THE RAMSEY PHASE-CHANGE HYPOTHESIS*

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Abstract. The series of papers by the late W. H. Ramsey developing on a mathematical and physical basis the phase-change theory of the terrestrial core are summarised, and the numerous remarkable successes of the theory in accounting for hitherto unexplained properties of the interior of the Earth are emphasised. Subsequent attack on the problem by a different approach, based on established seismic data, confirmed independently the correctness of the general Ramsey-theory, and enabled it to be developed in numerical detail to demonstrate the precise nature of the evolution of the Earth. This treatment shows that through radioactive heating the Earth, starting in all-solid form, develops conditions at its centre suitable for the change of phase to take place, and there then occurs the sudden Ramsey-collapse involving formation within a matter of minutes of a metallic liquid core of 'large' radius (in fact of 2042 km radius and just over 6% of the mass of the Earth). With further increase of temperature at the core-mantle boundary, evolution proceeds continuously along a stable series of configurations of increasing core-mass and core-radius, and *decreasing* overall surface-radius. A steadily decreasing moment-of-inertia accompanies the general contraction of the planet and at a rate in close agreement with that required by the intrinsic accelerative component of the angular velocity of the Earth revealed by the ancient-eclipse data. The total decrease of surface-radius since the initial sudden collapse has been by about 300 km, which involves a reduction of surface-area by some $50 \times 10^6\,\mathrm{km}^2$, and a redistribution by means of folding and thrusting of about $160 \times 10^9\,\mathrm{km}^3$ of outer-layers material, values quite adequate to account for some twenty separate periods of mountain-building during the age of the Earth. The iron-core hypothesis has nothing to offer by way of solution of any of these problems, and must be assigned negligible weight in comparison with the phase-change hypothesis.

It is now more than twenty-five years since the late W. H. Ramsey in a remarkable series of papers (Ramsey, 1948, 1949, 1950a, b, 1954) advanced powerful arguments for the phase-change interpretation of the nature of the terrestrial core. Prior to his time it seems gradually to have come to be believed that the Earth contained a large proportion of iron and nickel in order to explain the high mean-density of $5.52\,\mathrm{g\,cm^{-3}}$ arrived at when the constant of gravitation had been experimentally measured,** but the various arguments for this have little force when looked at with other than the eye of faith.

Thus iron itself was considered an abundant terrestrial element and therefore the most probable constituent, but it is far less abundant than supposed if the core is not iron, and the argument is seen to be in some degree circular. Strong support was also considered to come from the chemical compositions of meteorites, despite their origin being one of the most inscrutable and little-understood problems. Of meteorites actually *seen to fall,* less

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* Paper dedicated to Professor Hannes Alfvén on the occasion of his 70th birthday, 30 May, 1978.

** [That the density was between 5 and 6 times that of water was estimated by Newton from the relative masses of the planets, but he made no identification with iron. *Principia*, Book III, Propns VIII & X.]

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than 3% are irons, but as irons are much more readily recognised as meteorites than are stones most museum-specimens are irons, and so give the impression of being more abundant. However, the core of the Earth contains 31% of the total mass, and allowing for iron elsewhere, both free and in combination, would imply a percentage approaching 40. Thus from this point of view the meteorite-argument would if anything tell against an iron core. In addition, for meteorites reaching the Earth now, the core of some hypothetical planet similar to the Earth would be the last place from which they could be extracted, covered as they would be by some 3000 km of solid rock! Then again it was obviously tempting to ascribe the magnetic field of the Earth to the presence of iron, but it is now known that the effect of pressure on the Curie-points of iron and iron-nickel alloys is much too small to allow ferromagnetism to persist at the temperatures prevailing in the Earth. Of course a dynamo-mechanism within the core (driven by precession?) for the general magnetic field would not thereby be ruled out, indeed such seems essential, but a phase-change to metallic form, which is Ramsey's proposal, would provide material equally as suitable as iron for any such process. There is some evidence of almost an obsession with iron: even Eddington in his first attempts on the problem of stellar structure, for lack of any indication otherwise, adopted the molecular weight of iron for that of stellar material, and it occurs in the theory to the fourth power! But when the importance of ionisation was realised, the precise identification became less relevant, though his original supposition delayed recognition of the extent to which hydrogen was really the principal stellar constituent.

The Moon, despite its adjacency to the Earth, cannot contain a proportion of iron in the least comparable with that suggested by the mass of the terrestrial core (31%), for if it did the lunar radius would be some hundred kilometres less, and the same holds for Mars, the mean-density of which is as low as $3.9\,\mathrm{g\,cm^{-3}}$. But the most fatal blow is that seismic data show that, whatever the chemical composition of the core-material may be, its uncompressed density only slightly exceeds $6\,\mathrm{g\,cm^{-3}}$, whereas iron and nickel even in liquid form have densities 15% and 30% greater respectively. Attempts have been made to identify the core-material as iron by means of shock-wave data, but the physical arguments presented are thermodynamically quite unsound, and if accepted prima facie as correct would lead to the absurd conclusion that the mantle is made of aluminium. We shall not herein be concerned with the great outer planets, the mean-densities of which are so low that they must be composed mainly of the lightest gases, especially Jupiter and Saturn, though even in these phase-changes must be of importance so high are the internal pressures.

To describe briefly the nature of the phase-change idea, it may be noticed first that material cannot have infinite strength, and a breakdown of atomic structure is to be expected if sufficient pressure is applied. By discarding the outer shell of electrons, atomic volume can be much reduced, and the process involves the loss of at least part of the molecular and crystal bindings. But if the pressure is high enough, the work done by it during compression will more than compensate for the loss of binding, and when equilibrium is next achieved the substance will be a metal of high density. The transition from a molecular phase to a metallic phase is essentially different from mere ionisation as occurs in a stellar gas. Theoretical study of the transition had been made but only for solid hydrogen prior to Ramsey's work, and this showed that at a pressure of 0.7 x 10^{12} dyn cm^{-2} (and at negligible temperature) the density suddenly increases from about 0.4 g cm^{-3} in the normal state to 0.8 g cm^{-3} in the metallic phase. Other calculations have arrived at an even greater increase. The large jump in density is especially to be emphasised, for when - it comes to planetary structure an increase by a factor in excess of 1.5 is critical, as will be seen later. An increase by such a factor actually occurs across the mantle-core boundary in the Earth, and the claim of the phase-change hypothesis is that the core-material is chemically the same as that of the surrounding mantle.

