NUMERICAL MODEL OF THE MOON'S ROTATION

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Abstract. We have integrated numercially the differential equations for the Moon's rotation with respect to an inertial coordinate system, and the variational equations for (i) the six initial conditions of the rotation; (ii) the moment-of-inertia ratios β and γ; and (iii) the coefficients of the third-degree gravitational harmonics. When these integrations are used in conjunction with our current lunar-orbit and Earth-rotation models, and all of the relevant initial conditions and parameters are adjusted to fit five years of McDonald Observatory lunar laser ranging observations, the root-mean-square (rms) of the postfit range residuals is 28 cm. When we adjust the lunar-rotation initial conditions separately to fit the physical libration angles given by the numerical model of Williams (1975), we find an rms orientation difference over a six-year interval of ~ 0.03 arcsecond, after removal of a constant bias. A similar comparison of our model with the semi-analytical model of Eckhardt (1981) yields an rms orientation difference of ~ 0.2 arcsecond.

1. Introduction

The problems posed by the Moon's rotation, which is tightly coupled to its orbital motion, have challenged theorists at least since the time of Newton. With the techniques of laser ranging (Bender *et al.*, 1973) and very-long-baseline interferometry (Counselman *et al.*, 1972, 1973a, b; Slade *et al.*, 1977) now being used for observations of the Moon, the measurement uncertainties have been decreased drastically, to nearly one hundredth of a second of selenocentric arc, and the problems of theoretical representation have been confounded correspondingly. Especially difficult is the development of an analytical description (Eckhardt, 1970, 1981; Migus, 1980) of the Moon's rotation useful for the interpretation of these modern observations. Thus, interpretations (for example, King *et al.*, 1976; Stolz *et al.*, 1976; Williams, 1977; Calame and Mulholland, 1977) have been based on numerical integrations of differential equations that describe the Moon's rotation.

In this paper we discuss the numerical integration of the differential equations for the Euler angles that define the Moon's orientation with respect to an inertial coordinate frame. We compare this model of the Moon's rotation first with the available laser-ranging observations and then with two other theoretical descriptions: Williams' (1975) numerical integration of the differential equations for the 'Cassini' libration angles, and Eckhardt's (1981) semi-analytical model.

2. Numerical Model

Our numerical model is based on Euler's differential equations for the rotation of a rigid body about its center of mass. The coordinate system we used is described in Figure 1 and the resultant equations are given in detail by Cappallo *et al.* (1977). The only

