RESONANT STRUCTURES IN THE SOLAR SYSTEM

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Abstract. The resonance theory is discussed with respect to the Solar System with a view to show that every triad of successive planets in the Solar System follows Laplace's resonance relation. With rings now known to exist around three of the four major planets, scientists have begun to speculate about the possible existence of ring structure and one or two small planets going around the Sun itself. It is also believed that the ring systems may exist around the planets Neptune and Mars. In this paper an attempt is made to provide a basis to these beliefs using Laplace's resonance relation. The triads of successive innermost objects (rings and/or satellites) in the satellite – systems of Jupiter, Saturn and Uranus are also shown to follow Laplace's resonance relation.

1. Introduction

Resonance theory states that if n_1, n_2, n_3, ($n_i = 2\pi/T_i$, $n_1 > n_2 > n_3$), are mean motions of three planets in circular orbits, then a necessary (but not sufficient) condition for the frequent occurrence of mirror configuration (Dermott, 1973; and references given therein) is given by the equation

$$\alpha n_1 - (\alpha + \beta)n_2 + \beta n_3 = 0, \tag{1}$$

where α, β are mutually prime positive integers. It follows from Equation (1) that in a reference frame rotating with the mean motion of any one of the three planets, the relative mean motions n_i' of the other two are commensurate, and that in a frame I (that of the innermost planet), we have

$$n_2'/n_3' = (n_2 - n_1)/(n_3 - n_1) = \beta/(\beta + \alpha). \tag{2}$$

For a stable three-body resonance, the relative mean motion ratio $\beta/(\beta + \alpha) = 2/3$, and Equation (2) assumes the form

$$n_2'/n_3' = (n_2 - n_1)/(n_3 - n_1) = 2/3. \tag{3}$$

This is Laplace's resonance relation and the three successive orbits following this relation represent stable motion.

Our aim, here, is to discuss the resonance theory with respect to the Solar System with a view to find whether every triad of successive planets in the Solar System is in accordance with Laplace's resonance relation at least approximately, if not exactly. If such would be the state of affairs, then with the help of Laplace's resonance relation, we could venture to go inward to find whether there exist a few orbits within th orbits of Mercury, Phobos and Triton. The existence of such orbits would then strengthen the general belief that ring systems may exist around the Sun, Neptune and Mars. We shall also consider the

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triads of successive innermost objects in the satellite-systems of Jupiter, Saturn and Uranus with the hope that they may provide additional support to the aforesaid belief by submitting themselves to Laplace's resonance relation.

2. Resonant Structures in the Solar System

The application of relative mean motion ratio (RMMR) to the triads of successive planets in the Solar System yields the values set out in Table I. It is clear from Table I that the mode of RMMR is 2/3, and this may be taken as a constant connected with the Solar System. This also implies stability of the Solar System. Table I shows that the planets in the Solar System follow Laplace's resonance relation.

Suppose that an intra-Mercurial object (a planet or a ring) exists and that a triad-intra-Mercurial object, Mercury and Venus has a stable resonance, i.e., RMMR = 2/3. This restriction on RMMR of stable resonance on the orbits is slightly more stringent condition but from Table I, it is justified to take RMMR to be 2/3 for any triad of successive planets in the Solar System, and hence, for the above mentioned triad as well. Then by applying relation (3), we know the mean motion n_1 of the hypothetical intra-Mercurial object and hence its orbital period and distance.

