INDUCED VELOCITIES OF GRAINS EMBEDDED IN A TURBULENT GAS*

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Abstract. A theory is presented for the dynamics of dust particles in an incompressible turbulent fluid. Grain-gas coupling occurs through friction forces that are proportional to the mean grain velocity relative to the gas. This test particle theory is applied to the case of a Kolmogoroff spectrum in a protostellar cloud. The mean turbulence induced grain velocity and the mean turbulent relative velocity of two grains are calculated. Whereas the former should determine the dust scale height, grain-grain collisions are influenced by the latter. For a reasonable strength of the turbulence, the mean induced relative velocity of two particles turns out to be at least as large as the corresponding terminal velocity difference during gravitational settling.

1. Introduction

It is probable that astrophysical dust clouds are often in a turbulent state. This then leads to the question of the consequences for the dynamics of the grains embedded in these clouds. As long as the mass density of the dust is negligibly small compared to the total mass density there are essentially two effects: First of all, in a gravitational field, sedimentation is impeded to a varying degree depending on grain size, specific weight of the grain material, gas density, and temperature, that together determine the time-scale for frictional coupling. Secondly, relative velocities between grains are induced, even for like particles, that lead to or increase the probability of grain-grain collisions. Since such collisions are not only elastic but will lead to sticking at low and to shattering at high relative velocities, turbulence will introduce changes in the grain size spectrum. We will consider grains in a neutral gas.

In a dense protostellar cloud we expect random grain velocities to be still small enough so that sticking is the dominant process. The consequent increase in particle size however results in increased sedimentation so that turbulence in the gas might possibly lead to faster sedimentation than occurs in a laminar gravitational collapse. It may even directly lead to large solid bodies that later agglomerate into planets. With this latter idea in mind turbulent effects on grains were discussed for example by Cameron (1973).

In this paper we shall be concerned with the general problem of random grain velocities in a turbulent gas. However, the parameters used are chosen with regard to a collapsing

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cloud as discussed by Morfill *et al.* (1978) so that turbulent collision rates can be compared with the laminar ones.

2. Dynamics of Dust Particles

A dust particle immersed in a gas or a fluid is primarily characterized by its friction time τ_f. Assuming the mean free path of a gas molecule (mass m) to be large compared to the grain size, and its thermal velocity $v_{\rm th} = (2KT/m)^{1/2}$ to be large compared to the mean relative velocity between the grain and the gas, then

$$(\tau_f)^{-1} = \frac{2}{\sqrt{\pi}} \rho\, \sigma_s\, v_{\rm th}\, M^{-1},$$

where ρ, σ_s, and M denote the mass density of the gas, the geometric cross section, and the mass of a dust particle, respectively. Thus $\tau_f \sim r\,\rho_s$, where r is a linear dimension of the grain, and ρ_s its mass density. Taking $r = 10^{-4}$ cm, $\rho_s = 1\,{\rm g\,cm^{-3}}$, $\rho \sim 10^{-13}\,{\rm g\,cm^{-3}}$, and $v_{\rm th} \sim 10^5\,{\rm cm\,s^{-1}}$, typical values for τ_f range from 10^{10} to 10^{11} s during the early phase of the Jeans collapse of a $3M_\odot$ cloud (Tscharnuter, 1978). They are small compared to the typical collapse times of a few 10^{+13} s.

The equation of motion of a dust particle is

$$\frac{{\rm d}\mathbf{v}}{{\rm d}t} = \mathbf{a}_G(\mathbf{x},t) - \frac{1}{\tau_f}(\mathbf{v} - \mathbf{v}_G), \tag{1}$$

where $\mathbf{v}$ is the grain velocity, $\mathbf{a}_G$ its acceleration due to external volume forces, like gravity, and $\mathbf{v}_G$ is the mean gas velocity. The gas velocity in turn is given by

$$\frac{{\rm d}\mathbf{v}_G}{{\rm d}t} = \mathbf{a}_G - \frac{1}{\rho}\,{\rm grad}\, p, \tag{2}$$

where p is the gas pressure and we have neglected the momentum loss of the gas to the grains. Thus $\mathbf{v}_G$ is considered as a given quantity in the present context. In order to describe turbulent motions, let $\mathbf{v}_G$, p and correspondingly $\mathbf{v}$ have an average – i.e., a laminar, plus a randomly fluctuating component: $\mathbf{v}_G = \langle \mathbf{v}_G \rangle + \delta\mathbf{v}_G$ etc. By definition the ensemble average $\langle \delta\mathbf{v}_G \rangle$ over the fluctuating component of $\mathbf{v}_G$ is zero, as is true for δp and $\delta\mathbf{v}$. Inserting this into Equation (1) and subtracting the emsemble averaged equation we find that

$$\frac{\rm d}{{\rm d}t}\delta\mathbf{v} = -\frac{1}{\tau_f}\delta\mathbf{v} + \frac{1}{\tau_f}\delta\mathbf{v}_G, \tag{3}$$

which is a Langevin equation with $+\,(1/\tau_f)\,\delta\mathbf{v}_G$ as the random driving term.