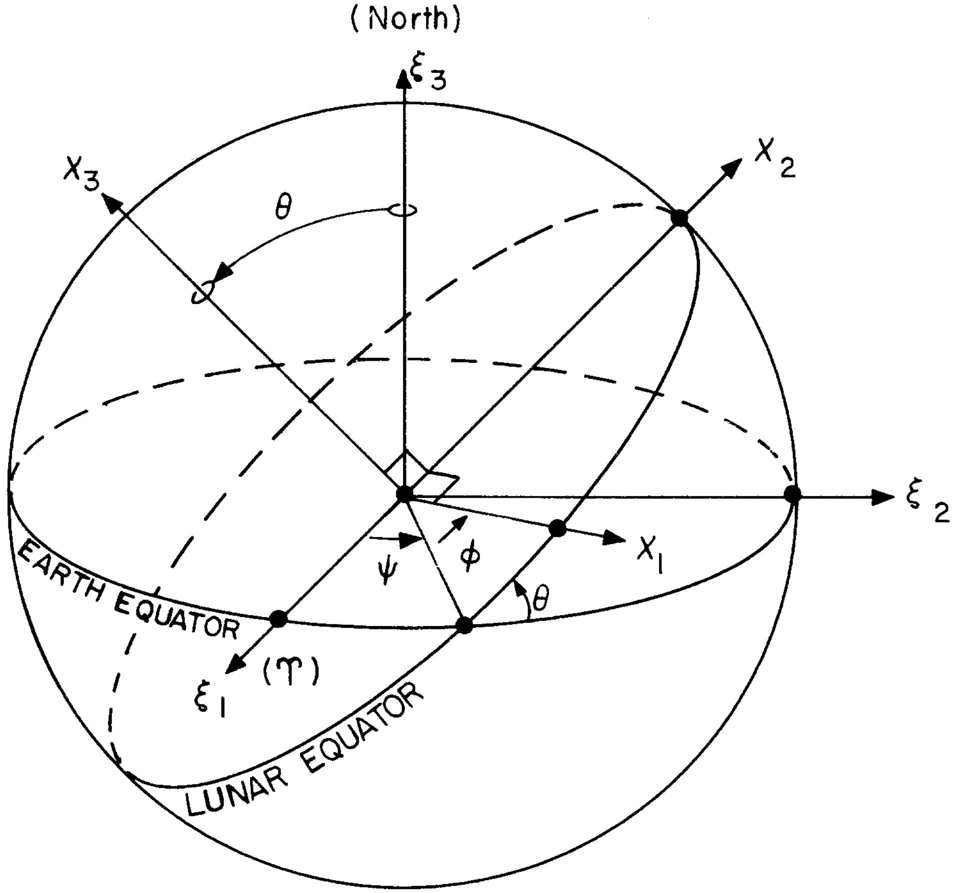

Fig. 1. Euler angles used to describe the orientation of the lunar body-fixed (x_1, x_2, x_3) axes with respect to the inertially fixed (ξ_1, ξ_2, ξ_3) axes defined by the man Earth equator and equinox of 1950.0. The longitude of the ascending node of the lunar equator is ψ; the inclination is θ; and the angle of rotation about the lunar north-polar axis is ϕ.

contributions to the torques that we have (so far) included in the integration of these equations are those due to the Earth and the Sun, with each of these bodies being treated as a point mass in the gravitational field of the Moon.

The adjustable parameters of the model include the six initial conditions, the lunar moment-of-inertia ratios β [$\equiv(C-A/$B] and γ [$\equiv(B-A)/C$], and the third- and higher-degree coefficients in a spherical harmonic expansion of the Moon's gravitational potential. The variational equations for the adjusted parameters [given in detail by Cappallo *et al.* (1977)] are integrated simultaneously with the equations of motion. This parallel computation of partial derivatives is more efficient and more accurate than the technique of finite-differencing of integrated motions, which has hitherto been used with numerical models of the Moon's rotation.

An eleven-point Adams-Moulton predictor-corrector method (Smith, 1968) was used to integrate both the equations of motion and the variational equations within the framework of the M.I.T. Planetary Ephemeris Program (Ash, 1965a, 1965b, 1972). The integration was done with steps of one-eighth day over the approximately six-year interval from Julian date 2 440 400 to 2 442 590. The orbital ephemerides of the Earth and the Sun, used for the evaluations of the torques acting on the Moon, had been obtained previously (King *et al.*, 1976; Ash *et al.*, 1971). We checked for round-off and truncation errors in our integration, and for consistency between our rotational equations and each of our variational equations by means of finite differencing, and found no evidence of any such errors as great as 10^{-3} arcseconds in the rotation, or one part in 10^4 in any partial derivative.

3. Comparison with Laser Observations

We compared our numerically-integrated model of the Moon's rotation with the laser ranging ('normal point') observations of the Apollo 11, 14, and 15, and the Lunokhod 2 retro-reflectors obtained from 1970 to 1975 by the McDonald Observatory (Abbott *et al.*, 1973; Shelus *et al.*, 1975; Mulholland *et al.*, 1975; Shelus, 1976). For this comparison, we adjusted simultaneously the six initial conditions of the Moon's rotation, the moment-of-inertia ratios β and γ, four coefficients of the third-degree terms in the Moon's gravitational potential,* the six initial conditions of the Moon's orbit about the Earth, five elements of the orbit of the Earth–Moon barycenter about the Sun (the longitude of the ascending node was fixed to define an origin of right ascension), the mass of the Earth–Moon system, the Earth's tidal lag angle, the three selenocentric coordinates of each lunar retroreflector, 126 parameters describing the variation of Universal Time (UT) (King *et al.*, 1978), the geocentric latitude and radius of the McDonald Observatory (the longitude was fixed to define the origin of UT), and a parameter representing a possible bias in the ranging observations in 1972 (Silverberg, 1975). An unfortunate consequence of simultaneous adjustment of so many parameters is the fact that errors in the lunar rotation model may have been masked to some extent by spurious adjustments to other (e.g., orbital) parameters. However, independent determinations of the other adjusted parameters with accuracies sufficient to enable us to use an *a priori* convariance matrix to advantage were not available.

The root-mean-square (rms) of the post-fit range residuals was 28 cm. This value is about twice the rms range-measurement error estimated by Shelus (1977). The post-fit residuals (King *et al.*, 1978) also have clearly discernible, systematic behaviour which is suggestive of deficiencies in our models, not only for the lunar rotation, but also for the lunar orbit and for the rotation of the Earth. Some effects which are not yet included in our model of the lunar rotation and which might contribute to the ranges residuals at the 15-cm level are discussed below in Section 6.

