RESONANCES AND CLOSE APPROACHES. I. THE TITAN–HYPERION CASE

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Abstract. The orbits of Titan and Hyperion represent an interesting case of orbital resonance of order one (ratio of periods 3/4), which can be studied within a reasonable accuracy by means of the planar restricted three-body problem. The behaviour of this resonance has been investigated by numerical integrations, of which we show the results in terms of the Poincaré mapping in the plane of the coordinates $\eta = \sqrt{[(2L-2G)]}\cos(\tilde{\omega}_H - t)$ and $\xi = -\sqrt{[(2L-2G)]}\sin(\tilde{\omega}_H - t)$, keeping a constant value of the Jacobi integral throughout all integrations. We find the numerical 'invariant curves' corresponding to low and high eccentricity resonance locking (which seem stable, at least during the limited time span of our experiments) and show that the observed libration of Hyperion's pericenter about the conjunction lies inside the stable high eccentricity region. If initial conditions are chosen outside the stable zones, we have no more stable librations, but a chaotic behaviour causing successive close approaches to Titan.

We discuss these results both from the point of view of the mathematical theory of invariant curves, and with the aim of understanding the origin of the resonance locking in this case. The tidal evolution theory cannot be rigorously tested by such experiments (because of the dissipative terms which change the Jacobi constant); however, we note that the time scale of chaotic evolution is by many orders of magnitude smaller than the tidal dissipation time scale, so that the chaotic regions of the phase space cannot be crossed by a slow and 'smooth' evolution. Therefore, our results seem to favour the hypothesis that Hyperion was formed via accumulation of the planetesimals originally inside a stable island of libration, while Titan was depleting by collisions or ejections the zones where the bodies could not escape the chaotic behaviour.

1. The Titan–Hyperion Resonance and How to Study It

The 3/4 resonance locking between the satellites of Saturn, Titan and Hyperion, can be described as a high eccentricity, simple *e*-type libration (Greenberg, 1973; Colombo *et al.*, 1974). This means that we have an integer combination of angle variables

$$\varphi = 3\lambda_T - 4\lambda_H + \tilde{\omega}_H, \tag{1}$$

which is librating about a mean value without doing a complete circulation. Here λ_T and λ_H are the mean longitudes of Titan and Hyperion respectively, while $\tilde{\omega}_H$ is the longitude of Hyperion's pericenter. Although the physical mechanism which assures the stability of

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this kind of orbital resonance is well known, the Titan–Hyperion case is still puzzling because up to now the origin of the resonance has remained an open problem, and the authors who have participated – until recently – in the discussion of this subject (Goldreich, 1965; Sinclair, 1972; Colombo and Franklin, 1973; Greenberg, 1973, 1977; Peale, 1976) have not presented any conclusive arguments supporting the proposed explanations. Therefore, we have tried to explore the dynamical behaviour of Hyperion's orbit (and of the orbits which are nearby to the real one in the phase space), in order to see what evolutionary models are really consistent with celestial mechanics. This task can be accomplished with very precise numerical experiments; but the results of such experiments are far more illuminating if presented in a geometric form which fits in the frame of the contemporary stability theory for dynamical systems, mainly as regards the results of the so-called KAM theory (after Kolmogorov, Arnold and Moser).

Since the relative inclination of the two orbits is very small ($6'$) and Titan's longitude of pericenter plays no role in the libration, the dynamical problem can be modelled quite well as a restricted three-body problem in which a secondary body (Titan) of mass $\mu = 1/4151$ revolves about a primary (Saturn) of mass $(1-\mu)$, and the third body (Hyperion) moves in the same plane as a zero-mass test particle (in fact, the mass of Hyperion is about 2×10^{-7} Saturn's mass). Within the scope of this model we neglect the eccentricity $e_T = 0.0289$ of Titan's orbit and the perturbations produced by other bodies; but as we shall see, the real libration of Hyperion is modelled very well in spite of these simplifying assumptions (all the quoted orbital elements are taken from the 'Explanatory Supplement to the Astronomical Ephemeris and the American Ephemeris and Nautical Almanac', Her Majesty's Stationery Office, London).

