SECULAR EFFECTS OF TIDAL FRICTION ON THE PLANET-SATELLITE SYSTEMS OF THE SOLAR SYSTEM

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Abstract. We separate the tidal evolution of a planet-satellite system with zero eccentricity in two phases: *phase 1* – from the formation of the system to satellite's corotation (satellite's corotation means that its spin angular velocity equals the orbital angular velocity); *phase 2* – after satellite's corotation.

We study the planet-satellite system during phase 1 with Darwin's graphical method and obtain an upper limit to satellite's Q which discloses whether or not it is corotating. Moreover we obtain some qualitative information about the future evolution of the corotating satellites.

The present work does not give any new result for the Earth-Moon case and for the Neptune-Triton case.

1. Introduction

The tidal evolution of a planet-satellite system with zero eccentricity can be separated in two phases: *phase 1* – from the formation of the system to satellite's corotation; *phase 2* – after satellite's corotation. Phase 2 has been studied by Darwin (1908) in the Earth-Moon case with a very simple graphical method.

We show that in the planet-satellite systems of the solar system it is possible to apply Darwin's method also to phase 1. Moreover, if the spin angular velocity of the planet does not change from the satellite corotation to date, we calculate the satellite's corotation distance and the time interval t_c necessary to reach such a distance, as function of satellite's Q. If we want $t_c < 10^9$ y, an upper limit to satellite's Q is obtained which says whether or not it is corotating.

The following satellites seem to corotate:

Phobos and Deimos.

Amalthea, Io, Europa, Ganymede, Callisto.

Janus, Mimas, Enceladus, Tethys, Dione, Rhea, Titan, Hyperion, Japetus.

Miranda, Ariel, Umbriel, Titania, Oberon.

Moreover we obtain some qualitative information about the future evolution of the corotating satellites.

In the Earth-Moon case and in the Neptune-Triton case, where the masses of planet and satellite are comparable, and planet's spin angular momentum (a.m.) is comparable with the orbital a.m., phase 2 dominates with respect to phase 1 and the present work does not give any new result.

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2. From the Formation of the Planet-Satellite System to Satellite's Stable Corotation (Phase 1)

We consider a planet-satellite system satisfying the following hypotheses:

Hypotheses A

(i) circular orbits ($e = 0$),

(ii) planet's despin rate equal to zero ($\dot{\omega}_p = 0$),

(iii) satellite's spin angular velocity parallel to the orbital angular velocity ($\vec{\omega}_s \| \vec{n}$).

In the solar system it seems reasonable to use the Hypotheses A during phase 1. The hypothesis (ii) is correct because in the solar system the planet's spin (a.m.) is always much greater than the satellite's spin (a.m.), and so $\dot{\omega}_p \ll \dot{\omega}_s$.

In these hypotheses the total a.m. and the total energy of the planet-satellite system, J and E, are given by

$$J = I_s\omega_s + I_p\omega_p + \frac{G^{1/2}Mm}{(M+m)^{1/2}}r^{1/2}; \tag{1}$$

$$2E = I_s\omega_s^2 + I_p\omega_p^2 - G^{2/3}Mm(M+m)^{-1/3}n^{2/3},$$

where I_s and I_p are the moments of inertia of satellite and planet, respectively; m and M, their masses; ω_s and ω_p, their spin angular velocities; n, the orbital angular velocity; and G, the universal gravitation constant.

It is better to use a new system of units obtained from the following conditions

$$\frac{Mm}{M+m} = 1, \quad I_s = 1, \quad \frac{G^{1/2}Mm}{(M+m)^{1/2}} = 1. \tag{2}$$

From (2) it follows that

$$G^{-2/3}Mm(M+m)^{-1/3} = 1, \tag{3}$$

and (1) becomes

$$J = \omega_s + \lambda\omega_p + r^{1/2}, \quad 2E = \omega_s^2 + \lambda\omega_p^2 - \frac{1}{r}, \tag{4}$$

where

$$\lambda = I_p/I_s. \tag{5}$$

In the new system of units ω_s, $\lambda\omega_p$ and $r^{1/2}$ are just the satellite's spin a.m., the planet's spin a.m. and the orbital a.m. of the system. Writing (4) we used the third Kepler's law in the new system of units – i.e.,

$$n^{-1/3} = r^{1/2}. \tag{6}$$

Owing to tidal friction on the satellite there is dissipation of energy. The total energy of the system decreases, the total a.m. is constant. The total a.m. does not change but there is a transfer between the satellite's a.m. and the planet's orbital a.m. (ω_p is constant in Hypotheses A)

With regard to this transfer of a.m. two cases are possible;

Case 1 (Figure 1) – the satellite's tidal bulges are pushed forward with respect to the line Mm – a fact giving rise to a torque on the satellite tending to decrease its spin angular velocity. The reaction of this torque on the planet will tend, with its component perpendicular to the line Mm, to increase the distance r.

