}} [3(\mathbf{r} \cdot \mathbf{r}^*)^2 - r^2 \cdot r^{*2}], \tag{1}$$

where s is the angle between the direction $\mathbf{r}^*$ to the Moon and the one $\mathbf{r}$ to the point where the potential is calculated. If we suppose that the deformation is synchronous with the stress, the theory of Love numbers yields to an additional Earth gravity field whose value at the Earth's surface is given by

$$U(\mathbf{R}_E) = \frac{k_2 Gm^*}{2r^{*5}} [3(\mathbf{R}_E \cdot r^*)^2 - R_E^2 \cdot r^{*2}], \tag{2}$$

where R_E is the equatorial radius of the Earth and k_2 the second degree Love number.

Outside of the Earth, the potential will be the exterior solution of the Laplace's equation, with the boundary value given by (2). That solution is unique and (cf. Kellogg, 1967) given by

$$U(\mathbf{r}) = U(\mathbf{R}_E) \frac{R_E^3}{r^3},$$

$$U(\mathbf{r}) = k_2 \frac{Gm^* R_E^5}{2r^{*5} \cdot r^5} [3(\mathbf{r} \cdot \mathbf{r}^*)^2 - r^2 \cdot r^{*2}]. \tag{3}$$

Equation (3) gives the additional potential in a suitable form to study the disturbance in the motion of an artificial satellite, due to the tidal bulge of the Earth. This method was generally used to obtain the value of k_2 (Lambeck *et al.*, 1974).

In this case, the retardation of the deformation has no important effect on the computation. But we are here interested in the evolution of the Moon, and the secular terms arise only if we take account of the delay in the response of the Earth.

For a rigid Earth, instantaneously distorted, we have the additional potential at time t, at point $\mathbf{r}$, with the Moon's coordinate $\mathbf{r}^*$, given by Equation (3)

$$U = F(\mathbf{r}, \mathbf{r}^*).$$

For an inelastic response of the Earth, we introduce the time delay

$$U = G(\mathbf{r}, \mathbf{r}^*) = F(\mathbf{r}_1, \mathbf{r}_1^*),$$

with

$$\mathbf{r}_1 = \mathbf{r}, \quad \mathbf{r}_1^* = \mathbf{r}^*(t - \Delta t) + \boldsymbol{\omega} \Delta t \times \mathbf{r}^*;$$

$\boldsymbol{\omega}$ representing the rotation vector of the Earth; and Δt, the time delay. Present value of

Δt is about 10 mn, so the quantities $(\mathbf{r}_1 - \mathbf{r})$ and $(\mathbf{r}_1^* - \mathbf{r}^*)$ are small enough to infer G from F by using a first-order expansion in Δt,

$$\mathbf{r}_1^* = \mathbf{r}^* - \mathbf{v}^* \cdot \Delta t, \qquad \mathbf{v}^* = \frac{\mathrm{d}\mathbf{r}^*}{\mathrm{d}t}.$$

All computations have been made and we obtained for the additional part of the potential in the first order in Δt the equation

$$V(\mathbf{r}, \mathbf{r}^*) = -3 \frac{k_2 G m^* R_E^5}{r^5 r^{*5}} \cdot \Delta t \left\{ (\mathbf{r} \cdot \mathbf{r}^*)[\mathbf{r}^* \cdot (\boldsymbol{\omega} \times \mathbf{r}) + \mathbf{r} \cdot \mathbf{v}^*] - \frac{(\mathbf{r}^* \cdot \mathbf{v}^*)}{2 r^{*2}} [5(\mathbf{r} \cdot \mathbf{r}^*)^2 - r^2 r^{*2}] \right\}. \tag{4}$$

The expression of the force acting upon a point mass (with mass unity) as well as the tidal torque is easily carried out from the Equation (4)

$$\mathbf{F} = \operatorname{grad}_r V, \qquad \mathfrak{T} = \mathbf{r} \times \mathbf{F},$$

$$\mathbf{F} = 3 \frac{k_2 G m^* R_E^5}{r^5 r^{*5}} \Delta t \left\{ 5 \frac{\mathbf{r}}{r^2} \left[(\mathbf{r} \cdot \mathbf{r}^*)[\mathbf{r}^* \cdot (\boldsymbol{\omega} \times \mathbf{r}) + \mathbf{r} \cdot \mathbf{v}^*] - \frac{(\mathbf{r}^* \cdot \mathbf{v}^*)}{2 r^{*2}} \cdot [5(\mathbf{r} \cdot \mathbf{r}^*)^2 - r^2 r^{*2}] \right] - [\mathbf{r}^* \cdot [\mathbf{r}^* \cdot (\boldsymbol{\omega} \times \mathbf{r}) + \mathbf{r} \cdot \mathbf{v}^*] + (\mathbf{r}^* \times \boldsymbol{\omega} + \mathbf{v}^*)(\mathbf{r} \cdot \mathbf{r}^*)] + \frac{(\mathbf{r}^* \cdot \mathbf{v}^*)}{r^{*2}} [5 \mathbf{r}^* (\mathbf{r} \cdot \mathbf{r}^*) - \mathbf{r} r^{*2}] \right\},$$