The repeated application of this idea gives us all the stable resonant orbits within the orbit of Mercury. Table IIa shows orbital distances and periods of all hypothetical intra-Mercurial objects that one may get using this procedure. The hypothetical intra-Mercurial object at a distance of 0.23 AU lies well outside the synchronous orbit for a planet orbiting the Sun and, therefore, may have survived there without disruption, whereas, all other hypothetical intra-Mercurial objects lie well within the synchronous orbit for a planet orbiting the Sun. Tidal forces would tend to drag those objects toward the Sun, until they break up at the Roche radius $\simeq 2.46(\rho_\odot/\rho)^{1/3} \times R_\odot$, where $\rho_\odot$ is the mean density of the Sun, ρ is the density of the object going around the Sun and $R_\odot$ is the radius of the Sun. The general formula for Roche limit $\simeq 2.46(\rho_p/\rho_s)^{1/3} \times R_p$, where ρ_p is the density of the primary, ρ_s is the density of the secondary and R_p radius of the primary. For an object going around the Sun and consisting of the material of density $\simeq 2\,\mathrm{g\,cm^{-3}}$ (refractory material such as graphite has this density), one finds Roche limit $\simeq 2R_\odot$, and for an object consisting of silicates of density $\simeq 3.5\,\mathrm{g\,cm^{-3}}$ the Roche limit moves inward to $\simeq 1.6R$. Such an argument naturally gives rise to a ring system around the Sun analogous to those found around the major planets. Brecher *et al.* (1979) arrive at a consistent picture of the primordial ring of refractory material, possibly of graphite, allowed to reside around the Sun. According to them, it must lie at a distance $\gtrsim 4R_\odot$, its total mass $M \lesssim 6 \times 10^{25}\,\mathrm{g}$* it could consist in that orbit of at most $N \lesssim 10^6$ objects of minimum radius $A \simeq 10\,\mathrm{km}$. They are also hopeful of the observation of such a ring system around the Sun. The hypothetical intra-Mercurial objects at distances of 0.14, 0.088,

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* If we apply Bruman's formula (1968) to determine the mass residing within the orbit of the intra-Mercurial planet corresponding to the distance 0.23 AU, the same figure can be arrived at.

TABLE I

Resonance in the triads of successive planets in the Solar System

Triad	RMMR $\beta/(\beta + \alpha)$
Mercury Venus Earth	$3/4 \simeq 2/3$
Venus Earth Mars	$4/7 \simeq 2/3$
Earth Mars Asteroid	$5/8 \simeq 2/3$
Mars Asteroid Jupiter	$2/3$
Asteroid Jupiter Saturn	$5/7 \simeq 2/3$
Jupiter Saturn Chiron	$2/3$
Saturn Chiron Uranus	$2/3$
Chiron Uranus Neptune	$4/7 \simeq 2/3$
Uranus Neptune Pluto	$3/4 \simeq 2/3$

0.05, 0.03, 0.019 AU (Table IIa) lie beyond $4R_{\odot}$. At the same time they are also within the synchronous orbit for a planet orbiting the Sun. If they consist of graphite and have minimum radius $A \simeq 10\,\mathrm{km}$, then they might have survived both vaporisation and drag for the age of the Solar System forming a ring system around the Sun. At a distance of only $2R_{\odot}$, even an object as large as it is allowed by considerations of possible tensile strengths ($A \simeq 300\,\mathrm{km}$) would evaporate, no matter what its composition, in less than 10^9 years. Therefore, the last two hypothetical intra-Mercurial objects at distances of 0.012 and 0.007 AU (Table IIa) going around the Sun are not expected to be traced out. Rawal (1978) has shown that there may be a planet, yet undiscovered, going around the Sun within the orbit of Mercury. Brecher *et al.* (1979) have shown that there may be a ring structure, yet undiscovered, going around the Sun within the orbit of Mercury. Here

TABLE II

Orbital distances and periods of all hypothetical intra-Mercurial, intra-Phobosian and intra-Tritonian objects obtained using Laplace's resonance relation

a. Intra-Mercurial objects	Dist. in AU	Dist. in Million km	Revolution period in days
HIMO$_1^a$	0.230	34.50	40
HIMO$_2$	0.140	21.00	20
HIMO$_3$	0.088	13.20	10
HIMO$_4$	0.050	07.50	05
HIMO$_5$	0.030	04.50	02.5
HIMO$_6$	0.019	02.85	01.25
HIMO$_7$	0.012	01.85	00.66
HIMO$_8$	0.007	01.05	00.30
b. Intra-Phobosian objects		Dist. in km.	Revolution period in days
HIPO$_1^b$		6000	0.1279
c. Intra-Tritonian objects			
HITO$_1^c$		171,300	1.983
HITO$_2$		96,310	0.8368
HITO$_3$		57,710	0.3883
HITO$_4$		35,520	0.1875

[a] HIMO$_n$ = Hypothetical Intra-Mercurial object No. n.
[b] HIPO$_1$ = Hypothetical Intra-Phobosian object No. 1.
[c] HITO$_n$ = Hypothetical Intra-Tritonian object No. n.

it is shown that there may be one or two small planets and a ring structure, all yet undiscovered, going around the Sun within the orbit of Mercury.