* We estimated only J_3, C_{32}, S_{32} and S_{33}, since the current set of lunar ranging observations is relatively insensitive to the other third-degree coefficients. See, for example, King *et al.* (1976).

4. Comparison with LLB-5

Although, in a sense, comparison with observations provides the ultimate test of a model, it is important to compare our numerical model of the Moon's rotation with other theoretical models. In this section we compare ours with the numerical 'LLB-5' model developed by Williams (1975).* This comparison is non-trivial not only because Williams' and our models were developed independently, but also because LLB-5 was generated by integration of the differential equations governing the (small) *Cassini* angles that describe the libration of the Moon with respect to a frame that rotates with the 'mean' lunar orbit, whereas our model was generated by means of an integration in inertially-referred (Euler-angle) coordinates. The orbital ephemerides used in the two rotation integrations were also derived independently.

To effect the comparison, we integrated the equations of our model using the same values of the lunar moment-of-inertia ratios and gravitational-potential harmonic coefficients as had been used in the generation of LLB-5. To obtain initial conditions for this integration, we first calculated values for the Euler angles and their rates directly from the initial values of the LLB-5 Cassini angles and rates, according to the definitions of these angles. The results of our Euler-angle integration were then transformed to Cassini angles at one-day intervals and compared with the corresponding values from LLB-5. This comparison showed differences between the models at the level of a few tenths of an arcsecond, that reflected mainly the inconsistencies between the fundamental coordinate frames to which Williams' and our angles are referred. These coordinate systems, which are defined operationally here by our orbital ephemerides, are known to differ in overall orientation at epoch $\sim$1975 by a few tenths of an arcsecond; there are also indications of a possible relative angular velocity of the order of $0''.01$ per year (King *et al.*, 1978).

In order to eliminate such inconsistencies as a cause of apparent differences between our lunar rotation models, we subtracted a constant offset, or 'bias', from each of the Cassini angles given by LLB-5, and we adjusted these three biases and the six initial conditions of our integration to fit our model to LLB-5 in a least-squares sense. Specifically, we minimized the sum of the squared differences in the three Euler angles, evaluated at one-day intervals over the six-year span.

The differences between our adjusted model and LLB-5 are shown in Figure 2. The estimated Cassini-angle biases, and the rms differences with the biases subtracted, are given in Table I. The greatest rms differences, in ρ and $I\sigma$, correspond to differences of about 4 cm in range to the lunar-surface retro-reflector farthest from the sub-Earth point, so that one neither expects, nor finds (Cappallo *et al.*, 1977), a significant difference between the two models' abilities to fit existing laser ranging observations. The relatively large bias differences between the models are masked, in fits to observations, by differences in the values obtained for the selenocentric coordinates of the retro-reflectors.

* An early version of the latter model, called 'LLB-3', was described by Williams *et al.* (1973).