$$\mathfrak{T} = -3 \frac{k_2 G m^* R_E^5}{r^5 r^{*5}} \Delta t \left\{ (\mathbf{r} \times \mathbf{r}^*)[\mathbf{r}^* \cdot (\boldsymbol{\omega} \times \mathbf{r}) + \mathbf{r} \cdot \mathbf{v}^*] - 5 \frac{\mathbf{r}^* \cdot \mathbf{v}^*}{r^{*2}} (\mathbf{r} \times \mathbf{r}^*)(\mathbf{r} \cdot \mathbf{r}^*) + (\mathbf{r} \cdot \mathbf{r}^*)[(\mathbf{r} \cdot \boldsymbol{\omega}) \cdot \mathbf{r}^* - (\mathbf{r} \cdot \mathbf{r}^*) \boldsymbol{\omega} + \mathbf{r} \times \mathbf{v}^*] \right\}.$$

But in our case, the tide-raising object – the Moon – is also the body of which the motion is studied; namely $\mathbf{r}^* = \mathbf{r}$. Then, the two previous equations become simpler; and taking account of the fact that the mass of the Moon is different from unity, we obtain

$$\mathbf{F} = -3 \frac{k_2 G m^2 R_E^5}{r^{10}} \cdot \Delta t [2\mathbf{r}(\mathbf{r} \cdot \mathbf{v}) + r^2 (\mathbf{r} \times \boldsymbol{\omega} + \mathbf{v})], \tag{5}$$

$$\mathfrak{T} = -3 \frac{k_2 G m^2 R_E^5}{r^8} \Delta t [(\mathbf{r} \cdot \boldsymbol{\omega}) \mathbf{r} - r^2 \boldsymbol{\omega} + \mathbf{r} \times \mathbf{v}], \tag{6}$$

The Equations (5) and (6) held for the additional force and torque, caused by the time delay, acting upon the Moon.

For a circular and equatorial orbit, we find the classical result, in putting

$$(\omega - n) \Delta t = \delta,$$

$$\mathfrak{T} = \frac{3k_2 G m^2 R_E^5}{2r^6} \sin(2\delta),$$

where δ = phase lag.

The torque acting upon the Earth is exactly the opposite of the torque given by Equation (6), with the hypothesis of an isolated two-body problem.

3. Dynamical Equations

As soon as we know the tidal torque, it is easy to write the general equations of the motion for the Moon's and Earth's angular momenta.

If $\mathbf{H}_M$ and $\mathbf{H}_E$, denote respectively, the Moon's and the Earth's angular momenta, we have

$$\frac{d\mathbf{H}_M}{dt} = \mathfrak{T}, \tag{7}$$

$$\frac{d\mathbf{H}_E}{dt} = -\mathfrak{T}. \tag{8}$$

Thus, there are six first-order differential equations. Three integrals for the constancy of the total angular momentum are easily exhibited

$$\mathbf{H}_M + \mathbf{H}_E = \mathbf{H}, \tag{9}$$

which reduces the system to a third-order one. The variables $\mathbf{H}_E$ and $\mathbf{H}_M$ are related to the variables $\boldsymbol{\omega}$, $\mathbf{r}$, $\mathbf{v}$ by the equations

$$\mathbf{H}_M = m\mathbf{r} \times \mathbf{v},$$

$$\mathbf{H}_E = C \cdot \boldsymbol{\omega},$$

where C is the principal momentum of inertia of the Earth around its rotation axis. With the help of Equation (9), Equation (6) can be written as

$$\mathfrak{T} = -\frac{3k_2 G m^2 R_E^5}{Cr^8} \Delta t \cdot \left[(\mathbf{r} \cdot \mathbf{H})\mathbf{r} - r^2 \mathbf{H} + r^2 \left(1 + \frac{C}{mr^2}\right) \mathbf{H}_M \right]. \tag{10}$$

All the variable parameters which concern the Earth rotation evolution have vanished and the solution of Equations (7) and (8) is reduced to the integration of Equation (7) and algebraic computation with the help of (9). In these equations we have neglected the mass of the Moon with respect to that of the Earth.