It was shown by Ramsey on general arguments that for most substances the critical pressures for transition to a metallic phase to occur are of the order of 10^{12} dyn cm^{-2}, rather as ionisation-potentials are of much the same order for many elements, though there are elements for which both metallic and ordinary phases exist at low pressure, as for instance tin and arsenic. Theoretical calculations of the critical pressures for complex substances such as terrestrial rocks are beyond present capabilities, but recently there has emerged from the work of Mao and Bell (1976) and others the highly interesting possibility of reaching experimentally *static pressures* equal to those occurring in the core, which range from 1.37 x 10^{12} dyn cm^{-2} at the boundary up to about 3.6 x 10^{12} dyn cm^{-2} at the centre. This is done by means of a diamond-anvil press, as the instrument or apparatus is termed, and as yet pressures up to only about 0.5 x 10^{12} dyn cm^{-2} have been utilised, but these already show that molecular bonds are destroyed for some substances. The consequences of moderately high pressures, such as the foregoing, are first being studied, but the achievement of still higher pressures are considered attainable, and it is expected that before not too long conditions comparable with those obtaining at the core-mantle boundary will be achieved. The apparatus is so constructed that the diamond-elements compressing the sample are transparent to laser-beams, and thus temperature can also be adjusted to simulate likely conditions in the core.

One of the most remarkable features of the core-mantle boundary is its extreme sharpness, which is difficult if not impossible to reconcile with a change in composition, though readily understandable within the phase-change theory. The reflecting power of the interface for seismic waves is found to be wellnigh perfect, and as Jeffreys (1945) has shown this is not what would be expected if the density merely increased continuously over a small depth, which is what a change in composition would involve. Ramsey himself showed that on this latter basis, at the temperature prevailing in the core (possibly approaching 10 000 K), the density-jump would not be sharp at all but instead spread over a depth of some 300 km or more, and this would be quite inconsistent with the observed properties of seismic waves at the interface. Ramsey also pointed out that if it were by any means possible to achieve so complete and sharp chemical separation, as the seismic data require, then the elements of the core-material should be very rare elsewhere in the Earth, which of course is known not to be the case for iron at the surface. The formation of an iron core requires the postulate of an initially molten Earth, but it now seems certain that the terrestrial planets and the Moon accumulated in all-solid form from solid dust-particles, giving no opportunity for any serious differentiation to occur. Nor could the present known liquid core be the remaining uncooled portion of an originally all-molten Earth, for conduction is so slow that cooling to this depth could not yet have taken place. On the phase-change theory, radioactive energy-release, aided by the consequent gravitational energy-release through contraction of the planet, is the agency, under the high-pressure conditions, responsible for the liquid nature of the core. As the elements concerned have half-lives comparable with the age of the Earth, the core is still growing, as will later be seen.

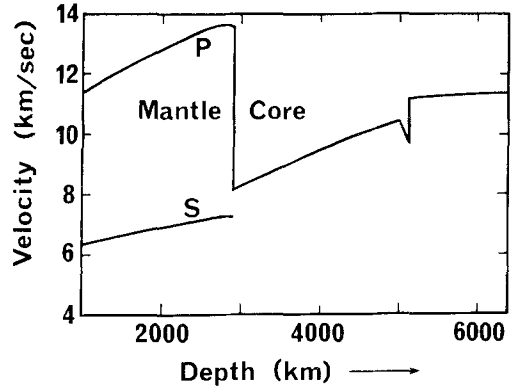

Fig. 1. Showing the unexpected behaviour of the seismic wave-velocities in the mantle as the core-boundary is approached. The rate of increase not only diminishes, but the velocities show a slight decrease immediately outside the boundary for the P- and S-waves. The anomaly does not occur on the high-pressure side in the core. The P-waves behave similarly at the boundary of the small inner core (right-hand side of diagram).

These phenomena are explicable if the boundary corresponds to a phase-change, but not if it were a change of composition.

Yet another remarkable success of the phase-change theory is in explaining what hitherto had been curious anomalies in the velocities of P- and S-waves in the mantle near the core-boundary. Both velocities fail to go on increasing at the rate expected as the boundary is approached, while strangely no such irregularity is found to occur in the velocity of the P-waves immediately inside the boundary. (Figure 1) The phase-change theory readily accounts for these apparent anomalies by showing that immediately outside the core-boundary the value of the bulk-modulus will be lowered by about 1%, and the discrepancy diminish exponentially with height above the boundary, partly because the temperature decreases outwards but mainly because the excitation-energy increases. The rigidity will show a similar discrepancy of comparable amount. Moreover the anomaly occurs only on the lower-pressure side of the transition, for on the high-pressure side the excitation-energy is necessarily always very large as a result of the large change in volume which is characteristic of the phase-change to metallic form. These seemingly strange properties of the seismic velocities were already well known but had remained without satisfactory explanation, and a principal triumph of Ramsey's work was in at long last providing a physical explanation of them. The phenomenon recurs at the boundary of the small core-within-the-core of radius 1250 km and where the pressure has risen to about 3.27×10^{12} dyn cm^{-2}, suggesting that this too may be a further phase-change.

When it comes to applying the phase-change hypothesis to the mechanical equilibrium of the Earth (and other terrestrial planets), further impressive evidence in its favour emerges. In the first place, Ramsey established the remarkable result that, if the increase of density associated with the change of phase is by a factor greater than 1.5, there would occur with gradually increasing mass (or increasing internal temperature with constant mass), a sudden catastrophic collapse affecting the planet as a whole, and the formation within a matter of minutes from the onset of a central core of 'large' radius. For the Earth or a comparable planet (Venus), the initial size of core is about 2000 km radius, while the sudden decrease of outer radius is by about 70 km, though the Ramsey-theory could give only rough values.