Figure 1 shows an integrated librating orbit similar to Hyperion's one in the reference frame rotating with the mean motion of Titan. The three characteristic lobes of the curve are due to the fact that, because of the 3/4 resonance, between two successive conjunctions Hyperion must complete three revolutions (with corresponding apocenters and pericenters) in the inertial reference frame. Obviously, the conjunction always occurs near the apocenter, so that too close encounters between the two satellites are not allowed and the resonance assures a stability which otherwise could not exist (Roy, 1979).

The most suitable coordinate set for our study is the usual set of action-angle or Delaunay synodic variables $L = \sqrt{a}$, $G = \sqrt{[a(1-e^2)]}$, $l =$ mean anomaly, $g = \tilde{\omega}_H - t$ (hereinafter unsubscripted elements will be Hyperion's ones, Titan's elements being $a_T = 1$, $e_T = 0$, $l_T = t$), but in order to avoid the singularity corresponding to $e = 0$ we introduce the Poincaré synodic variables

$$\begin{cases} \xi = -\sqrt{[(2L-2G)]}\,\sin g, \\ \lambda = l + g, \\ \eta = \sqrt{[(2L-2G)]}\,\cos g, \\ \Lambda = L, \end{cases} \tag{2}$$

with λ reducing to the true anomaly for $(L-G) \to 0$, i.e. for $e \to 0$.

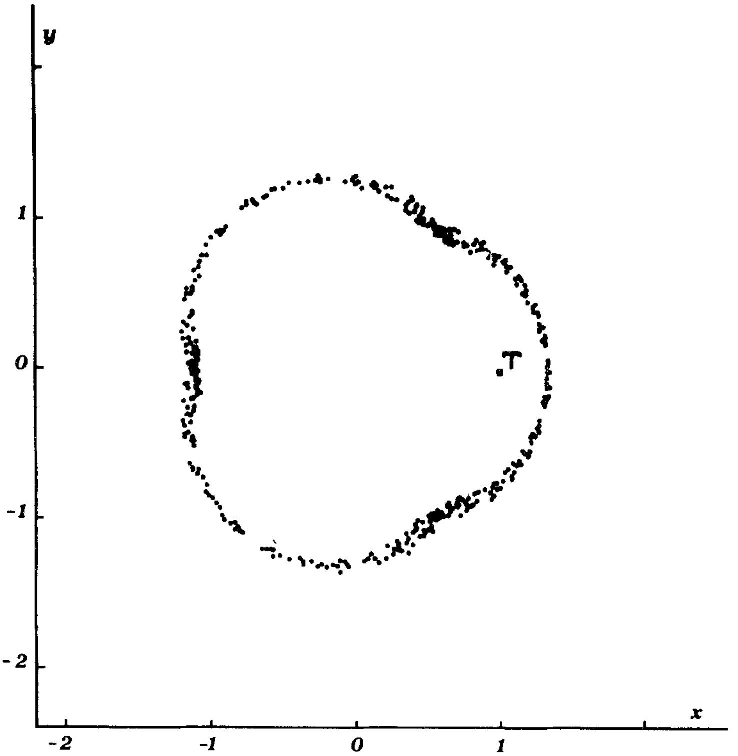

Fig. 1. An orbit similar to Hyperion's one in the reference frame rotating with Titan. The libration is shown by the spreading of points about the resonant periodic orbit which would appear as a closed line.

The topological structure of the solutions is determined by a fact known from theoretical proofs by Arnold and Moser and from numerical experiments by Contopoulos, Hénon and others (for a review see Moser, 1973): some solutions can form invariant tori which surround (in the phase space) the periodic orbits of first and also of second kind. In order to see how these invariant manifolds can bound and control the possible motions, we have to present the global dynamics in a way which reduces the dimensionality and allows to display the results in an easily understandable graphical form. This can be done by using a method going back to Poincaré, that of surfaces of section (or Poincaré maps): we choose an angle variable – here the mean synodic longitude λ – and take a section of the phase space defined by a constant value of that angle – here $\lambda = 0$ or multiple of 360°. This means that we record the elements of the perturbed motion once every conjunction, because $\lambda = l + \tilde{\omega}_H - t = l + \tilde{\omega}_H - l_T = 0$ indicates conjunction in the sense of mean anomalies. Since the restricted three-body problem has an exact first integral, the Jacobi integral H, by restricting the analysis to a level 3-manifold of H the surface of section (defined by $\lambda = 0$, $H =$ const.) becomes bidimensional. Hence the long term behaviour of an orbit may be described by plotting the successive points of intersection of the orbit