Hitherto, we did not need to choose a frame of reference because all formulas were expressed in terms of vectorial notation. To derive a set of scalar differential equations, we are forced to project the previous equations on a coordinate system. The best choice is made by using the z-axis lying along the total angular momentum; that allows us to carry out the so-called elimination of nodes. The ascending node of the orbit of the Moon and the descending node of the Earth's equator are blended. Furthermore, the constancy of $\mathbf{H}$ involves that the frame of chosen reference is inertial.

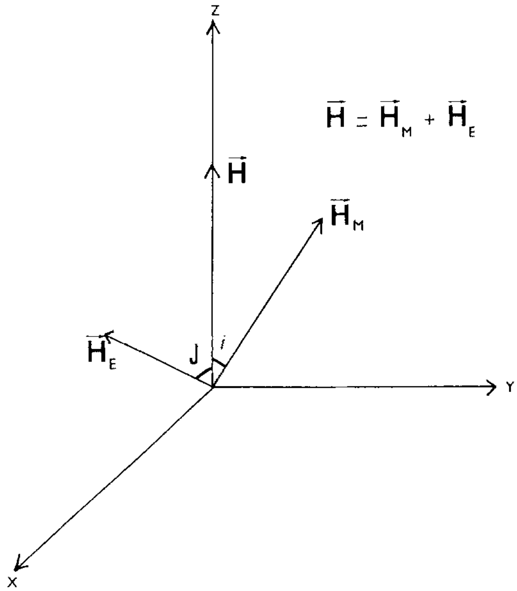

Fig. 1. Coordinate system.

We call i the inclination of the Moon's orbit on the xy-plane and J the inclination of the Earth's equator on the same plane. With this coordinate system, we have the inclination of the Moon's orbit on the equatorial plane equal to $I = i + J$. Hereafter, the xy-plane will be called the absolute plane. Besides, between this set of variables, the following relations hold:

$$\left[\begin{array}{l} H_{\mathrm{E}} \cos J + H_{\mathrm{M}} \cos i \;=\; H, \\ H_{\mathrm{E}} \sin J - H_{\mathrm{M}} \sin i \;=\; 0. \end{array}\right. \tag{9'}$$

We shall solve this system in terms of (H, H_{M}, i) to find (H_{E}, J). Let us derive two scalar equations for the variation of the semi-major axis and for the inclination of the Moon on the xy-plane:

$$\begin{aligned} \mathbf{H}_{\mathrm{M}} \cdot \frac{\mathrm{d}\mathbf{H}_{\mathrm{M}}}{\mathrm{d}t} &= \frac{1}{2}\frac{\mathrm{d}H_{\mathrm{M}}^2}{\mathrm{d}t} = \mathfrak{T} \cdot \mathbf{H}_{\mathrm{M}} \\ &= -3\,\frac{k_2 G m^2 R_{\mathrm{E}}^5}{C r^8}\,\Delta t \Bigg[(\mathbf{r}\cdot\mathbf{H})(\mathbf{r}\cdot\mathbf{H}_{\mathrm{M}}) - r^2(\mathbf{H}\cdot\mathbf{H}_{\mathrm{M}}) \\ &\qquad + r^2\left(1 + \frac{C}{m r^2}\right) H_{\mathrm{M}}^2 \Bigg], \end{aligned} \tag{11}$$

$$\begin{aligned} \mathbf{H} \cdot \frac{\mathrm{d}\mathbf{H}_{\mathrm{M}}}{\mathrm{d}t} &= \frac{\mathrm{d}(\mathbf{H}\cdot\mathbf{H}_{\mathrm{M}})}{\mathrm{d}t} = \mathfrak{T} \cdot \mathbf{H} \\ &= -3\,\frac{k_2 G m^2 R_{\mathrm{E}}^5}{C r^8}\,\Delta t \cdot \left[(\mathbf{r}\cdot\mathbf{H})^2 - r^2 \cdot H^2 + r^2\left(1 + \frac{C}{m r^2}\right)(\mathbf{H}\cdot\mathbf{H}_{\mathrm{M}}) \right]. \end{aligned} \tag{12}$$

But since we are looking for the secular evolution of the Moon's orbit, an averaging of the right-hand sides of these equations has to be carried out. We have

$$\mathbf{r}\cdot\mathbf{H}_{\mathrm{M}} = \mathbf{r}\cdot(\mathbf{r}\times\mathbf{v}) = 0.$$