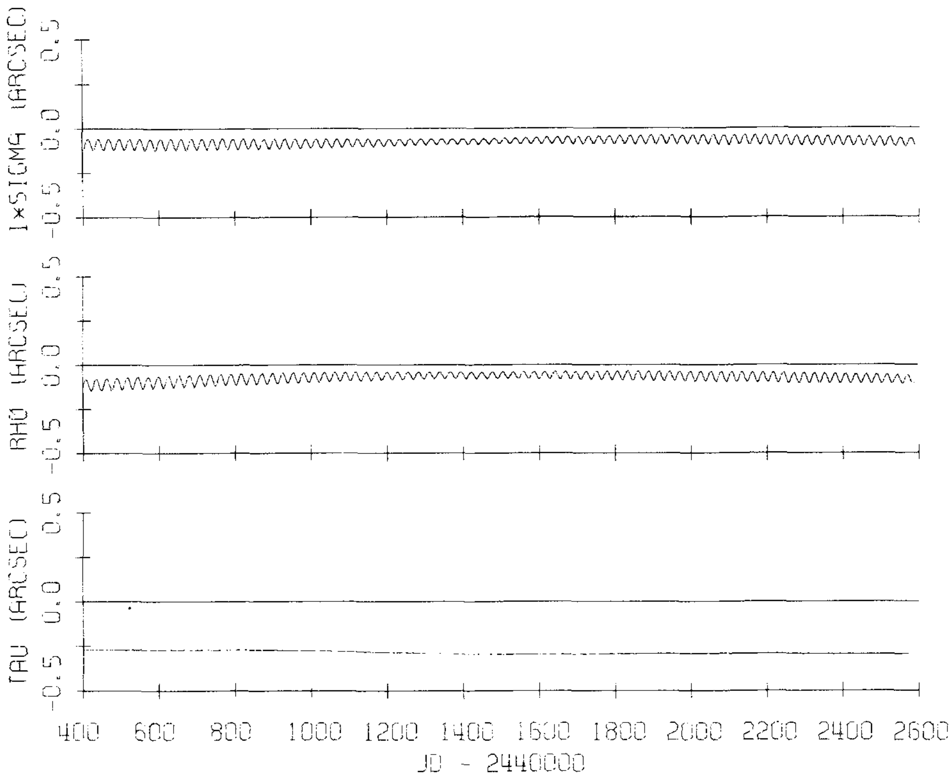

Fig. 2. Differences in Cassini libration angles between Williams' LLB-5 and our model, in the sense ours minus his. See text.

TABLE I.

Estimated biases and root-mean-square (rms) differences with the biases removed, for the Cassini angles describing lunar librations in longitude (τ), latitude (ρ), and node ($I\sigma$), resulting from the comparisons of our lunar rotation model with the LLB-5 model of Williams (1975) and the series-500 model of Eckhardt (1981). At the surface of the Moon, 0''.01 is equivalent to ~8 cm.

Cassini Angle	Williams		Eckhardt	
	bias	rms	bias	rms
τ	0''.286	0''.009	0''.164	0''.069
ρ	0''.069	0''.024	0''.023	0''.114
$I\sigma$	0''.083	0''.021	0''.091	0''.129

5. Comparison with Eckhardt's Model

A comparison of our numerical model with an analytical model of the Moon's rotation is important because of the possibility, with an analytical model, of distinguishing readily between free and force librations of the Moon. Such a comparison can also reveal an error in either model unless, importantly and unfortunately, the error happens to have a signature resembling that of one or more of the modes of the free libration.

With this limitation in mind, we have compared our numerically-integrated model with Eckhardt's (1981) series 500 semi-analytic model. We adjusted our initial conditions and the Cassini-angle biases to fit the Cassini angles given by Eckhardt's model, following the procedure described in Section 4, but with one additional step: Since, in Eckhardt's libration theory, the effects of the so-called planetary and additive terms in the lunar orbit are neglected, we added 66 corresponding correction terms, calculated by Williams (1975) [see also Williams *et al.*, 1973] to Eckhardt's theory before making the comparison with our model. These terms range in amplitude from $\sim 0\overset{''}{.}005$ to $14''$ of selenocentric arc and in period from near-monthly to 271 years. We also added to Eckhardt's theory seven terms calculated by Williams (1975) to compensate for the fact that the argument of the node in Eckhardt's development is reckoned from a moving equinox rather than a fixed direction in inertial space.

The post-fit Cassini-angle differences are plotted in Figure 3; the rms differences and estimated biases are given in Table I. A spectral analysis of the six-year span of computed ρ- and $I\sigma$-differences between our model and Eckhardt's, shows significant 'power' to be present at periods of ~ 27.0 and ~ 27.3 days, but has insufficient resolution to distinguish among several possible causes of the differences.

The differences between our model and Eckhardt's are significantly larger than those between our model and LLB-5 and they are comparable in size to the differences given by Migus (1980) between his analytic model and Eckhardt's series-500 model. There is a constant latitudinal difference of $\simeq 0\overset{''}{.}16$ between the theories of Migus and Eckhardt (see Migus' Figure 2). This difference is equivalent to a 27.2-day term of similar amplitude in ρ and $I\sigma$. Another possible source of the differences between our model and Eckhardt's is an error in the planetary and additive corrections which we applied to his model: Two terms in Williams' list, with periods of 27.10 and 27.44 days and amplitudes of $0\overset{''}{.}11$ and $0\overset{''}{.}15$, respectively, are not present in an analogous list calculated by Migus (1977). Williams (1975) himself has suggested that the linearizing procedure used for the calculation of the planetary and additive corrections may not have been adequate.

There are also differences at the several-tenths-of-an-arcsecond level, both between Eckhardt's and Migus' basic theories, and between Williams' and Migus' planetary and additive corrections, in terms with periods near that of the free libration in longitude, ~ 2.9 years.* Errors in these terms are masked in a comparison with a numerical theory by adjustment of the latter's initial conditions.