3. Are There Rings Around Mars and Neptune?

Let us probe into the general belief that ring systems may exist around Mars and Neptune. For this, we should find the stable resonant orbits, if any, between Mars and its satellite Phobos and between Neptune and its satellite Triton. Following the method which was applied to find intra-Mercurial objects, one gets Table IIb and Table IIc showing hypothetical intra-Phobosian and intra-Tritonian objects respectively.

According to relation (3), between Mars and the satellite Phobos, there could be a satellite at a distance of $\simeq 6000$ km from the centre of the planet Mars (Table IIb). This hypothetical satellite lies inside the synchronous orbit for a satellite orbiting the planet, and therefore, tidal forces would tend to drag it toward Mars, eventually breaking it up at its Roche radius forming a ring. Over the age of the Solar System, this process might have converted this object into a ring structure around the planet. In case, it has not been converted into a ring structure till now, then it is expected to be under slow disruption due to tidal forces and might have formed, or might be forming, a thin ring around the planet. If this hypothetical object would have density $\lesssim 10.34\,\mathrm{g\,cm^{-3}}$, then its Roche limit would have been $\gtrsim 6000$ km and in that case, it would have already been broken up forming a ring structure around the planet.

It is to be noted that the satellite Phobos at a mean distance of 9380 km has density $\sim 2\,\mathrm{g\,cm^{-3}}$ (Fielder, 1978, Veverka *et al.*, 1978). Its Roche limit is ~ 10 000 km. Therefore, Phobos lies within its Roche limit. Phobos has not been torn off forming a ring structure around the planet leads one to suspect that perhaps it lies just outside its Roche limit or just at the edge of its Roche limit. In any case, the possibility of its being under the slow disruption due to tidal forces cannot be ruled out (Soter and Harris, 1977), and hence it might have formed, or might be forming a thin ring around the planet. We note that the ring of Mars has not been detected so far. Is it that the ring has disappeared due to various drags? The density of Phobos is $\sim 2\,\mathrm{g\,cm^{-3}}$ and that of Mars $\sim 3.94\,\mathrm{g\,cm^{-3}}$, we take radius of Mars to be 3400 km and constant factor appearing in the formula for calculating Roche limit to be 2.46 (some take it to be 2.44), and, therefore, its Roche limit comes out to be 10 000 km. Is it that the density of Phobos is slightly higher and/or radius and density of Mars are slightly lower than the ones quoted above or else that the constant factor appearing in the formula for calculating Roche limit is 2.44 and not 2.46 so that the Roche limit comes down to a lower value so that Phobos has remained there so far without being torn off? Phobos seems to be a very interesting and puzzling satellite.

According to relation (3), between Neptune and the satellite Triton, there could be four satellites of Neptune at distances shown in Table IIc. The two hypothetical satellites between Neptune and Triton at distances of ~ 171 300 and 96 310 km lie well outside the synchronous orbit for a satellite orbiting Neptune and, hence, they may well be there. The third and the fourth hypothetical satellites at distances of ~ 57 710 and 35 520 km lie within the synchronous orbit and, therefore, tidal forces would tend to drag them toward Neptune, eventually breaking them up at their respective Roche radius forming a ring. Over the age of the Solar System, this process might have converted these objects into a ring structure around the planet. In case, they have not been converted into a ring structure till now, then they are expected to be under slow disruption due to tidal forces and might have formed or might be forming thin rings around the planet. If the third hypothetical object would have density $\lesssim 2.24\,\mathrm{g\,cm^{-3}}$ and the fourth would have density $\lesssim 6\,\mathrm{g\,cm^{-3}}$, then their Roche limits would have been $\gtrsim 57\,710$ km and 35 520 km respectively, and in that case, they would have already been broken up forming ring structures around the planet. From the above discussion, one may conclude that the planet Neptune may have two or three more satellites and a ring system, yet undiscovered, going around it within the orbit of Triton and the planet Mars may have a ring structure, yet undiscovered, going around it within the orbit of Phobos.