The solution of Equation (3) can be expressed as

$$\delta\mathbf{v}(t) = \frac{1}{\tau_f}\int_0^t {\rm d}t'\, \exp\{-(t-t')/\tau_f\}\,\delta\mathbf{v}_G(\mathbf{x}(t'), t') +$$

$$+\,\delta \mathbf{v}(t = 0) \exp\{-t/\tau_f\}, \tag{4}$$

where $\mathbf{x}(t')$ is the position of the dust particle at time t'. Thus for spatially varying $\delta \mathbf{v}_G$, solution (4) constitutes a nonlinear integral equation that can be solved only approximately.

For this purpose we consider $\delta \mathbf{v}_G$ to be represented by spatial Fourier components (eddies) $\mathbf{w}(k, t)$ corresponding to velocity variations over the sale $2\pi/|k|$: i.e.,

$$\delta \mathbf{v}_G(\mathbf{x}, t) = \int_{-\infty}^{+\infty} \mathrm{d}k \mathbf{w}(\mathbf{k}, t)\, e^{i\mathbf{k}\mathbf{x}}. \tag{5}$$

The time variation of $\mathbf{w}(\mathbf{k}, t)$ first of all consists of a factor $\exp\{-i\mathbf{k}[\langle \mathbf{v}_G\rangle + \int_{k_0}^{k} \mathrm{d}k'\, \mathbf{w}(\mathbf{k}', t)\, e^{i\mathbf{k}'\mathbf{x}}]\, t\}$ due to the convection of the eddy $\mathbf{k}$ by the average gas flow and the larger eddies $\mathbf{k}'$ (with $|\mathbf{k}'| < |\mathbf{k}|$) on which it rides. In addition there exists an intrinsic time dependence of $\mathbf{w}(\mathbf{k}, t)$ describing the statistical correlations for the eddy $\mathbf{k}$; it is expressed in the form of an eddy life-time τ_k.

In the trajectory integral of Equation (4) the convective time dependence of $\delta \mathbf{v}_G$, as seen from the moving grain, enters through the factor $e^{i\mathbf{k}\mathbf{x}}$ in Equation (5). The two types of convective time dependences can be combined into an expression (r.h.s. of Equation (4)) of the type $\exp\{i\mathbf{k}\mathbf{V}_{\mathrm{rel}}(|\mathbf{k}|)t'\}$, where $\mathbf{V}_{\mathrm{rel}}$ is the speed of the grain relative to the eddy $\mathbf{k}$. The technical details of the derivation will be published elsewhere. The result is that in the sense of an r.m.s. value, averaged over the random phases of the Fourier components $\mathbf{w}(\mathbf{k}, t)$, the modulus of the quantity $\mathbf{V}_{\mathrm{rel}}$ can approximately be written as

$$V_{\mathrm{rel}}(k) \simeq \left\{ V_L^2 + \int_{k_0}^{k} \mathrm{d}k' P(k') \frac{\tau_f}{\tau_f + \tau_{k'}} \right\}^{1/2}, \tag{6}$$

where $P(k)$ is the power spectrum of the fluctuations $\delta \mathbf{v}_G$, which is assumed to be isotropic and therefore only depends on $k \equiv |\mathbf{k}|$: i.e.,

$$(\delta \mathbf{v}_G)^2 = \int_{k_0}^{\infty} \mathrm{d}k P(k). \tag{7}$$

The largest scale of the turbulence is given by $2\pi/k_0$; the grains can have an average (laminar) speed relative to the gas of magnitude V_L, which may for example be due to an average sedimentation effect.

With this quantity $\mathbf{V}_{\mathrm{rel}}(k)$ we can now approximately solve Equation (4) by dividing the eddies into two classes (at given particle radius, i.e. τ_f).