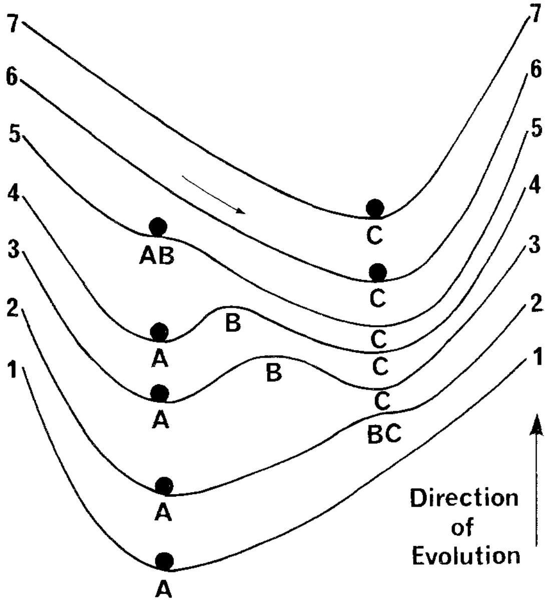

Fig. 2. A heavy bead on a slowly evolving quartic curve. At first, curve 11, the cubic giving the equilibrium positions has only 1 real root, and there exists only the stable position A. Later at stage 22, an inflexion BC appears, the roots having become all real with 2 equal; the bead will remain at the minimum A, though BC is a possible equilibrium position. Next, on each of 33 and 44, three different equilibrium positions exist, with A stable, B unstable, and C stable, but left to itself the bead will remain at A. At 55, A and B have now become coincident, and the bead remains there in neutral equilibrium. With the slightest further evolution, positions A and B *cease to exist* (it is not a question of instability), the bead will then fall rapidly towards C, and dissipation will bring it to rest at C eventually in stable equilibrium. Thereafter as the curve continues to evolve slowly, the bead will occupy the sole possible equilibrium position C.

To illustrate the theory leading to these conclusions and the nature of the instabilities involved, let us consider a bead under gravity on a wire in the form of a quartic curve lying in a vertical plane (Figure 2). The points where the bead can rest in equilibrium will be determined by the roots of the derived cubic equation, and if the quartic contains some slowly changing parameter, then at first the cubic could have only *one* real root, while for a range of the parameter it could come to have *three* real roots, and then later still it could revert to just *one* real root, as illustrated in Figure 2. Thus to begin with on curve 11 there will be only one possible equilibrium position A for the bead, stable for the form of curve adopted, but as the curve evolves to 22, an inflexion BC would appear corresponding to two new coincident equilibrium positions B and C, which would next proceed to separate into two more possible positions, shown at B and C on 33 and 44, besides the original A. Clearly position B is unstable while C is stable, but although the bead could rest at either of these positions if specially placed there, left to itself it will continue to remain at the minimum A so long as that position persists. But with still further evolution of the curve to 55, A and B will come into coincidence at another point of inflexion, and then next as for 66 will *disappear altogether* as possible equilibrium positions. It is to be emphasised, with Ramsey's results in mind, that it is not a matter of positions A and B becoming unstable in some sense, but of completely *ceasing to exist*. The bead must now move dynamically and fall towards the only existing equilibrium position C, and frictional forces will cause it to come to rest there on 66 and 77 with further evolution of the curve.

The foregoing results are exactly analogous to those that Ramsey established for an idealised planet capable of undergoing a phase-change. He postulated a (spherically symmetrical) planet of material that at static pressure less than a critical value p_c has constant uniform density ρ, while at pressure greater than p_c it switches to uniform density $\lambda\rho$, where $\lambda > 1.5$. If the total mass is gradually increased (corresponding to the evolution of the quartic curve), then the central pressure will also increase. At first only a single configuration exists, as with the bead on the wire, and it will be a 1-zone planet of density ρ. But when the mass m passes beyond a certain critical value M_1, Ramsey showed that *three* possible configurations come to exist: one having central pressure less than p_c and therefore single-zoned of density ρ, and two having central pressure greater than p_c and therefore two-zoned with cores of density $\lambda\rho$. In Figure 3, these are denoted by A, B, and C, and exactly as with the bead, since A is initially stable, when all three positions exist A and C are stable, but B is unstable. However, the planet in fact, although it could conceptually be placed in form B or form C and be in equilibrium, will evolve along the series OA, and continue to do so with further increase of mass until A and B merge together at Y for a certain value M_2 of the mass. Then, upon the slightest addition of mass, these two configurations *cease to exist at all*, and there remains state C as the only possible form of equilibrium available. Just as with the bead, the planet will rapidly move to this position by collapsing down, and in so doing develop what Ramsey termed a 'large' core having radius a sizable fraction of the overall radius of the planet. The associated *linear series of configurations* are shown schematically in Figure 3 in which the total

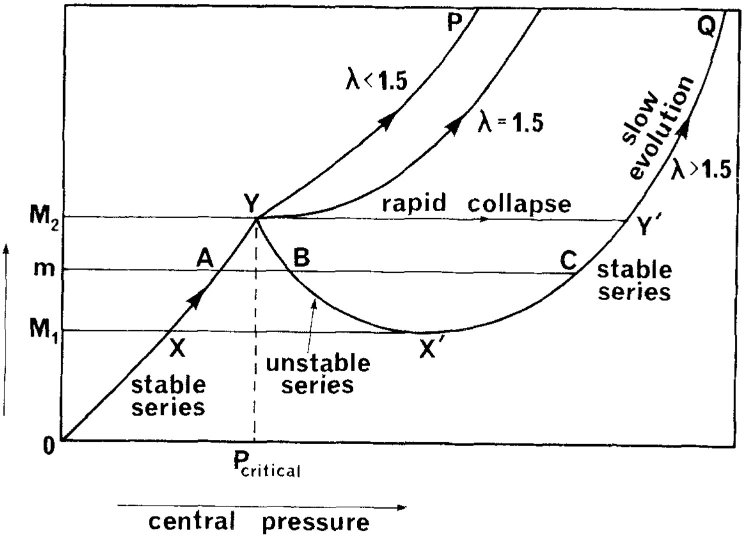

Fig. 3. Schematic diagram of the linear series of configurations associated with the phase-change to high density.

For $\lambda < 1.5$, evolution proceeds along OY before core-formation starts and then along YP with core steadily growing from zero mass. Both series are of thoroughly stable forms.

For $\lambda > 1.5$, evolution is at first steady by stable forms along OY. But between X and Y three possible equilibrium forms exist, A, B, and C, of which A with no core and C with 'large' core are stable, and B with 'small' core is unstable. If the series begins on OX, evolution proceeds only along the A-series to Y with no core. If the mass m exceeds M_2 by however little, the forms A and B *cease to exist*, and the system must leap to Y′ on the C-series, the motion $Y \rightarrow Y'$ corresponding to rapid collapse to a configuration with 'large' core. Thereafter the planet evolves stably along Y′Q with steadily growing core.

mass m as ordinate is plotted against central pressure. The time taken for the planet to reach configuration C, after A and B have ceased to exist, will be as Ramsey put it: "comparable with the time taken for a free particle at the surface to fall 100 km (the order of the decrease of outer radius) under gravity; that is, a matter of minutes, or at most hours." In actuality, the collapse would of course be retarded by dissipation, which would lengthen the time taken beyond that for free fall, hence Ramsey's "at most, hours". Calculations for the actual Earth making full allowance for compressibility, show that gravitational energy in excess of 5×10^{37} erg is released in the collapse, and that more than 80% of this goes into forming the core, which to begin with has mass about 6% of the whole Earth and radius just over 2000 km.