With $e = 0$, $r = a$ (semi-major axis) only the term $(\mathbf{r}\cdot\mathbf{H})^2 - r^2H^2$ requires some computation: i.e.,

$$\langle(\mathbf{r}\cdot\mathbf{H})^2 - r^2H^2\rangle = \langle H^2\cdot z^2 - r^2H^2\rangle = -H^2\cdot\langle x^2+y^2\rangle.$$

The formulas of the elliptic two-body motion lead to

$$\langle x^2+y^2\rangle = \frac{a^2}{2}(1+\cos^2 i) + O(e^2).$$

By chosing for unity of angular momentum the angular momentum of a satellite having the Moon's mass and revolving at the Earth's surface (grazing satellite), we then define three dimensionless variables

$$X = \frac{H_{\mathrm{M}}^2}{GMm^2R_{\mathrm{E}}}, \qquad Y = \frac{\mathbf{H}\cdot\mathbf{H}_{\mathrm{M}}}{GMm^2R_{\mathrm{E}}}, \qquad T = \frac{H^2}{GMm^2R_{\mathrm{E}}}; \tag{13}$$

but

$$H_{\mathrm{M}} = m(GMa)^{1/2}, \qquad \mathbf{H}\cdot\mathbf{H}_{\mathrm{M}} = H\cdot H_{\mathrm{M}}\cos i,$$

then

$$X = \frac{a}{R_{\mathrm{E}}}, \qquad Y = T^{1/2}X^{1/2}\cos i. \tag{14}$$

We gather (11), (12), (13) and (14) to obtain

$$\frac{\mathrm{d}X}{\mathrm{d}t} = 4\frac{K\Delta t}{X^7}\left[-X^2+XY-\alpha\frac{M}{m}\right], \tag{15}$$

$$\frac{\mathrm{d}Y}{\mathrm{d}t} = \frac{K\Delta t}{X^8}\left[(T-2Y)X^2+XY^2-2\alpha\frac{M}{m}Y\right], \tag{16}$$

where K is a compact form for

$$\frac{G\pi^2k^2(m/M)^2}{\alpha P^2},$$

with M = Earth's mass, $\alpha = C/MR_{\mathrm{E}}^2$, and P = period of revolution of an Earth-grazing satellite.

The present approximate values of these variables in the Moon's case are:

$$X \simeq 60, \qquad Y \simeq 71, \qquad T \simeq 85,$$

$$\alpha \simeq 0.33, \qquad \frac{M}{m} \simeq 81.3, \qquad k_2 \simeq 0.3;$$

$K = 0.58\ 10^7/10^6$ yr with the value of $\Delta t = 10$ mn. The last number is chosen in order to obtain the present rate of the recession of the Moon ($3\ \mathrm{cm\,yr^{-1}}$). The value of Δt determines the time scale of this problem and may have been very different in the past.

Let us put $Z = \cos i$ and perform the change of variable $Y = T^{1/2}\cdot X^{1/2}\cdot Z$ in

Equations (15) and (16). Finally, the two relevant equations which describe the evolution of the Moon's orbit are expressed in terms of X and Z as

$$\begin{aligned} \frac{\mathrm{d}X}{\mathrm{d}t} &= 4K\frac{\Delta t}{X^7}\left[-X^2 + T^{1/2}X^{3/2}Z - \alpha\frac{M}{m}\right], \\ \frac{\mathrm{d}Z}{\mathrm{d}t} &= \frac{K\Delta t T^{1/2}}{X^{13/2}}(1 - Z^2)\,. \end{aligned} \tag{17}$$

The two first-order equations (17) may be solved and the solution is substituted in Equation (9′) in order to compute simultaneously the evolution of the speed of the rotation of the Earth as well as its spin axis orientation.

Let us delay the exact integration to discuss the general features of the solution in the next section.

4. Heuristic Discussion

In the two equations (17), the right-hand side does not contain a time variable and the differential system is autonomous. So, the easiest way to obtain valuable information about the solutions consists in studying the phase space. For that, a very effective method is to eliminate the time and to find out how the trajectories in the (X, Z) phase plane look.

For the system (17), we have

$$\frac{\mathrm{d}Z}{\mathrm{d}X} = \frac{T^{1/2}X^{1/2}(1 - Z^2)}{4\left(-X^2 + ZT^{1/2}X^{3/2} - \alpha\dfrac{M}{m}\right)} \tag{18}$$

Now, as long as the right member of (18) can be determined for the point (X, Z), it provides the slope and only one trajectory goes through this point.

This procedure breaks down in case of indeterminacy which occurs at points of equilibrium, through which many trajectories may go. Therefore, we locate the equilibrium points and determine whether these equilibriums are stable or not.