Class 1 comprises all eddies in which a particle gets stopped by friction before it either crosses the eddy or before the eddy decays. This means that $k \ll k^*$, where

$$\frac{1}{\tau_f} = \frac{1}{\tau_{k^*}} + k^* V_{\mathrm{rel}}(k^*). \tag{8}$$

Clearly the particle under consideration 'sees' the eddies of class 1 as essentially a spatially homogeneous but time dependent medium, i.e. δv_G is homogeneous in space but varies in time. Class 3 comprises all eddies $k \gg k^*$. The particle interacts weekly with class 3 eddies.

Class 2 comprises the eddies $k = 0(k^*)$ and is not easily tractable. We omit it by approximating class 1 and class 3 by $k < k^*$ and $k > k^*$, respectively. With these approximations and assuming the phases of the eddies to be uncorrelated with each other we can calculate $\langle(\delta v)^2\rangle$ corresponding to the mean square of induced particle velocity

$$\langle(\delta v)^2\rangle = \int_{k_0}^{k^*} dk P(k) \frac{\tau_k}{\tau_k + \tau_f} + \int_{k^*}^{\infty} dk P(k) \frac{\tau_k}{\tau_k + \tau_f} \times \\ \times \frac{\operatorname{arctg} \left\{ \dfrac{k V_{rel}(k)\, \tau_k\, \tau_f}{\tau_k + \tau_f} \right\}}{\left\{ \dfrac{k V_{rel}(k)\, \tau_k\, \tau_f}{\tau_k + \tau_f} \right\}}. \tag{9}$$

We can also calculate the mean square $\langle(\Delta\delta v)^2\rangle$ of the induced relative velocity between two particles (denoted by the index 1, 2)

$$\langle(\Delta\delta v)^2\rangle = \int_{k_0}^{k^*(1)} dk P(k) \frac{\tau_k(\tau_f(1) - \tau_f(2))^2}{[\tau_k + \tau_f(1)]\,[\tau_k + \tau_f(2)]\,[\tau_f(1) + \tau_f(2)]} + \\ + \int_{k^*(1)}^{k^*(2)} dk P(k) \frac{\tau_k}{\tau_k + \tau_f(2)} + \\ + \left\{ \int_{k^*(1)}^{\infty} dk P(k) \frac{\tau_k}{\tau_k + \tau_f(1)} \frac{\operatorname{arctg} \left\{ \dfrac{k V_{rel}(k,1)\, \tau_k\, \tau_f(1)}{\tau_k + \tau_f(1)} \right\}}{\left\{ \dfrac{k V_{rel}(k,1)\, \tau_k\, \tau_f(1)}{\tau_k + \tau_f(1)} \right\}} \right. \\ \left. + (1 \to 2) \right\} \tag{10}$$

where $(1 \to 2)$ indicates interchange of index 1 and 2, while $k^*(1) < k^*(2)$ was assumed.

To obtain the total mean square relative velocity we have to add the term $[V_L(1) - V_L(2)]^2$ to the right-hand side of Equation (10).

3. Results

We shall now apply the foregoing results to the case of a collapsing protostellar cloud (Tscharnuter, 1978; Morfill *et al.*, 1978). For this we take typical parameters of the 'hydrostatic core' which develops in the inner region. The values chosen are $\rho \sim 10^{-13}$