Seismic data show that $\lambda > 1.5$ for the Earth, lending strong support for the Ramsey theory. Were $\lambda \leqslant 1.5$, no such collapse would occur. For example, for the first phase-change associated with the so-called 20°-discontinuity at 413 km depth, where the pressure is only 0.14×10^{12} dyn cm^{-2}, the value of λ is less than 1.3. Therefore a small planet such as Mars, (in which the central pressure is less than 0.3×10^{12} dyn cm^{-2}), if of similar composition to the Earth, would evolve continuously along OYP with a kink in the curve at Y, where core-formation would commence, but the core in this case would grow continuously from zero and not start as a 'large' core. Only a single configuration is possible at any stage, of one zone before Y and of two zones thereafter, and no sudden collapse occurs.

As aforesaid, Ramsey adopted the simplifying assumption of two states of densities ρ and $\lambda\rho$ for the material, but it was at the same time shown by Lighthill (1950) that the general scheme of results remains completely unchanged if the material is compressible and the density increases continuously up to that associated with the critical value of the pressure, when again the density jumps by a factor λ. For $\lambda > 1.5$, three forms of equilibrium are again possible for a range of values of the total mass. This generalisation of the theory by Lighthill removed any doubt that might otherwise have been felt about Ramsey's results as possibly a product of the specialised assumptions. The whole theory provides a beautiful example of Poincaré's theory of linear series of equilibrium configurations and their exchange of stabilities, and that the theory of planetary evolution is representable in such elegant mathematical form must of itself lend cogent evidence of its likely applicability (by the Dirac principle).

Ramsey further showed that the critical masses M_1 and M_2 (Figure 3) depend on the internal temperature, and that evolution with increasing tempeature (at constant mass) will take the same form as for increasing mass. Owing to radioactive energy-release throughout the interior, temperature would have been gradually rising since the formation of the Earth, but so slow is conduction that it is doubtful if more than a small proportion, if that, of the heat so generated in the central regions has yet reached the surface to escape. As a consequence, these central regions must be practically isothermal and at a temperature of several thousand degrees, though the precise value is difficult of determination for the reason that at the high pressures prevailing the density depends only on the pressure, as the seismic data show, any temperature-dependence remaining negligible in this regard.

For the Earth, λ is in fact importantly greater than 1.5, so the foregoing theory is indeed applicable, and this means that the density of the liquid phase ρ_L represented by the core-material is *greater* than of the solid phase ρ_S represented by the mantle-material. On the other hand, for materials at low pressures it is usually the case that ρ_S is greater than ρ_L, and as a result the factor in the thermodynamic Clapeyron-equation for the melting-point, namely $(\rho_S - \rho_L)/\rho_L$ has sign implying a melting-point temperature *increasing* with depth. Now it is established that the Earth is solid down to 2898 km depth, and if the situation continued on down with no other phase possible, it would mean that melting could not occur anywhere. But when in fact conditions (of temperature and pressure) become such that a new liquid phase is possible, then it is such that $\rho_L > \rho_S$, and the melting-point thereon decreases with depth, which is obviously an essential requirement for the existence of a liquid core. A change of composition across the boundary to iron, for which $\rho_S > \rho_L$, would not escape the difficulty for iron is solid at the surface. Thus the fact that the core is liquid (apart possibly from the small innermost core within it) is readily explicable on the phase-change hypothesis and indeed would seem to demand it.

With the simplifying assumptions made, no precise value can emerge for the radius of the initial 'large' core, and only when use is made of the actual seismic data to allow for compressibility can this be found. It is not clear whether Ramsey intended identification of the 'large' core of configuration C with the present core, but in any event his next step was a theoretical investigation of compressibility. It had already been shown by Bullen (1940, 1942) that, except at discontinuities, the bulk-modulus k and pressure p must be linearly related in a form $k = k_0 + \alpha p$, where k_0 and α are adjustable constants. Curiously enough, however, despite the extremely different natures of the mantle and core, Bullen contrived some kind of meaned-up values for k_0 and α to obtain k as being roughly the same linear form in both zones. Ramsey (1950b) pointed out the unsatisfactory nature of this step, and proved that the values of k_0 are very different in the two regions, being as low as 0.54×10^{12} dyn cm^{-2} for the liquid core, and 2.1×10^{12} dyn cm^{-2} in the mantle. This showed that to use some sort of mean for k_0 and offset any error by adopting a common value for α, not necessarily correct for either zone, was a quite invalid procedure. The values of k_0 found by Ramsey bring out a feature of the highest importance, which is that the core-material is far more compressible at the relevant pressures than mantle-material (k measures the *in*compressibility), because of the large difference in the two values of k_0 (and the only slight difference in the α-values). These results emerged from considerations of the seismic data, and Ramsey next set himself to examine what conclusions could be reached by means of the atomic theory of solids (Ramsey, 1949, 1950b).

Since the material can be considered as a crystal for which the attractive and repulsive forces between neighbouring ions balance at zero pressure, the strong resistance to compression means that the repulsive forces must increase much the more rapidly with decreasing distance as compression occurs. For a wide range of power-laws for the repulsion, suggested by materials of known properties, Ramsey showed that α in both mantle and core must take a value between 3 and 4, and he went on to show that, even if the core and mantle represented *different materials*, their values of α could not differ by more than unity, but if they are *different phases of the same material* then the two values of α would differ by considerably less than unity. Thus it was demonstrated theoretically that a relation of the form $k = k_0 + \alpha p$ would be expected to exist for the two zones, but with different values of k_0 and substantially equal values of α. Since in homogeneous material $k = \rho \mathrm{d}p/\mathrm{d}\rho$, it followed that $k \propto \rho^{\alpha}$, and hence, if ρ_0 is the density that the material would have at zero-pressure, then $k/k_0 = (\rho/\rho_0)^{\alpha}$, where k_0 is the appropriate value for the core. Making use of the numerical values for ρ and k just inside the core-boundary (of unstated source however, but prior to 1949), Ramsey found that $\alpha = 3.8$, $k_0 = 0.54 \times 10^{12}$ dyn cm^{-2} (as quoted earlier herein), and $\rho_0 = 4.8$ g cm^{-3} for the uncompressed density. (It was later shown by Ramsey (1950b) that if the core is well mixed by convection, as is almost certainly the case, then α would be 3.5.)