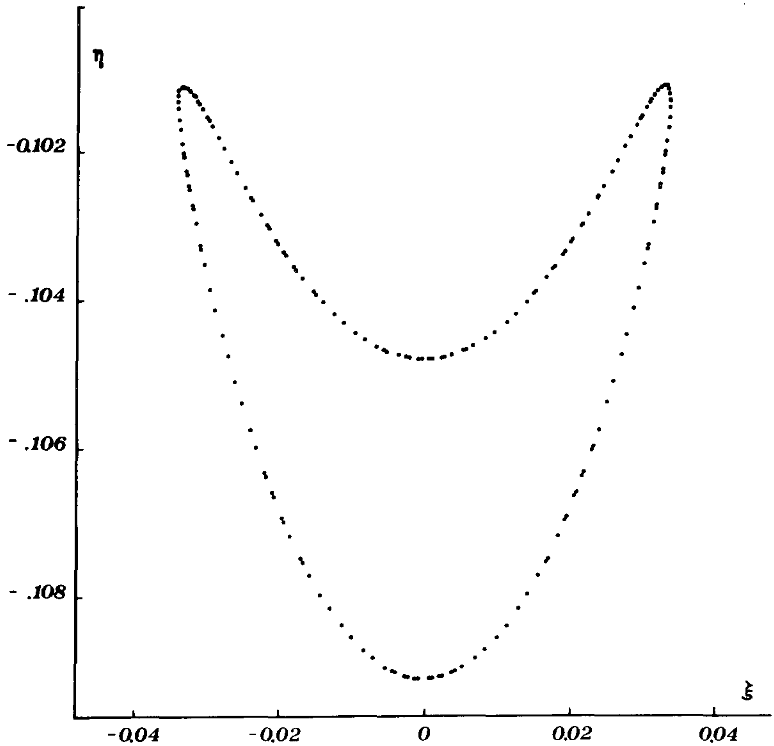

Fig. 2. The same orbit of Figure 1 represented in the phase space by the method of surfaces of section. The polar angle with respect to the origin in the ξ, η plane is the angle between Titan and Hyperion's pericenter, the radius is $a^{1/4}e$ (neglecting higher order terms in e).

itself with the chosen surface of section. It follows that, by this method, fixed points correspond to periodic orbits and invariant curves to invariant tori.

2. The Numerical Experiments

We have developed the computer program ORBIT2 with the aim of performing the following tasks: to solve numerically the equations of motion of the restricted three-body problem with a high degree of accuracy; to change coordinates from Cartesian ones (easier to use for the numerical integration) to Poincaré synodic variables; to take by interpolation the intersection with the surface of section; to display the obtained points in a graphical output (video or plotter). The program worked interactively, allowing us to change initial conditions keeping a constant value of H, to display the resulting orbits separately or together, to get graphical and/or numerical outputs. The main difficulty was connected with the requirement of a high precision throughout the integrations, avoiding that accumulated numerical errors could affect the results in a relevant way. It is well known (Brouwer, 1937) that the numerical error along track accumulates more than