We obtain these points by setting the second part of system (17) equal to zero. Their coordinates are the roots of the equations

$$Z = +1, \qquad F(X) = +1,$$

$$Z = -1, \qquad F(X) = -1;$$

with

$$F(X) = T^{-1/2}\left[\frac{\alpha\dfrac{M}{m}}{X^{3/2}} + X^{1/2}\right]. \tag{19}$$

The case $Z = -1$ is ruled out because of the positive value of the right-hand part of the

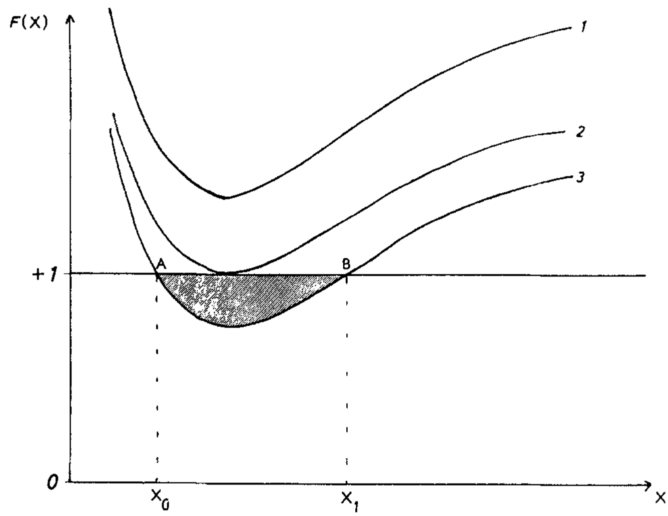

Fig. 2. Locus of point of the closest approach.

last equation. In the plane $(X-Z)$ the representative curve of the $F(X)$ function owns a smaller value and in certain cases intersects the line $Z=1$. Typical aspect of these curves can be seen in Figure 2. An equilibrium point exists only if the minimal value of $F(X)$ is smaller than unity, which is expressed by the condition

$$\frac{256}{27}\alpha\frac{M}{m}\leqslant T^2. \tag{20}$$

Physically this criterion is related to the possibility for the system to reach the synchronous rotation of the planet with the satellite's revolution. Let us now pay attention to some limiting cases.

First, assume that the major part of the angular momentum is generated by the planet's rotation. Hence, we neglect the orbital motion

$$T\simeq\frac{(C\omega)^2}{GMm^2R_{\mathrm{E}}}=\left(\alpha\frac{M}{m}\right)^2\frac{P^2}{D^2},$$

where D is the planet's rotation period.

In the solar system, the Neptune-Triton pair belongs to this class. Then, Equation (20) becomes

$$\left(\alpha\frac{M}{m}\right)^3\frac{P^4}{D^4}\geqslant\frac{256}{27}. \tag{21}$$

This inequality is fully satisfied for the Neptune-Triton system.

Conversely, we now neglect the planet's angular momentum with respect to that of the

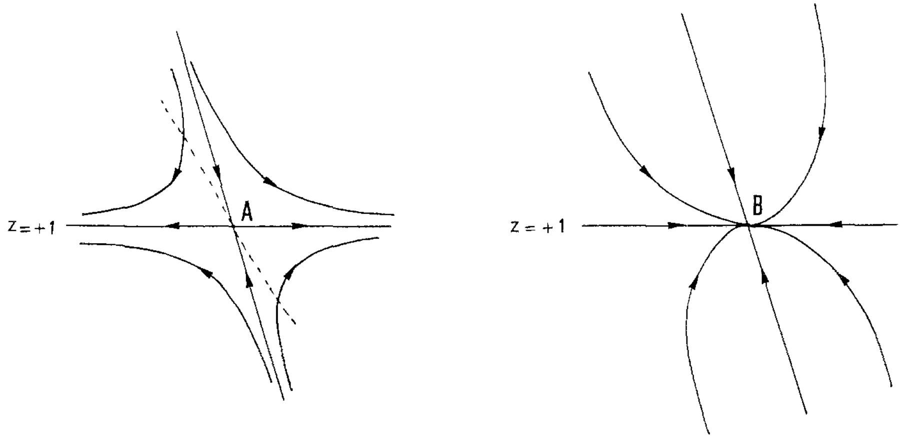

Fig. 3. Behaviour of the trajectories in the vicinity of the singular points.

satellite and Equation (19) becomes

$$\left(\frac{a}{R_E}\right)^2 \geqslant \frac{256}{27}\alpha\frac{M}{m}. \tag{22}$$

In the Earth–Moon system, 80% of the angular momentum is embodied in the orbital motion of the Moon, the criterion can be used and it is also satisfied.