Ramsey emphasised that this density is far lower than that of iron or nickel even in liquid form, and that if the core-material were to undergo a change in structure as pressure is released, this could only result in some decrease of ρ_0. The figure rules out any possible identification with iron and nickel. Calculations by the present author (Lyttleton 1965a) based on the later Jeffreys-Bullen seismic data led to somewhat different numerical results, though not altering the general conclusion. Thus the solution yielded $\alpha = 3.52$, $k_0 = 1.29 \times 10^{12}$ dyn cm^{-2}, $\rho_0 = 6.1$ g cm^{-3}. For the mantle, Ramsey had found $\alpha = 3.7$, $k_0 = 2.1 \times 10^{12}$ dyn cm^{-2}, whereas a similar least-squares solution based on the more accurate (1962) data yielded $\alpha = 3.39$, $k_0 = 2.22 \times 10^{12}$ dyn cm^{-2}, and $\rho_0 = 4.0$ g cm^{-3}. Although ρ_0 and k_0 for the core, turn out larger than Ramsey's original values, identification with iron-nickel remains impossible, and since the values of α are almost equal the important conclusion emerges that the liquid-core material is far more compressible at any relevant pressure than is the mantle-material. The consequences of this for the evolution of the Earth will be made clear later.

The foregoing discussion summarises the papers of Ramsey on the phase-change hypothesis published in the period 1948–1951, but in spite of their sound physical and mathematical basis, and their success in accounting for longstanding difficulties, as explained above, the ideas seem to have received scant attention. It was not until the early 1960s, when interest in planetary physics received an enormous impetus from the space-program, that the present author approached the problem from a different standpoint akin to that developed for stellar structure. In this, the central problem can be stated in brief as: Given a mass of stellar material with physical properties known, how bright will it be and how large? In other words, what will be its luminosity and radius? For planets there is no vast outward flux of radiation, and so the central problem becomes: Given a mass m of planetary material, how large will it be when in (spherically symmetrical) equilibrium. This is a far easier problem than that for a star, yet apart from a few numerical integrations agreeing with properties such as the mass and moment of inertia, no systematic general theory had been developed. Nevertheless intricate theories of convection within the *solid mantle* (sic) were being offered before this zero-order problem had even been attempted. The solution involves finding the pressure p, the density ρ, and included mass $m(r)$, at general distance r from the centre, subject to obvious boundary-conditions. The surface-radius r_s is the value for which $p(r_s) = 0$, but to determine this the whole march of internal values of p and ρ have to be found.

Since pressures throughout almost the entire Earth (and even the Moon), apart from the shallow outermost layers, far exceed the strengths of materials, the distribution must satisfy the mechanical equation of hydrostatic equilibrium: namely,

$$\frac{\mathrm{d}p(r)}{\mathrm{d}r} = -Gm(r)r^{-2}\rho(r); \tag{1}$$

and the mass-increment is given by $\mathrm{d}m(r) = 4\pi r^2 \rho(r)\,\mathrm{d}r$. Clearly there is also needed a specific equation-of-state expressing the physical (elastic) properties of the material, and it is precisely this that can be derived from the seismic data. These show that the Earth consists of 3 main zones, and in each of these the bulk-modulus k is a (different) linear function of the pressure, thus $k = a + bp$. Since in any homogeneous zone $k = \rho\,\mathrm{d}p/\mathrm{d}\rho$, the relation for k integrates to give a pressure-density relation

$$p = \frac{a}{b}\{(\rho/\rho_u)^b - 1\}, \tag{2}$$

wherein ρ_u arises as a constant of integration and is clearly the density at zero-pressure. The values of a and b, and hence of ρ_u, can be found by a least-squares fit of $k = a + bp$ to the seismic data for each zone, and the solutions show that within the uncertainties of the data it is adequate (but not essential) to adopt $b = 3.5$ the same in each zone. For computational purposes, a mathematical representation of the data, such as (2), is again not essential, but it is of value in enabling the importance and interplay of the various parameters to be displayed analytically.

The following numerical results emerge:

Zone 1: liquid core $a_1 = 1.34 \times 10^{12}\,\mathrm{dyn\,cm^{-2}}, \quad \rho_{u1} = 6.26\,\mathrm{g\,cm^{-3}},$ (3.i)

Zone 2: solid mantle $a_2 = 2.15 \times 10^{12}\,\mathrm{dyn\,cm^{-2}}, \quad \rho_{u2} = 3.98\,\mathrm{g\,cm^{-3}},$ (3.ii)

Zone 3: outer shell $a_3 = 1.17 \times 10^{12}\,\mathrm{dyn\,cm^{-2}}, \quad \rho_{u3} = 3.30\,\mathrm{g\,cm^{-3}}.$ (3.iii)

When these are used to compute a 3-zone Earth-model of mass 5.976×10^{27} g, the surface-radius is 6371.2 km; and the core-mass m_c is 1.876×10^{27} g, while the core-radius is $r_c = 3476.7$ km, in close agreement with the known values. Thus the values (3), together with $b = 3.5$ in each zone, provide reliably accurate quantities for discussion of the evolution of the Earth, as will be seen. The calculated pressure at the core-mantle boundary comes to $1.36 \times 10^{12}\,\mathrm{dyn\,cm^{-2}}$, and at the 20°-discontinuity $0.141 \times 10^{12}\,\mathrm{dyn\,cm^{-2}}$. Before proceeding, it may be of interest to mention that when these calculations were first made in the early 1960s and the value $\rho_{u1} = 6.26\,\mathrm{g\,cm^{-3}}$ emerged for the uncompressed density of the core-material, the writer thought some arithmetical mistake had entered, since he himself had uncritically accepted up to that time the general belief that the core was iron. But no amount of re-working could detect any error and 'break' the result. It was then that the meaning of the Ramsey work of a decade before came back to mind, and it immediately became clear that the mantle-core discontinuity must be a phase-change.