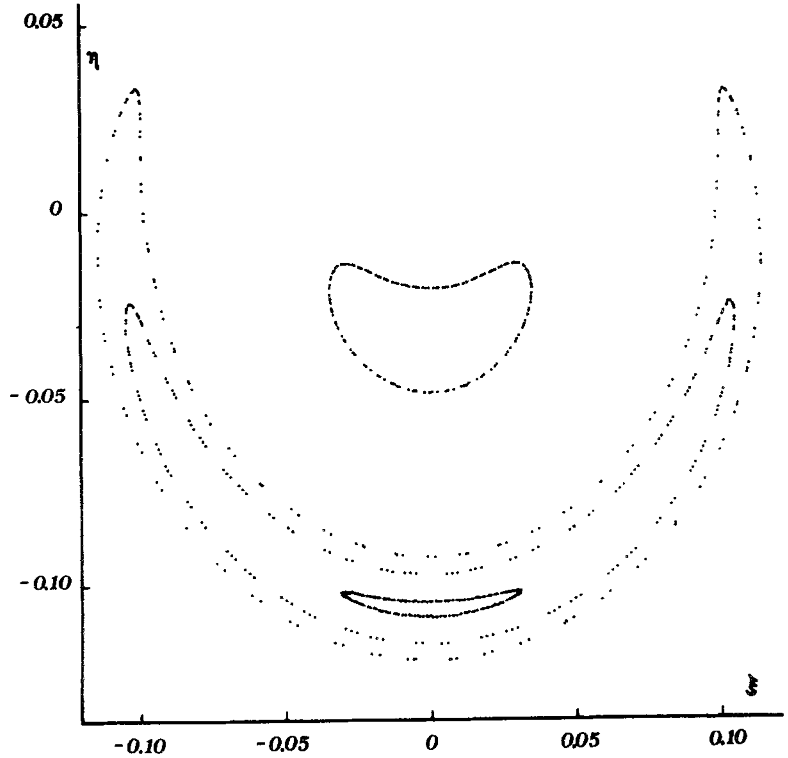

Fig. 3. Four superimposed librations with the same value of the Jacobi integral ($H = -1.507\,880$): below, three librations of the high eccentricity type; above, one of the low eccentricity type.

linearly with time, so that very long integrations become exceedingly expensive (because of the very small stepsize needed) or of poor reliability. The problem was solved with a method of rather high order (Adams–Moulton eighth-order predictor–corrector) with automatic stepsize control. The accumulated numerical error was tested by means of two-body experiments: we can estimate that for the worst points (i.e., at the end of the integration interval) the accumulated error was smaller than 10^{-3} radians in the angle g (an along-track variable because of the rotating reference frame) and smaller than 10^{-6} in the radius $\sqrt{(2L - 2G)} = \sqrt{(\xi^2 + \eta^2)}$. Therefore the figures we are going to discuss can be considered reliable, since the error is always smaller than the point indication used by the plotter. The value of the Jacobi integral was selected to be $H = -1.507\,880$ on the basis of the known elements of the real Hyperion's orbit, and this value was conserved by the numerical integration with an error smaller than 3×10^{-7}.

Figure 2 shows an orbit (i.e., the intersections of an orbit with the surface of section) in the ξ–η plane very similar to the observed one of Hyperion, which librates stably with an amplitude of about 36° about the mean value of φ (180°), with a libration period of 1.75 yr (Woltjer, 1928), that corresponds to about 10 synodic periods. This means that we expected to find, for suitable initial conditions, an invariant curve (section of an invariant