In both previous cases, the system (17) owns two equilibrium points given by the roots of the equation

$$X^2 - T^{1/2}X^{3/2} + \alpha\frac{M}{m} = 0; \tag{23}$$

the value of Z being unity.

Hereafter, we name the point A $(X_0, 1)$ and B $(X_1, 1)$ with (X_0, X_1) the positive roots of Equation (23), and $X_0 \leqslant X_1$. Another feature of the trajectories is seen in Figure 2. Inside the dashed area bounded by $Z = +1$ and $F(X)$ the slope $\mathrm{d}Z/\mathrm{d}X$ is positive; it is negative elsewhere. The curve $F(X)$ is the locus of the points where $\mathrm{d}Z/\mathrm{d}X$ is infinite. Then, along the trajectories which intersect this curve, there is a minimal value of X, reached for the value $Z = F(X)$. The meaning of the closest approach can be understood easily. In the circular case of a satellite in the equatorial plane, the minimal approach corresponds to the synchronous rotations of the satellite and the planet. For a nonzero inclination, Equation (23), with the help of the first two integrals (9′), is transformed in

$$\omega \cos I = n.$$

It is a generalization of the synchronous condition as was pointed out by MacDonald (1964).

The study of the associated linear system in the vicinity of the points A and B, allows

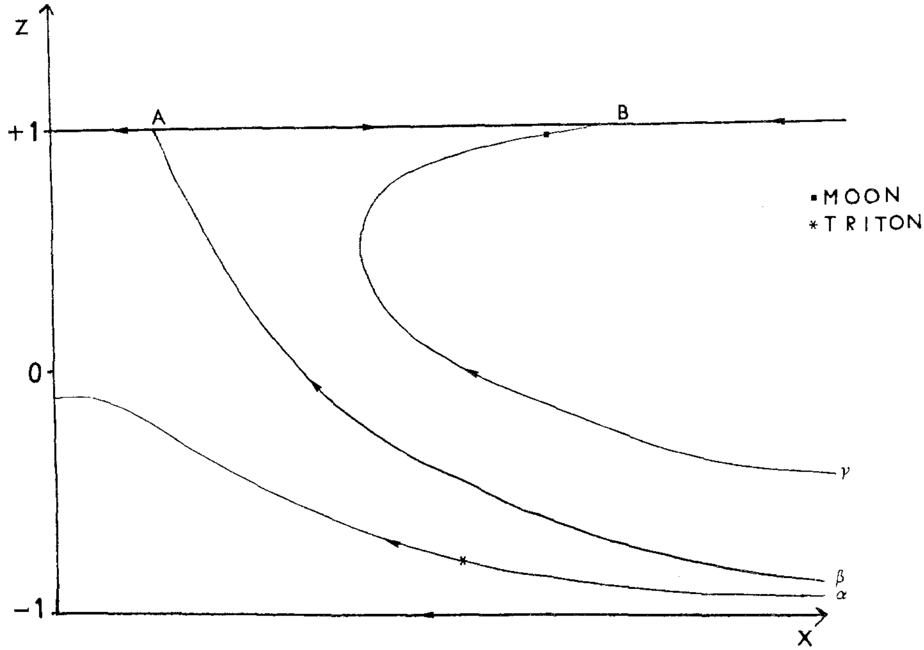

Fig. 4. Qualitative set of trajectories. Cosine of the inclination on the absolute plane versus the semi-major axis. (The unity is one Earth radius.)

us to determine the character of these equilibrium points. After some straightforward manipulations in linear equations, we conclude that the point A is an hyperbolic point whereas B is a node, a quasi-degenerate node for the Moon, the eigenvalues being given by

$$\lambda_1 = \frac{4}{X^7}\left[-2X + \tfrac{3}{2}X^{1/2}T^{1/2}\right],$$

$$\lambda_2 = -2\frac{T^{1/2}}{X^{13/2}}.$$

The value of X being X_0 or X_1 according to be in question A or B. In the Figure 3, we have crudely plotted some trajectories in the neighbourhood of A and B. The dashed line represents the curve $F(X)$.

All information now available allows us to sketch in Figure 4 the set of trajectories in the phase space, restricted to the domain included between $Z = 1$ and $Z = -1$, which only has a practical interest.

In this diagram, we have represented the curve α, β and γ; α and γ are samples of two classes, whereas β is unique.

In Figure 4, the orientation indicated on the trajectories agrees with a positive sense of time towards the future, and we have considered the criterion (20) to be satisfied.