Modern theories of the origin of the solar system and of the terrestrial planets in particular provide cogent reasons for the conclusion that the Earth began its existence in all-solid form (1972, 1973). Such an initial state would mean no liquid core, and therefore a planet of two zones only: an inner solid mantle surrounded by a solid outer shell. Solution of (1) and (2), using the values of (3.ii) and (3.iii), gives a central pressure of $1.63 \times 10^{12}\,\mathrm{dyn\,cm^{-2}}$, about 20% greater than the present value at the base of the mantle for which (3.ii) is empirically verified, and therefore only a moderate extrapolation of the relation $k_2 = a_2 + bp$ is involved. The surface-radius r_s of this all-solid initial Earth is found to be 6741.5 km, and therefore some 370 km *greater* than the present radius. Quite evidently this implies that subsequent evolution taking the form of contraction of the

Earth as a whole must have taken place, and the proposal is that this is brought about through steady rise of temperature of the interior as a result of radioactive energy-release, but the evolution involves two separate stages: first a sudden collapse of the Ramsey style, and thereafter steady contraction along a series of stable forms.

On the Ramsey theory, for the phase-change to occur at a given pressure would require a certain associated temperature to be reached (the total mass of the planet now remaining constant), and at first the form solid throughout would persist at a low temperature of probably no more than a few hundred degrees. The only change then going on within the Earth would be the gradual but inexorable rise of internal temperature: there would be no thermal expansion since the pressure-density relation (2) is independent of temperature. At the present time, whatever the precise value of the temperature at the boundary of the core may be, it must be such that the phase-change occurs at a pressure of 1.37×10^{12} dyn cm^{-2}, and therefore a central pressure of 1.634×10^{12} dyn cm^{-2} would require a substantially *lower* temperature for the phase-change to commence. Thus the situation, in that p_c (all-solid Earth) is greater than 1.37×10^{12} dyn cm^{-2} the present value at the core-mantle boundary, conforms to the requirements of the phase-change theory. Conditions suitable for the phase-change would first arise at the centre, since pressure is highest there, and the Earth would take up a state represented by the point Y in Figure 3, a stage after the formation of the Earth that might take of order 10^9 yr to be reached.

Now calculations based on the values (3) show that the central density in this all-solid Earth would be 5.9 g cm^{-3} whereas the liquid-core phase at the same pressure would have density 10.2 g cm^{-3}, and therefore the value of the crucial λ would be 1.73, strongly in excess of the critical 1.5. Accordingly, moments after reaching state Y, the whole Earth would undergo rapid dynamic collapse to the state Y$'$ of Figure 3 and form a large core. It is here that improvement on the Ramsey discussion first emerges, for use of the actual seismic values (replacing Ramsey's simplifying assumptions) shows that the initial radius of this 'large' core is 2042 km, far smaller than the present value of 3477 km (as calculated on the same numerical values). The mass of the initial core is 0.37×10^{27} g, or 6.2% of the total mass of the Earth. The calculated surface-radius r_s is 6671.4 km, and this means a sudden decrease by about 70 km. The gravitational energy released in the collapse is about 5.9×10^{37} erg, which may be compared with the figure of 10^{25} erg estimated for the total average annual energy released by earthquakes at the present time (though no association is being implied by the comparison).

The procedure by which the foregoing results are obtained brings out points of high interest (1965a). Following methods used in the theory of stellar structure, it was decided first to compute a series of static models of gradually increasing core-mass m_c, starting at $m_c = 0$ and continuing on up by small increments to the present value $m_c = 1.876 \times 10^{27}$ g and beyond. The Ramsey theory shows that, since $\lambda = 1.73$, these could not at any rate at first be actual physical models in equilibrium, but conceptually their elements could be regarded as at rest everywhere with the equation of mechanical equilibrium satisfied, and the internal conditions then calculated. It is by such a numerical procedure that the

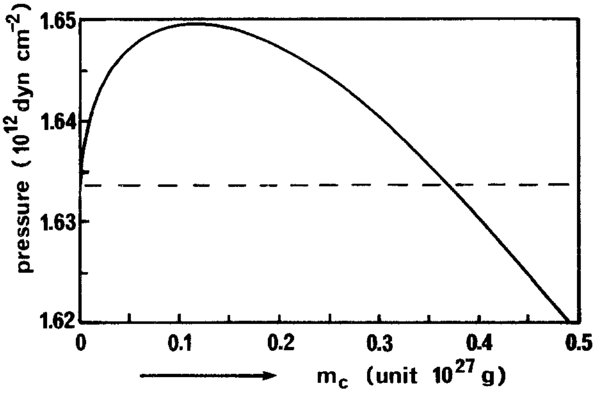

Fig. 4. Showing the initial *increase* of pressure p_{12} at the core-mantle boundary with growing core-mass m_c. For the all-solid planet, $m_c = 0$ and $p_c = p_{12} = 1.634 \times 10^{12}$ dyn cm^{-2}. The curve rises tangentially to the ordinate, increases to a maximum of 1.649×10^{12} dyn cm^{-2}, and thereafter decreases monotonically. It falls to the original central value at $m_c = 0.37 \times 10^{27}$ g (radius 2042 km) to give the initial 'large' core. This is the way in which the Ramsey-collapse is revealed by the series of models of increasing m_c-values.

Ramsey-collapse manifests itself in an entirely different and remarkable way, for when a small value of m_c is adopted and the static model for it is calculated, the pressure p_{12} at the resulting core-mantle boundary is found to *exceed* the value 1.634×10^{12} dyn cm^{-2} formerly prevailing at the *centre* of the original all-solid planet. Since the central regions are isothermal, the conditions at this boundary will therefore be more favourable for producing the phase-change than before at the all-solid centre because of the higher pressure, and so further conversion will occur and the core-mass m_c will increase. But the interface-pressure increases only up to a certain value (about 1.649×10^{12} dyn cm^{-2}) when it begins to decrease, which it thereafter continues to do monotonically. In Figure 4 is shown how this interface-pressure p_{12} between the liquid core and solid mantle changes as m_c increases. The curve is tangential to the pressure-ordinate, so the initial rate of rise is infinite; p_{12} then rises to a maximum as m_c increases, and thereafter decreases. Now for the value $m_c = 0.37 \times 10^{27}$ g this interface-pressure p_{12} has fallen back to the central value 1.634×10^{12} dyn cm^{-2} in the original all-solid planet. For still larger values of m_c, the pressure p_{12} becomes less and therefore not sufficient to bring about the phase-change *at the existing temperature*. Thus, followed by means of static models, the growth of core ceases at this stage, and in agreement with the Ramsey-theory gives a 'large' core of radius 2042 km, as earlier stated. It is in this way that the Ramsey-collapse is revealed and re-presents itself by this series of models of increasing core-mass. (For a planet with $\lambda < 1.5$, the curve of p_{12} would start off tangentially downwards and p_{12} be always decreasing for increasing m_c).