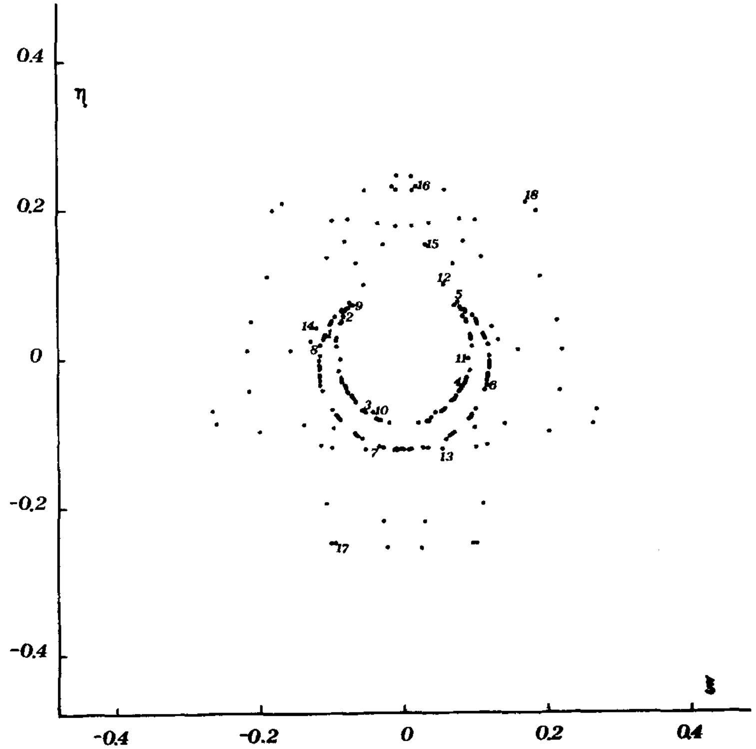

Fig. 4. A high-amplitude libration which results in a chaotic behaviour. The numbered points, representing consecutive conjunctions, show the gradual wandering away from the 'false invariant curve'.

torus) on which after ten successive iterations of the Poincaré map the representative point would return very close to the initial position. This theoretical prediction was confirmed by the numerical experiment: the orbit of Figure 2 has a libration period of ~ 10.1 synodic periods and an amplitude of ~ 37°, in good agreement with the observed values. We remark that these results are satisfying because both the observations and the mathematical model (neglecting a 0.029 eccentricity of Titan) are approximate.

Then we started to look for librations of larger amplitude (by changing the initial values of ξ and η and keeping H constant). Figure 3 shows four superimposed orbits, which suggest the existence of an 'ordered' region of high eccentricity librations that occupies all the 'crescent' zone of the Figure. The libration of highest amplitude, which is stable for many libration periods, has an amplitude of about 213° and a period slightly smaller than 10 synodic periods. Also in Figure 3 we can see a low eccentricity libration (the heart-shaped curve), which is stable for a long time too. On the other hand, if initial conditions are chosen just a little outside the regions bounded by the high and low eccentricity librations of Figure 3, the situation changes radically. In Figure 4 we see that for a 'long' time the orbit seem to lie on an invariant torus: the minor satellite librates until the conjunction ($\lambda = 0$) occurs with maximum amplitude of the librating argument

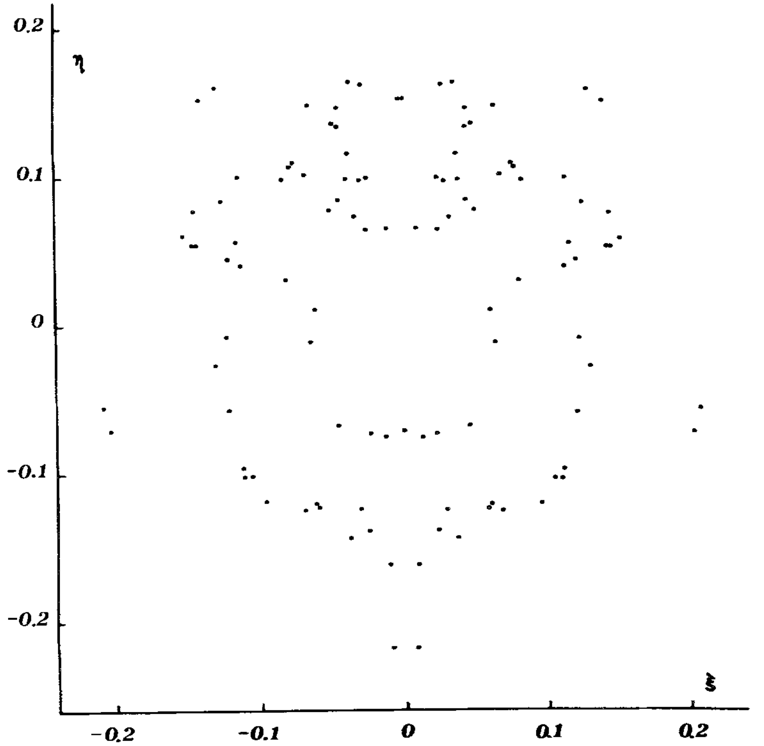

Fig. 5. A chaotic orbit which surrounds the ordered zone.