Some interesting remarks can be made. If at the outset the satellite is retrograde and located in the equatorial plane, it stays in that plane, approaching the planet to end by collapsing on its surface.

For a slightly different initial condition ($Z \simeq -1$) the dramatic end remains, whereas the orbital plane leaves the equatorial plane.

A numerical investigation showed us that in the current state, Triton is represented by an α trajectory. So the semi-major axis is decreasing, while the period of rotation of Neptune is increasing. The orbital plane moves away from the equatorial plane.

The γ curve is the most interesting for us, because the past and the future evolutions of the Moon's orbit are linked to this curve.

Today, the recession of the Moon from the Earth is an evidence for its evolution; no catastrophic end has to be feared unless the act of the Sun (which will become relatively more important with respect to the Earth's one as the Moon is approaching B) changes it to an α evolution. From the position of the Moon on the γ curve, two important conclusions can be drawn. First the semi-major axis and the inclination of the orbital plane have suffered from considerable variations in the past and the minimal approach of the Moon has already occurred (no allowance is made for the Moon's age!)

Secondly, the present position of the Moon is so close to the point B that a slight modification of the initial condition – that is, the today state – involves a very different result for the past, especially in the value of the minimal distance.

The trajectory β is the only way to reach the hyperbolic point A. It is a critical curve which divides the phase plane in two regions; one filled by the α like curves, the other by the γ like curves. So, the situation of the initial point with respect to the β curve determines the kind of evolution.

It remains the trajectory $Z = +1$ which leads to the well-known result for the evolution of the Moon, assumed to be orbiting in the equatorial plane in a direct motion. The three usual cases are visible. For an initial state between the origin and A, the planet attracts the satellite which falls in spiraling on the primary. In all the other cases, the point B constitutes the final state.

5. Numerical Integration

The Equations (17) are numerically integrated in detail in the case of the Moon and some runs are devoted to Triton's one. The routine used is AMC1 (Adams–Moulton–Cowell for first-order system) prepared by N. Borderies and L. Castel at C.N.E.S. in France.

The principal problem in this integration lies in the choice of initial conditions. The present state of the Earth–Moon system is far from being an exact picture of the tidal evolution. The gravitational disturbance of the Sun has gradually obliterated the evolution of the isolated two-body system. From a mathematical point of view, the complete solution of Equation (17) requires two kinds of data:

(1) the coordinates (X, Z) of the starting point, and (2) the values of the different parameters which are present in Equation (17).

Our goal is trying to find the orbit of the Moon in the distant past, we are, therefore,

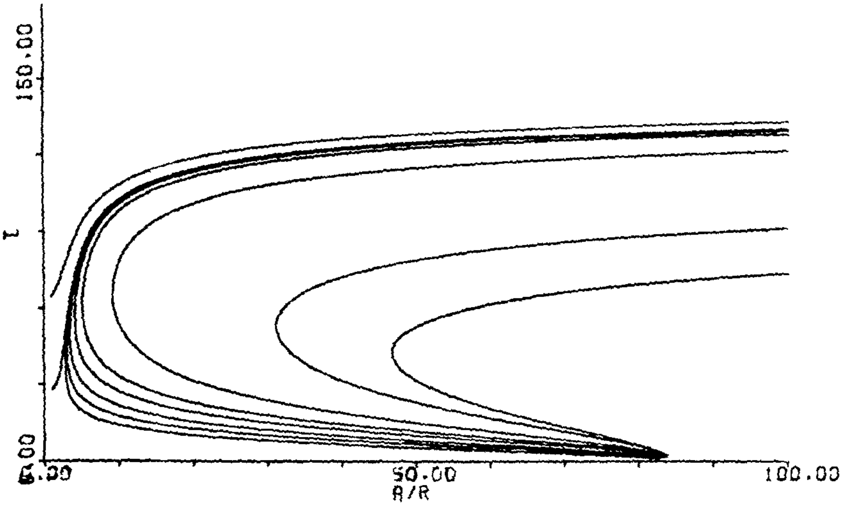

Fig. 5. Variation of the inclination of the Moon on the absolute plane versus the semi-major axis. (The unity is one Earth radius.)