4. Resonance in the Triads of Successive Innermost Objects (Rings and/or Satellites) in the Satellite-Systems of Jupiter, Saturn and Uranus

Table III shows the distances, periods and RMMRs of the successive innermost objects constituting the triads in the satellite systems of Jupiter, Saturn and Uranus. From Table III

TABLE III

Resonance in the triads of successive innermost objects (rings and/or satellites) in satellite systems of Jupiter, Saturn and Uranus

System	Triad	Dist. in 1000 km	Revolution period in days	RMMR $\beta/(\beta+\alpha)$
a. Jupiter	JNS	131	0.2044	
	V (Amalthea)	181	0.498	2/3
	1979 J2	221	0.66	
b. Saturn	Ring C*	72	0.2273	
	Ring B	106	0.4059	$\simeq 2/3$
	Ring A	130	0.5544	
	Ring B	106	0.4059	
	Ring A	130	0.5544	$\simeq 2/3$
	SNS	147	0.6656	
	Ring A	130	0.5544	
	SNS	147	0.6656	2/3
c. Uranus	Ring	37	0.2115	
	UHS_1	68	0.5286	$\simeq 2/3$
	Miranda	131	1.414	

* Average distance.

it is clear that such a triad has its RMMR 2/3, i.e., the relation (3) holds. Conversely, if one assumes that relation (3) holds, one can check the period and hence the distance of any one of the objects in the triad.

In the Jovian system (Table IIIa), we have an object, designated here as JNS, at a distance of 131 000 km from the centre of the planet. The distance of this object is interpreted here to be the representative of the system consisting of a ring and small bodies (satellites) 1979 J1, 1979 J3 of Jupiter recently discovered by Voyagers in 1979 (Jewit *et al.* (1979), Johnson (1979), Thomsen and Van Allen (1979), Smith and Doose (1979), Synnott (1980), Owen *et al.* (1979)).

Stone (1980) has reported that a new satellite designated 1979 J2 has been discovered on images obtained during the encounter of Voyager with Jupiter. 1979 J2 has an orbital period of ~ 16 h and a mean distance of ~ $3R_j$ from Jupiter.

We have considered a triad of successive innermost objects consisting of JNS, V(Amalthea) and 1979 J2 (Table IIIa).

It is to be noted that Amalthea is just at the synchronous orbit of a satellite orbiting Jupiter. Tidal forces may tend to drag it toward the planet. In this process it might be losing or may lose some matter which might be contributing or may contribute to the ring system around the planet. Some how, if it would go within the synchronous orbit, then in that case, tidal forces would tend to drag it toward the planet, eventually breaking it up at its Roche radius forming a ring.

In the Saturnian system (Table IIIb), we have an object, designated here as SNS, at a distance of 147 000 km from the centre of the planet. The distance of this object is interpreted here to be the representative of the system consisting of several small bodies (satellites) and a ring system of Saturn recently discovered by Pioneer 11 in 1979 (Anderson *et al.* (1980), Dollfus (1968), Dollfus and Brunier (1980), Fillius *et al.* (1980), Fountain and Larson (1977, 1978), Franklin *et al.* (1971), Gehrels and Van Allen (1979), Gehrels *et al.* (1980), Rawal (1978), Simpson *et al.* (1980), Simpson (1979), Smith and Doose (1979), Stone (1980), Trainor *et al.* (1980), Van Allen *et al.* (1980)).

We have considered various triads of successive innermost objects consisting of Ring C, Ring B, Ring A, SNS and Janus (Table IIIb). In the Uranian system (Table IIIc), we have an object at a distance of 37000 km from the centre of the planet (see Rawal 1978). The distance of this object is interpreted here to be the representative of a ring system consisting of several thin rings of the planet discovered in the year 1977 (Bhattacharyya and Kuppuswamy, 1977; Elliot *et al.*, 1977). In Table IIIc, UHS_1 stands for Uranian Hypothetical Satellite No. 1 at a distance of 68000 km from the centre of the planet (see Rawal (1978), Hughes (1977), Steigmann (1978), Goldreich and Tremaine (1979), Elliot *et al.* (1977), Aksnes (1977), Dermott and Gold (1977)).

We have considered a triad of successive innermost objects consisting of Ring, UHS_1 and Miranda (Table IIIc). It is interesting to note that if the UHS_1 hypothetical object has a density of $\lesssim 1.5\,\mathrm{g\,cm^{-3}}$, then its Roche limit would have been $\gtrsim 68\,000$ km, and in that case, it would have already been broken in the form of a ring structure around the planet. If this hypothetical object would have density $\gtrsim 1.5\,\mathrm{g\,cm^{-3}}$, then it may well be there.

Agreement of really existing innermost objects in the satellite systems of Saturn, Jupiter and Uranus with Laplace's resonance relation seems to support the concepts of the existence of intra-Mercurial, intra-Phobosian and intra-Tritonian objects (rings and/or satellites) because the latter concepts are also brought forth by the same relation.

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