φ (i.e., with the minimum value of g) at the right top of the crescent. At this instant the conjunction occurs near the pericenter and it is in fact a close approach to Titan (point number 5); then the orbit begins to oscillate outside the 'false' invariant curve, two more close encounters occur (points number 9 and 12) and finally the orbit is pushed far away from the libration zone towards a very close approach to Titan (point number 16) followed by a 'chaotic' evolution (the symmetry $\xi \to -\xi$ is artificial, due to the fact that we plotted together two symmetric orbits to better exhibit the 'false' invariant curve).

If initial conditions are chosen farther away from the ordered region, as shown by Figure 5 a more unstable libration results, which is very quickly destroyed by perturbations and followed by the 'chaotic' behaviour with multiple close approaches to Titan.

3. Discussion of the Results

For a deep understanding of the preceding results, it is necessary to describe an 'order 1' resonance as a bifurcation of a family of periodic orbits, in which a non-resonant family of near-circular 'first kind' periodic orbits bifurcates into three different families for a critical value of the parameter which characterizes the family (the Jacobi integral in our case). Among these families, two are linearly stable (i.e., have pure imaginary Liapounov

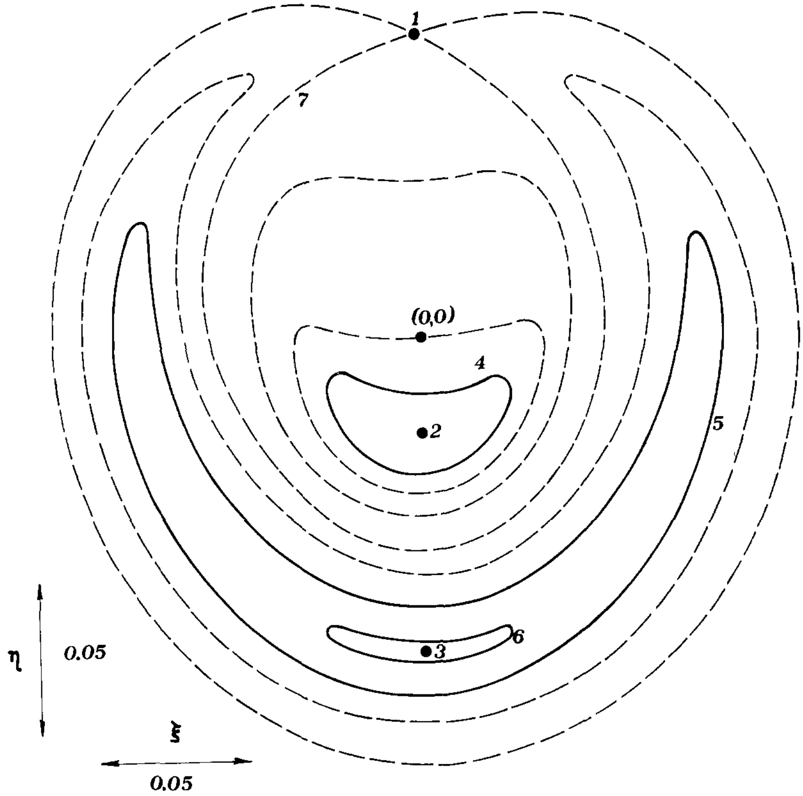

Fig. 6. Interpretation of numerical results (qualitative sketch): 1: hyperbolic fixed point (unstable periodic orbit); 2: elliptic fixed point (stable periodic orbit of first kind); 3: elliptic fixed point (stable periodic orbit of second kind); 4: largest possibly stable low eccentricity libration; 5: largest possibly stable high eccentricity libration; 6: 36° libration; 7: singular level curve of the 'second integral', which is destroyed by perturbations (as all other dashed lines).

characteristic exponents) and their position can be easily guessed, by looking at Figure 3, at the centers of the stable librations. But the most important for our problem is the third family, which is composed of unstable periodic orbits (with two characteristic exponents real and of opposite sign) and lies on the positive η axis (that means conjunction at pericenter, an unstable configuration). Such a periodic orbit gives on the Poincaré map an hyperbolic fixed point which is characterized by a stable and an unstable invariant curve; both these curves must surround the two stable periodic orbits. Figure 6 provides a qualitative description of the bifurcation phenomenon by showing the level lines of a 'second integral', i.e., the system of invariant curves which should result if the dynamical problem were integrable.