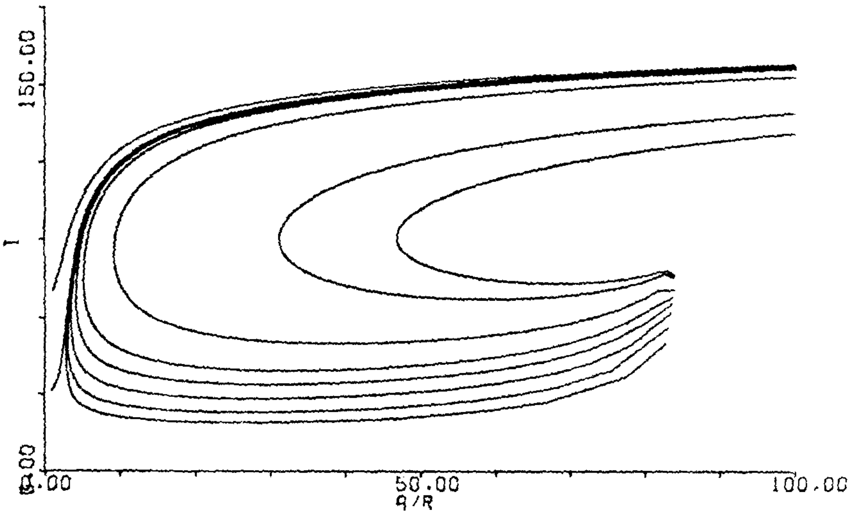

Fig. 6. Variation of the inclination of the Moon on the Earth equator versus the semi-major axis. (The unity is one Earth radius.)

looking for the present value which would be taken by the parameters, if their evolution has only been caused by tidal interaction between the Earth and the Moon. For example, it is not sure that the best value for the present duration of the day is 24 h; other effects such as core-mantle coupling, tidal interaction with the Sun, variation of G, may produce an accelerating or decelerating mechanism.

TABLE I

Variation of the value of the closest approach for different initial inclinations. X is taken equal to 60 at the starting point

$\cos i$	a/R	$\cos i$	a/R
0.998	2.7	0.988	8.6
0.996	3.2	0.975	17
0.993	4.0	0.960	34

The same question is asked for the inclination of the Moon on the absolute plane. The present value of the mean inclination of the Moon's orbit on the Earth's equator is given by a simple averaging process:

$$\langle \cos I \rangle = \cos i_1 \cos \epsilon,$$

with i_1 and ϵ being, respectively, the inclination of the Moon's orbit on the ecliptic plane and the ecliptic obliquity.

So blending $\langle \cos I \rangle$ and $\cos \langle I \rangle$, which is a valuable approximation, we find $I \simeq 24°$. By use of Equation (9′), we obtain $T = 85$, the period of rotation of the Earth being taken equal to 24 h, and the semi-major axis to 60.3 Earth's radii.

In order to take account of the bad determination of the present state of the Earth–Moon system (in the sense explained in the previous paragraph), we tried many computer runs with different assumptions about the values of I, T, Z. As foreseen after the heuristic discussion, Z is the more critical parameter. The current value is $Z = 0.998$ and we scanned about this value to obtain Figures 5 and 6.

The results plotted on these figures are given with $T = 85$ and no important modification occurs if one changes this value. The integration was performed backward and forward in the time and stopped when $a/R_E = 100$ in the past and in the future near $a/R_E = 0$ for α curves and a/R_E near the synchronous state which would be reached at the point B for the γ like curves.

The inclination of the Moon on the absolute plane tends to zero (the greatest part of the angular momentum being borne by the orbital motion of the Moon); whereas, with respect to the equator, the inclination seems to increase.

In fact, if we shorten the step of integration, the variation of this inclination is inverted and goes to zero at the point B. However, as long we neglect the Sun's effect, there is no point in studying the trajectories very close to the point B. The coordinates of B being $X = 84.4, Z = +1$, we found from a rough numerical calculation that the Sun will change drastically our results for $X > 84.1$.

An examination of the trajectory patterns shows the very important effect caused by a small change in the Z value. In Table I, we read the different values of the closest distance in the case of different Z values.

Let us assume a retarding mechanism in the rotation of the Earth which would have occurred during a long interval of time; this is not included in our equations. Then the angular velocity of the Earth would be greater in the past than the calculated one.

The same conclusion could be drawn for the closest approach. The retardation of the Earth due to the Sun goes in this respect.

Hence, it is not absurd to consider that the genuine position of the Moon has never been so close to the Earth as the formal lower limit derived from the present state; and the difficulty arising from the Roche limit would be avoided.

6. Concluding Remarks

As shown in this study, it is quite imaginable that in the past the Moon was orbiting in a retrograde way. A close approach of the Earth without involving a path crossing the Roche limit is found as a solution of our simple model.

From the above research, we cannot say the time-scale problem and the minimal approach problem are solved. But the same medicine may be used for both. The new problem to be solved is formulated in the question: how could the inclination of the Moon on the absolute plane be more radically increased? We have only speculative answers to that question and in a forthcoming paper we intend to test their validity.

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