It is to be emphasised that it is the independence of temperature of the pressure-density relationships (2) in the various zones that warrants and validates these calculations of earlier mass-distribution models of the Earth. On the other hand temperature *is* concerned, as Ramsey showed, in bringing about the conditions necessary for the phase-change, and this in turn results from radioactivity. With further increase of the core-mass, the pressure p_{12} at the interface, as already seen (Figure 4), falls below that at the centre of the original all-solid Earth, and evolution from this point on proceeds by thoroughly stable forms, corresponding to configurations C on Y′Q in Figure 3, as the temperature continues to rise steadily. The rising temperature offsets the decreasing interface-pressure and maintains conditions suited to the phase-change. Thus the 'large' core configuration Y′ of the Ramsey-theory (Figure 3) cannot be identified with the present core, and further growth to the present mass 1.876×10^{27}g and radius 3476 km must have occurred since the initial collapse. The accompanying *decrease* of surface-radius r_s to date has been by just about 300 km, the consequences of which will be reverted to later. Since the half-lives of the radioactive elements responsible: uranium, potassium, thorium: are comparable with the age of the Earth, the process of evolution must still be continuing, with the core steadily increasing in mass and volume, and the surface-radius diminishing.

Ramsey himself, as seen, could not arrive at any specific value for the 'large' core radius, but the above-described calculations obviously imply steady evolution with growing core but with contraction of the Earth as a whole after the initial catastrophic collapse. They also imply a steady *decrease of the moment of inertia*, and hence an intrinsic accelerative contribution to the rate of change of the angular velocity of the Earth to conserve angular momentum. This result, arrived at in the early 1960s, proved to be yet another triumph of the phase-change theory; for a few years later it was established by Dicke (1966) that just such a rotatory acceleration existed. When all tidal effects of the Sun and Moon retarding the rotation of the Earth are allowed for, the ancient-eclipse data show that there is a residual unexplained acceleration, a result later confirmed in an extensive study by Muller (1975), and the numerical values found by these authors were in close agreement with the already-predicted value (1976). It may be noted in passing that this accelerative component cannot possibly be explained on the iron-core hypothesis, for which radioactive heating might cause slight thermal expansion and deepening of the level of the 20°-discontinuity, both of which would produce expansion and minutely small *increase* of moment of inertia and reduction of angular velocity.

But perhaps the greatest achievement of the phase-change theory, emerging when developed numerically by means of the seismic data, is in providing a mechanism giving an amount of contraction at long last sufficient to meet the geologists' demands where mountain-building is concerned. Here is what one renowned standard work on physical geology (Leet and Judson, 1961) has to say: "One generalization on which all geologists agree is that mountain-building involves a reduction in the surface-area of the globe – a shortening of the distance between points of the surface – and that all mountain-building is the product of a single mechanism – *squeezing by horizontal compression*. But when it comes to what *causes* the squeezing, there is no general agreement." Then again, Lees (1953), who spoke with highest authority based on lifelong experience as a field-geologist, was inflexible in his conclusion that the Earth must at one time have been several hundred kilometres larger in radius than at present. The real problem of mountain-building, let it be made clear, is not simply to account verbally for the one most recent period, but to find explanation for possibly as many as *twenty* such eras of worldwide orogeny. Within the past 1500×10^6 yr there have occurred at least six major periods, four of these in the last mere 500×10^6 yr (Holmes, 1965). Since over 99% of the mass and volume is at pressures far exceeding the strength, contraction resulting from steady growth of the core will proceed continuously throughout almost the entire Earth irrespective of how the shallow outer layers may have to react. In these, for a time the strength may prevail, but eventually stresses will and must build up causing the layers to try to fit down onto the smaller interior, and since their material is solid, this can be achieved only by folding and thrusting.

Contraction by 300 km in radius, means a reduction in the circumference by nearly 2000 km. Now granite, for example, can be compressed elastically by about 1/800 lengthwise before it fails, and thus if compression shortened the 40 000 km terrestrial circumference by 50 km, the material along it would have to yield somewhere to relieve the stress, if only by being crushed or sheared. On this simple basis, 40 such shortenings could be accounted for. However, reduction of length along every great-circle would occur, and this means reduction of surface-area, measured horizontally. Radial contraction by 300 km means about 50-*million* square kilometres of areal reduction, and since the material is solid this can only be achieved by compressional folding of the surface-layers, or by thrusting of one layer over another, according to local circumstances. The total *volume* of material involved in such redistribution in the age of the Earth will be effectively the total reduction of volume (since no material can stand out more than a few kilometres above the average surface), and this is approximately $4\pi r_s^2 \Delta r_s$, where r_s is the present radius and Δr_s the 300 km contraction. This gives the impressive figure of 160 *billion* (10^9) cubic kilometres, and if this is divided between say 20 eras of mountain-building, it will provide a volume to be re-disposed somehow in each era of about 8×10^9 km^3. A typical mountain-range of averaged dimensions 5 km high, 200 km broad, and 10 000 km long, would have volume 10^7 km^3, and so the contraction in a *single* era would involve redistribution of enough material for 800 such ranges, which would seem abundantly adequate for all presently known mountain-systems. Thus the phase-change theory leads on without further assumption to account for mountain-building on an entirely adequate scale. On the other hand, as scarcely needs saying, the iron-core hypothesis can offer nothing whatever towards solution of the problem.

For several decades, by assuming an initially molten Earth, the thermal-contraction hypothesis has been advocated by Jeffreys, but all along it has been widely considered to yield quite insufficient contraction. Owing to complexities and uncertainties, it has been possible for the theory of itself to carry on being maintained, but once it is recognised that the Earth instead began all-solid, the hypothesis obviously becomes nugatory. Moreover, it has long been known that the Moon has no mountains of folded and thrusted types, and in recent years the surface of Mars has been inspected in hitherto undreamt-of detail (and that of Mercury also to a lesser extent), with no sign whatever of mountains of these types. Yet on the thermal contraction hypothesis, Mars should show such features perhaps to even greater extent than the Earth, since gravity is considerably less opposing upheaval. (Volcanic mountains, which do exist on Mars, are not of course concerned in the present connection.)