These lines are the same that can be obtained by averaging out the short-period terms of the perturbation, i.e., the terms depending not only on the librating argument φ and on the action variables, but also on the non-librating argument (Sinclair, 1972). But

Hamiltonian systems are 'generically' not integrable (Poincaré, 1892; Moser, 1955), so that the short-period terms are not negligible, and taking into account their effect the dashed curves become wildly oscillating. In particular the stable and the unstable curve of the hyperbolic fixed point, which in Figure 6 appear as a unique singular curve, split into two curves which 'generically' will intersect transversally each other at some points (see Arnold, 1976; Appendix 7). These are homoclinic points; if the intersection is transversal, there is an infinite number of them, hence their topological closure is a non-trivial recurrent set (Poincaré, 1899, Ch. XXXIII; Smale, 1967; Moser, 1973, Ch. 3). Without entering more deeply into such a complex mathematical subject as homoclinic points and hyperbolic sets, from a qualitative point of view what happens is that around the singular level curve of the second integral (Figure 6) a 'chaotic region' must be necessarily generated by the amplified resonant perturbative terms. This corresponds to the small divisors (produced by the resonance) which cause the series to diverge both in the classical perturbation theory and in the Birkhoff normal form theory. On the other hand, the KAM theory prescribes that around 'generic' elliptic fixed points (corresponding to linearly stable periodic orbits) we must have some invariant curves resisting to perturbations, therefore bounding two 'ordered regions' where the stability of librations is assured for every time span. But neither the stability theorem of KAM theory, nor the instability theorem about homoclinic points give any quantitative indication of practical value about the width and the boundaries of 'chaotic' and 'ordered' regions. Therefore we cannot hope to get valuable information about the stability of librations of different amplitudes from analytical arguments. For instance, if we had supposed that short-period terms can be averaged out, we would conclude that stable librations up to 360° are possible, what is definitely false as we have seen.

It follows that, at the present state of the theory, only numerical experiments can be used to obtain informations about the resonant regions as regards stability or instability for different initial conditions. Of course no finite numerical integration can prove the existence of an invariant curve; but if the width of an annular 'chaotic' region could be bounded, using the observed width in a numerical experiment, the known numerical error and the time span of the experiment, then in the two-dimensional case the existence of two bounding invariant curves would be indicated. Unfortunately not even this seems possible because, as we have seen in Figure 4, the wandering away from a 'false invariant curve' is very slow at the beginning, then becomes very fast when the orbit passes through a 'hole' in the 'false invariant torus'. This fact corresponds to the theoretical argument that the time scale of this wandering is of the order of exp $(1/\mu^d)$ with $0 < d < 1$ (Arnold, 1976, Appendix 8). Therefore, every numerical experiment can prove the stability only for the time interval of the performed integration, which is necessarily very small on the astronomical time-scale (at maximum some thousands of revolutions). On the other hand, if the result of the numerical integration for a given orbit is a chaotic behaviour, then we can consider as proven for any time that at least in the region transited by that orbit the invariant curves have been disrupted by perturbations.

An important feature of this particular case (3/4 resonance with $\mu = 1/4151$) is that the chaotic region indicated by Figures 4 and 5 contains the singularity of collision with Titan, so that any initial condition outside the outermost presumed invariant curves gives rise very quickly (a few tens of synodic periods) to very close approaches to Titan. On the contrary, we do not determine if the same thing can happen to an orbit with initial conditions inside the presumed 'ordered region', perhaps after a very long lapse of time. Since Hyperion is librating with a 36° amplitude, we must argue that at least up to this amplitude invariant curves do exist; but we can say nothing about the stability for, say, 10^{11} revolutions (which correspond to the age of the system) of librations with an amplitude between 36° and 213°.