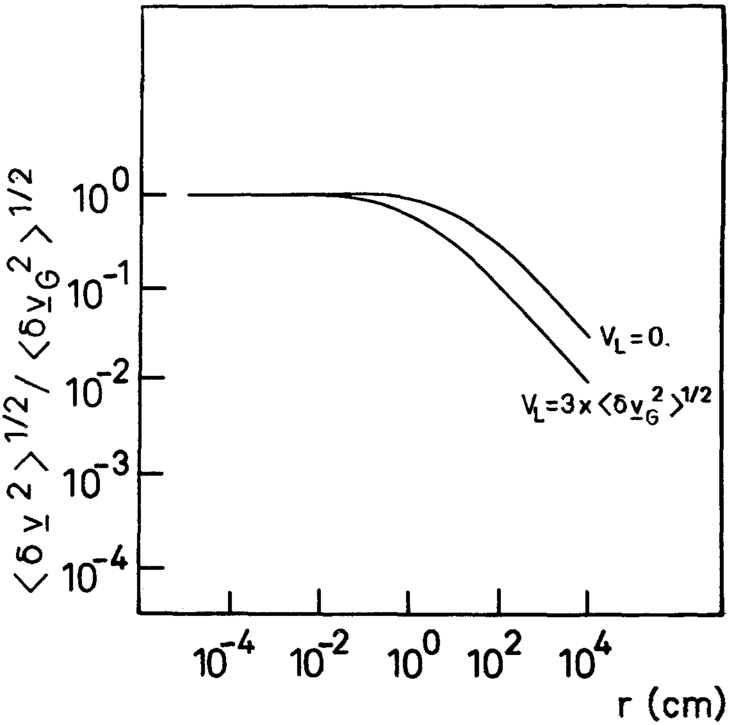

Fig. 1. R.m.s. turbulence induced velocity $\langle \delta v^2 \rangle^{1/2}$ of a dust particle of radius r, normalized to the r.m.s. turbulent gas velocity $\langle \delta v_G^2 \rangle^{1/2}$, for two values of the average velocity V_L of the grain relative to the gas. The mass density of a grain is taken as $1\,\mathrm{g\,cm^{-3}}$, the gas density as $10^{-13}\,\mathrm{g\,cm^{-3}}$; $k_0 = 5 \times 10^{-14}\,\mathrm{cm^{-1}}$ is chosen as the smallest wave number of the turbulence. The quantity $\langle \delta v_G^2 \rangle^{1/2}$ is assumed to be $c/3$, where c denotes the speed of sound.

$\mathrm{g\,cm^{-3}}$, $v_{th} \sim 10^5\,\mathrm{cm\,s^{-1}}$, and $k_0 \sim 5 \times 10^{-14}\,\mathrm{cm^{-1}}$, corresponding to a spatial extent along the rotation axis of about 8 AU at which the turbulence may be fed by, for example, the strongly nonuniform rotation. This 'hydrostatic core' denotes the region where free fall is stopped (essentially by an accretion shock) and within which – although it is still contracting – there exists an approximate state of equilibrium between pressure plus centrifugal forces and gravity. As an educated guess we assume a Kolmogoroff spectrum $P(k) \sim k^{-5/3}$ for the turbulence, with $\tau_k \sim k^{-1}.\ (kP(k))^{-1/2}$, and normalisation according to Equation (7). Hereby we have taken the inner scale $2\pi/k_i$ of the turbulence to be arbitrarily small; we can do this because of the fact that $k_0/k_i = 0(10^{-7})$ for the above parameters in an essentially neutral hydrogen gas.

Whereas the results up to Equation (1) were rather general, such a choice for the form of the turbulent spectrum introduces some arbitrariness. However, a number of astrophysical fluctuations appear at least to be given by power law spectra, in fact by chance or otherwise close to a Kolmogoroff spectrum. We mention here only the density and velocity fluctuations of the interstellar medium (Lee and Jokipii, 1976) and the velocity

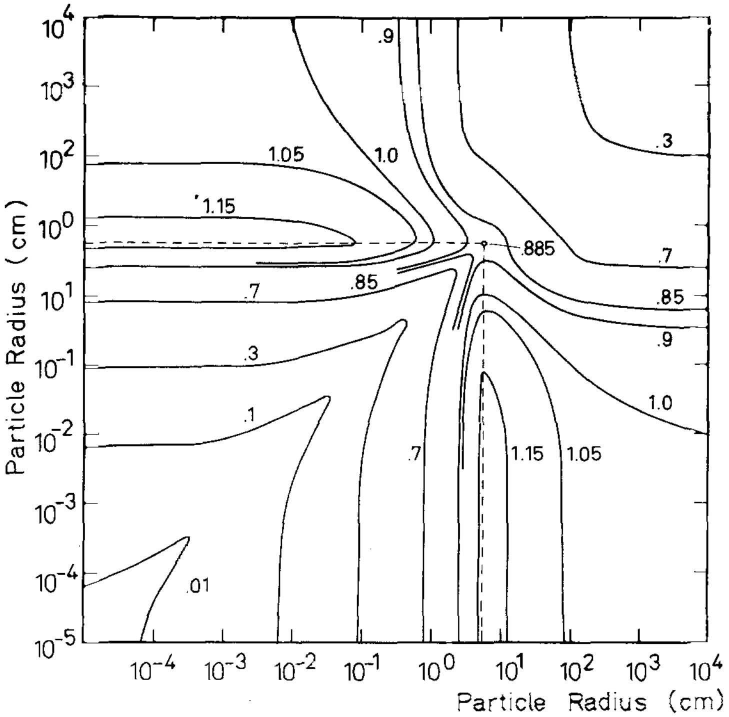

Fig. 2. Contour plot of normalized r.m.s. relative velocity between two particles in terms of their radii, given by the abscissa and ordinate, respectively. The normalization and the other parameters are the same as in Figure 1, retaining the value $V_L = 0$ only.