The Ramsey-hypothesis leads to further predictions of properties of the other terrestrial planets. Since the phase-change is to a metallic liquid form, and since also the origin of the main magnetic field of the Earth is attributed to driven currents in the core, no immediate presumption in favour of iron, as being a metal at ordinary pressures and temperatures, remains. On the other hand, the pressure at the centre of Mars is less than 0.3×10^{12} dyn cm^{-2}, a value reached in the Earth high up in the solid mantle, while the temperature is probably lower than in the Earth. Hence, no phase-change of the mantle-core variety can occur in Mars: there can be no metallic liquid core, and hence no dipole magnetic-field. This was predicted on the phase-change theory *before* the Mariner-IV magnetometer flown to Mars established that there was no measurable field. The observed radius of Mars itself makes plain that the planet cannot possess any core of material of properties similar to the terrestrial core in like proportion (31%) or even much less, for if the planet did the radius would be over 100 km less than observed. Far from contracting, slight expansion of Mars accompanied by rifting of the outer layers, with no mountain-building, were predicted in 1965a, but here on the basis of the Bernal phase-change to a denser crystal-form for the 20°-discontinuity in the Earth, a change that requires higher pressure to produce with rising temperature, the reverse of the mantle-core phase-change. Such rifting, in places on a gigantic scale (compared with terrestrial rifts), has since been observed on Mars.

The planet Venus has mass 0.814 of the Earth, while its solid surface-radius has been determined by radar-means as close to 6055 km. The central pressure must therefore exceed 1.4×10^{12} dyn cm^{-2} (the value for uniform density), while the internal temperature-distribution can be expected to be somewhat higher than in the Earth owing to the closer proximity to the Sun and the green-house effect of the deep atmosphere, thus giving rise to conditions suited to the phase-change. Making use of the seismic data for the Earth, it is calculated that at present Venus has a liquid core of radius 3032 km with pressure at its boundary 1.23×10^{12} dyn cm^{-2}, a surrounding solid mantle extending to radial distance 5540 km where the pressure is 0.16×10^{12} dyn cm^{-2}, and then a solid outer shell. These pressures accord with the phase-change requirements on the basis that the temperatures in Venus are somewhat higher than in the Earth: the core-mantle value of 1.23×10^{12} dyn cm^{-2} is slightly lower than in the Earth (1.37×10^{12}) and at the equivalent 20°-discontinuity 0.16×10^{12} dyn cm^{-2} slightly greater (0.14×10^{12}). Since its formation, Venus will have undergone evolution closely parallel with the Earth: starting in all-solid form, radioactively produced rise of temperature would bring about the sudden collapse and a 'large' core, followed thereafter by gradual growth of the core accompanied by contraction of the planet as a whole. All this would occur on a scale only slightly smaller than for the Earth, and Venus will have undergone a series of eras of mountain-building. Thus it can be predicted on the phase-change theory that folded and thrusted mountains are present on Venus, but they may or may not resemble terrestrial mountains in detailed appearance since weathering effects could be very different.

The planet Mercury alone seems to fail to conform to the uniformity of composition exhibited (on the phase-change theory) by Venus, the Earth, Mars, and the Moon. The currently claimed mean-density ($5.4\,\mathrm{g\,cm^{-3}}$) renders Mercury a most egregious object, and it has been attributed to the proximity to the Sun, yet the structure of Venus and its mean-density of $5.2\,\mathrm{g\,cm^{-3}}$ raise no suggestion that its closer proximity to the Sun (than that of the Earth) has any influence on the composition. However, the problem presented by the high density of Mercury in no way militates against the phase-change theory, for on any proposed mass and radius the central pressure would be far too low for the Ramsey phase-change to be of any relevance.

It is seemingly enigmatical that despite the numerous remarkable successes of the phase-change theory, and the great difficulties on all sides that the iron-core hypothesis encounters, yet the former has received little attention beyond disfavour, hostility, and obloquy; while the latter has been nurtured and defended by every imaginable device. The explanation is to be found in a number of factors nowadays rampant – and not only in the subject of geophysics. First, there is the perennial atmosphere of unquestioned acceptance of the iron core as an *established fact* (I fell in with the idea myself before I had occasion to study the matter closely), and it comes to be passed along as part of a folklore by word of mouth, by mere repetition and attitudes, gradually taking on the form of a religious tenet. The idea comes to seem so self-evidently true in its own right(?), that no serious person could do other than accept it as correct. The writer once enquired, genuinely seeking information of a well-known geophysicist, if he would say where the case for the iron-core had been explicitly set out and reasoned. The reply was that "it is all in my book". But the most thorough search yielded nothing amounting to more than a few casual and unsupported assertions. A second instructive instance arose when a speaker was asked if the sharpness of the core-boundary did not strongly favour the phase-change hypothesis? The immediate reply was: "I personally do not believe in a phase-change at the boundary." So that settles it; and there is nothing more to discuss. Heads get so turned round that, if a hypothesis is not believed in *a priori*, they are not prepared to enter discussion of it.

The situation is further aggravated and intensified by the automatic action of the 'Gold-effect', whereby once some hypothesis has for any reason achieved a modest majority of proponents and adherents as compared with some other hypothesis advocated by only a few, then purely motiveless procedural and organisational factors of a non-scientific kind will come into play and cause the first group to increase indefinitely in number to bring about a state of affairs in which the first hypothesis comes to dominate the field and seem to its supporters to represent 'truth'. Such a group then becomes an official club; and by means of power resulting purely from numerical strength, its members will give expression to their dislike of any alternative theory, while on the other hand snatching eagerly at anything that seems to support the official ideas (or be critical of the unofficial ideas) however erroneous they may turn out to be. There is also the proclivity, pointed out by Medawar (1972), that there are many who unconsciously resent any prospect of resolution of long-standing problems, especially by those outside the inner-ring (from which new advances will usually come); for they have become emotionally attached to the ideas they have grown up with and absorbed – even though their subject can be seen to be landed in confusion by them.

Such factors – too numerous to go into here – all spring from a deep misconception of the nature of science and of how knowledge is to be attained. As the vast outpourings of modern 'scientific literature' show, it is widely thought that purely verbal accounts and descriptive narrations ('scenarios') constitute scientific theory – when, in fact, these can contribute nothing to science, no matter what transports of uncritical enthusiasm inspire them, unless they can point to some testable predictions, or suggest crucial experiments unique to them. Moreover, there appears to be insufficient realisation of the weight that must be accorded to any theory that made successful prior predictions subsequently verified to be correct.

The validity of any hypothesis cannot be settled *a prior* by means of feelings or dislike that it may arouse, nor by aesthetic considerations, nor by moral judgments. A hypothesis is to be regarded as acceptable only to the extent that its adequately worked out theoretical consequences accord with known data, and much more so still to the extent that predictions emerging from it are found to be correct. The phase-change hypothesis closely meets these scientific requirements; the iron-core hypothesis does not.

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