4. The Origin of the Titan–Hyperion Resonance

The theory which explains various cases of resonance locking among satellites of the outer planets as the result of differential tidal evolution of orbits (Goldreich, 1965) has been widely applied in recent years (for a review see Peale, 1976), assuming a 'smooth' evolution from non-resonant orbits (with circulating critical argument) to resonant stable librations corresponding to commensurable periods. Within this context, the Titan–Hyperion resonance is an anomalous case: the large orbital distance of Titan implies that the tidal increase of its semimajor axis has been very small during the solar system's lifetime, even if Saturn's tidal dissipation coefficient is close to the lower limit derived by Goldreich and Soter (1966). This conclusion led Colombo and Franklin (1973) to explore the possibility that the resonance is a primordial configuration, by evaluating the relative volume in the phase space of initial conditions giving rise to a libration. This hypothesis is supported by Roy (1979), on the basis of an empirical stability criterium which implies the instability of Hyperion's orbit if the effects of the resonance are not taken into account.

Our results impose to the tidal hypothesis (or to any other theory based on small non-conservative terms) an important constraint: the evolution can really take place only inside 'ordered' regions. This because the time scale of tidal evolution is by many orders of magnitude longer than the time scale of dramatic orbital changes inside a 'chaotic' region, due to multiple close encounters with Titan (which finally cause a collision or the ejection of Hyperion into an 'irregular' orbit). The 'chaotic' region cannot be crossed by a 'smooth' and slow evolution, and inside it the equations commonly used in the tidal theory have no meaning, since the divergence of perturbative series results in the impossibility of isolating secular terms.

This argument does not exclude the possibility of a tidal evolution from circulation to libration, because in the presence of dissipative terms the Jacobi integral is no more constant. If the tidal evolution changes semimajor axes and eccentricities just in the same way these parameters change along the family of periodic orbits (parametrized by the Jacobi integral) which continues into the high eccentricity librations, then the evolution can take place always inside a 'tube' of ordered regions. However, the process of capture into libration surely does not occur automatically for every initial condition, and it would be necessary to verify the possibility of such an evolution and its probability by systematic numerical experiments (which cannot be easily replaced by analytical arguments, as we discussed before). We can conclude that in the case of the Titan–Hyperion system the tidal theory meets with relevant difficulties to explain the formation of the actual resonant configuration (even if it could account for some orbital evolution inside the ordered libration region).

Therefore we are led to examine in some more detail the mechanism for the so-called 'primordial' formation of the resonance. This mechanism was probably not a simple 'natural selection' process among a number of Hyperion-like objects with initial similar orbits. We rather suggest a model which is directly related to the process of accumulation of large solid bodies within the protosatellite cloud which surrounded Saturn. The growth of Titan's embryo had to cause, in the neighbouring region of the cloud, increasing gravitational perturbations, and our experiments show that when these perturbations became comparable with the present ones, a large region of the phase space around the libration zone was quickly depleted by collisions or ejections. On the contrary, the relatively small 'ordered' region (similar to the one shown in Figure 3) could continue to contain particles and small bodies in stable orbits. This material, confined in a narrow zone both in the physical space and in the phase space (i.e., with small relative velocities), could easily be accumulated into a single body whose present orbit must lie inside the same zone. We note that, while in the phase space the 'ordered' region has two detached components (low and high eccentricity librations), the corresponding orbits do intersect and therefore a single body can result from the accumulation process. Moreover, we can suggest a possible reason why Hyperion was formed at the 3/4 resonance and not at the 2/3 one, which is associated with a larger libration zone in the phase space (Colombo and Franklin, 1973): the effectiveness of the accumulation process was probably dependent not only on the amount of available solid material, but also on the existence of small relative velocities in the corresponding 'libration island'. A more limited island could originate a relatively massive object more easily and/or within a shorter time than a broader island with higher libration amplitudes.

The previous scheme is only qualitative: a more complete and quantitative theory for Hyperion's formation inside the 'libration island' could be formulated only on the basis of a detailed knowledge of the distribution of size and orbital elements among the bodies forming the protosatellite swarm.

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