spectrum in the solar wind (Coleman, 1966). A more important question is the total strength of the turbulence. Fortunately, at least for $V_L = 0$, this factor can be eliminated by normalizing induced grain speeds to turbulent gas velocities. Below we will use results derived from atomic bomb tests (Colgate, private communication). Although they may lie on the high side, they should not be grossly unrealistic either. According to Equations (6), (8), (9) and (10), the quantities $\langle(\delta v)^2\rangle^{1/2}$ and $\langle(\Delta\delta v)^2\rangle^{1/2}$ depend, for fixed V_L, on particle properties only through τ_f. Assuming the particles to be spherical with radius r, then $\tau_f \sim r\rho_s$. Thus, a plot of the right-hand sides of Equations (9) and (1) against r is in reality one against τ_f/ρ_s. In Figures 1 and 2 we chose $\rho_s = 1$. Such a value appears appropriate for grains in a protostellar environment such as considered here (Greenberg and Whipple, private communications). For given values of $\langle(\delta v)^2\rangle^{1/2}$ and $\langle(\Delta\delta v)^2\rangle^{1/2}$ the radius r corresponding to a grain with density ρ_s is then given by $r = r/\rho_s$ (ρ_s in g cm^{-3}).

In Figures 1 and 2, the turbulence induced grain speed $\langle\delta v^2\rangle^{1/2}$ and the turbulence induced relative velocity $\langle(\Delta\delta v)^2\rangle^{1/2}$ are normalized to the r.m.s. turbulent gas velocity $\langle(\delta v_G)^2\rangle^{1/2}$. For small grains, the normalised r.m.s. induced speed $\langle(\delta v)^2\rangle^{1/2}/\langle(\delta v_G)^2\rangle^{1/2}$ is equal to 1; almost all energy resides in eddies of class 1 and particles are tightly coupled to the gas. With increasing r however, k^* diminishes. When k^* reaches the minimum possible value k_0 (for our parameters around r = several cm), then $\langle(\delta v)^2\rangle^{1/2}$ starts to decrease, asymptotically $\sim 1/r \sim 1/\tau_f$. This behaviour (for $V_L = 0$) is not strongly changed if we chose $V_L = 10^5$ cm s^{-1}, a very high value, which is of the order of the free fall speed of the protostellar cloud.

Figure 2 is a contour plot of $\langle(\Delta\delta v)^2\rangle^{1/2}/\langle(\delta v_G)^2\rangle^{1/2}$ as a function of the radii of the two particles. Obviously the picture must be asymmetric around the 45° line. Relative to a very large grain, $r \gg 10$ cm, a very small grain has a normalised r.m.s. velocity of about 1, since the large particle decouples from the gas almost completely, whereas the small grain is tightly coupled to the gas, as mentioned earlier. If, however, the larger particle has a radius such that $k^* = k_0$, then the turbulence induced normalised r.m.s. relative velocity has a maximum (dotted line) which for a small partner particle can exceed 1. This is so because the uncorrelated effects of eddies with $k > k^*$ add up quadratically.

For similar particles the first two terms on the right-hand side of Equation (10) disappear and only the uncorrelated effects from $k > k^*$ remain. These are small for small particles because then $k^* \gg k_0$. For large particles they are again small because now τ_f is large. The maximum value ~ 0.885 appears for $k^* = k_0$. It should be pointed out that relative velocities between similar particles are not possible in a laminar gas flow except near stagnation points (e.g. near the midplane of a collapsing disk). If we assume $\langle(\delta v_G)^2\rangle^{1/2} \sim c/3$, where c is the velocity of sound of the gas (Colgate, private communication) and insert $c \sim 10^5$ cm s^{-1} as a typical value, then we see that the r.m.s. relative induced velocity between particles of radii $r \sim 10^{-3}$ cm and $r \sim 10^{-2}$ cm is of the order of 3×10^3 cm s^{-1}. This is at least as high as typical values for the corresponding difference $V_L(r = 10^{-3}) - V_L(r = 10^{-2})$ of the laminar terminal velocities (Morfill *et al.*, 1978). Thus, particle collisions induced by turbulence may indeed play a significant role in the dynamics of dust in a protostellar environment if the gas is essentially neutral. In the case of dust in an H II region, where the grains are charged, τ_f becomes very much smaller due to Coulomb drag. Then relative velocities between grains induced by turbulence are probably negligible.

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