* Accurate calculation of such terms is obviously crucial to the use of an analytic theory for the detection of free libration.

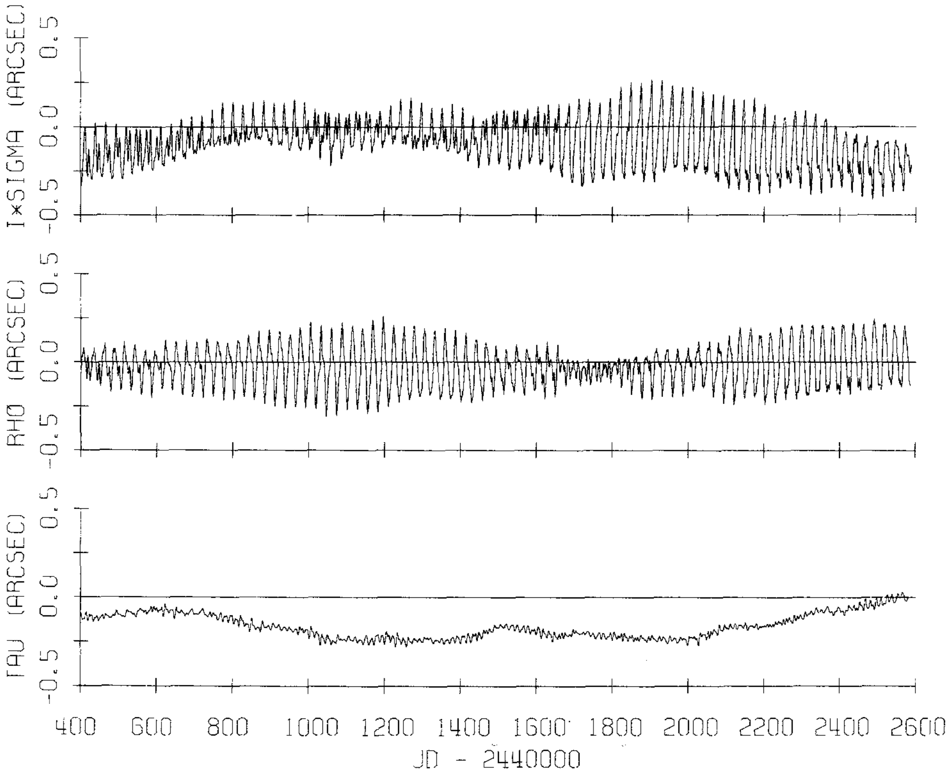

Fig. 3. Same as Figure 2, except for Eckhardt's series 500 model.

6. Summary and Conclusions

Our numerical model of the Moon's rotation, used in conjunction with our current models for the Earth's rotation and for the Earth's and the Moon's orbital motions, fits lunar laser ranging observations over a five-year period within 28 cm (rms). Our rotation model is consistent with the LLB-5 numerical model of Williams (1975) to within $\sim 0\overset{\prime\prime}{.}01$ (rms) in longitude and $\sim 0\overset{\prime\prime}{.}02$ in latitude and node, except for biases in the Cassini libration angles. Most of these differences may be due to differences between the orbital ephemerides used by Williams and by us in the generation of our respective rotation models. In both models, the Moon was treated as a rigid body with no nonzero gravitational harmonics beyond third degree, and subject to torques from only the Sun and the Earth, with these bodies represented by point masses.

Our model differs from the series 500 semi-analytical model of Eckhardt (1981) by $\sim 0\overset{\prime\prime}{.}07$ (rms) in longitude and $\sim 0\overset{\prime\prime}{.}12$ in latitude and node, after the subtraction of constant biases and the application of Williams' (1975) corrections to Eckhardt's model. These relatively large differences may represent the effects of errors either in the corrections we have applied, or in Eckhardt's basic theory.

The Moon is, of course, not a rigid body, nor is the Earth a point mass. We are therefore in the process of including the effects of lunar elasticity and internal dissipation and also those of the Earth's oblateness, in our equations of motion and in the variational equations that we integrate simultaneously. We have already incorporated the effects of the lunar gravity harmonics of fourth and higher degree, but so far we have kept these coefficients set to zero and have not attempted to adjust them to fit observations. We now also have the capability of integrating the Moon's rotational and orbital equations of motion concurrently, a procedure which improves computational efficiency and treats rigorously the cross-coupling between the two sets of equations. With the completion of these modifications, we hope to have a lunar rotation model with accuracy exceeding that of the best modern observations.

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