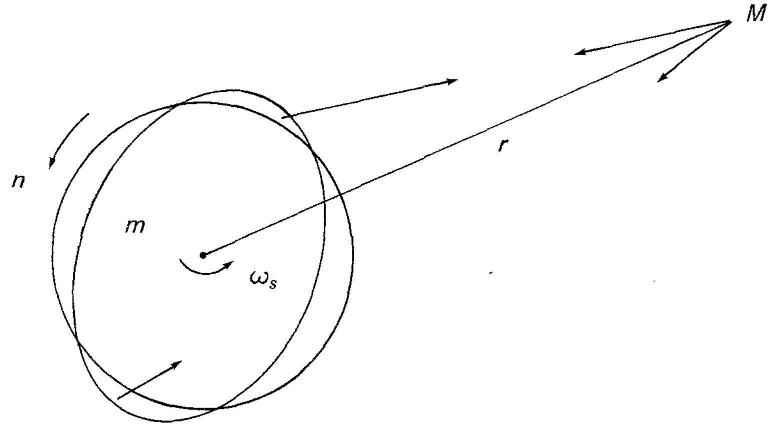

Fig. 1. M and m are the masses of planet and satellite, r is the distance between their centers, ω_s is the spin angular velocity of the satellite, n the orbital angular velocity. The plane of the sheet is the orbital plane. The figure is not on scale.

Case 2 (Figure 2) – the satellite's tidal bulges lag behind the line Mm, causing a torque on the satellite which tends to increase its spin angular velocity. The reaction of this torque on the planet will tend, with its component perpendicular to the line Mm, to decrease the distance r.

We shall have Case 1 or Case 2 depending on the ratio ω_s/n: if ω_s is greater than n + 50% n Case 1 will take place, otherwise Case 2 will take place (cf. Goldreich and Soter, 1966).

The component along the line Mm of the tidal forces on M does not modify the orbital a.m., but contributes to dissipate energy if the eccentricity of the system is not zero.

If we put

$$\bar{J} = J - \lambda\omega_p \tag{7}$$

and assume the orbital a.m. $r^{1/2}$ as independent variable, if we set $r^{1/2} \equiv x$, Equation (4) become

$$\omega_s(x) = \bar{J} - x, \quad 2E(x) = (\bar{J} - x)^2 + \lambda\omega_p^2 - \frac{1}{x^2}. \tag{8}$$

The condition $dE/dx = 0$ is safeguarded by the equation

$$x^4 - \bar{J}x^3 + 1 = 0, \tag{9}$$

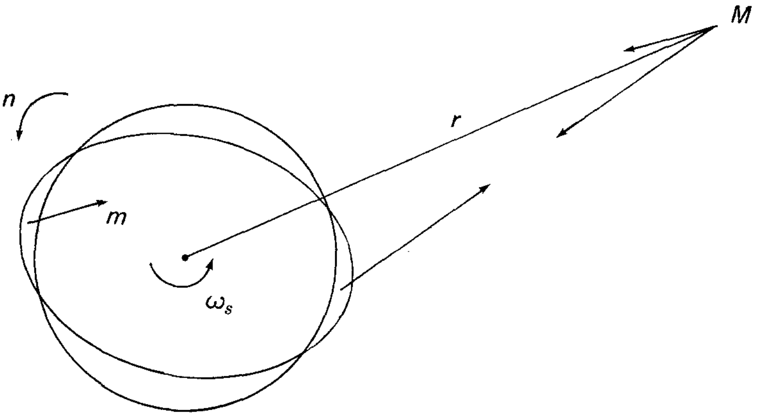

Fig. 2. M and m are the masses of planet and satellite, r is the distance between their centers, ω_s is the spin angular velocity of the satellite, n the orbital angular velocity. The plane of the sheet is the orbital plane. The figure is not on scale.

which represents nothing but the satellite's corotation. In fact, if we want the satellite to corotate, $\omega_s = n$, and the conservation of total a.m., using (6), we have the system of equations

$$\omega_s = 1/x^3, \quad \bar{J} = \omega_s + x, \tag{10}$$

which gives directly (9).

Depending on the value of $\bar{J}$ there are three cases for the solutions of (9):

Case a – $\bar{J} > \bar{J}_{\min} = 1.75$. (9) has two real roots x_1 and x_2 corresponding, respectively, to unstable and stable satellite's corotation (see Figure 3)

Case b – $\bar{J} = \bar{J}_{\min} = 1.75$. (9) has only one real root $x^* = \frac{3}{4}\bar{J}$. The satellite's corotation at $x = x^*$ is clearly unstable. (Figure 4)

Case c – $\bar{J} < \bar{J}_{\min} = 1.75$. (9) has no real root (Figure 5).

In Case a, from the values x_1 and x_2, knowing M, m and $\bar{J}$, it is possible to calculate the numerical values, in cm, of the distances d_1 and d_2 of unstable and stable corotation of the satellite. Moreover it is easy to see that

$$0.75\bar{J} < x_2 < \bar{J}, \tag{11}$$

i.e.,

$$\frac{G^{1/2}Mm}{(M+m)^{1/2}} d_2^{1/2} \simeq (I_s\omega_s)_{t_c} + \left(\frac{G^{1/2}Mm}{M+m} r^{1/2}\right)_{t_c} = J - (I_p\omega_p)_{t_c}, \tag{12}$$

where the subscript t_c means that the corresponding quantity is calculated at the corotation time of the satellite. Thus if we verify that

(α_1) – the planet's spin angular velocity does not change from the satellite's corotation to date;

$$(\omega_p)_{t_c} \simeq (\omega_p)_{t_a}$$

or

$$(\omega_s + x)_{t_c} \simeq (\omega_s + x)_{t_a},$$

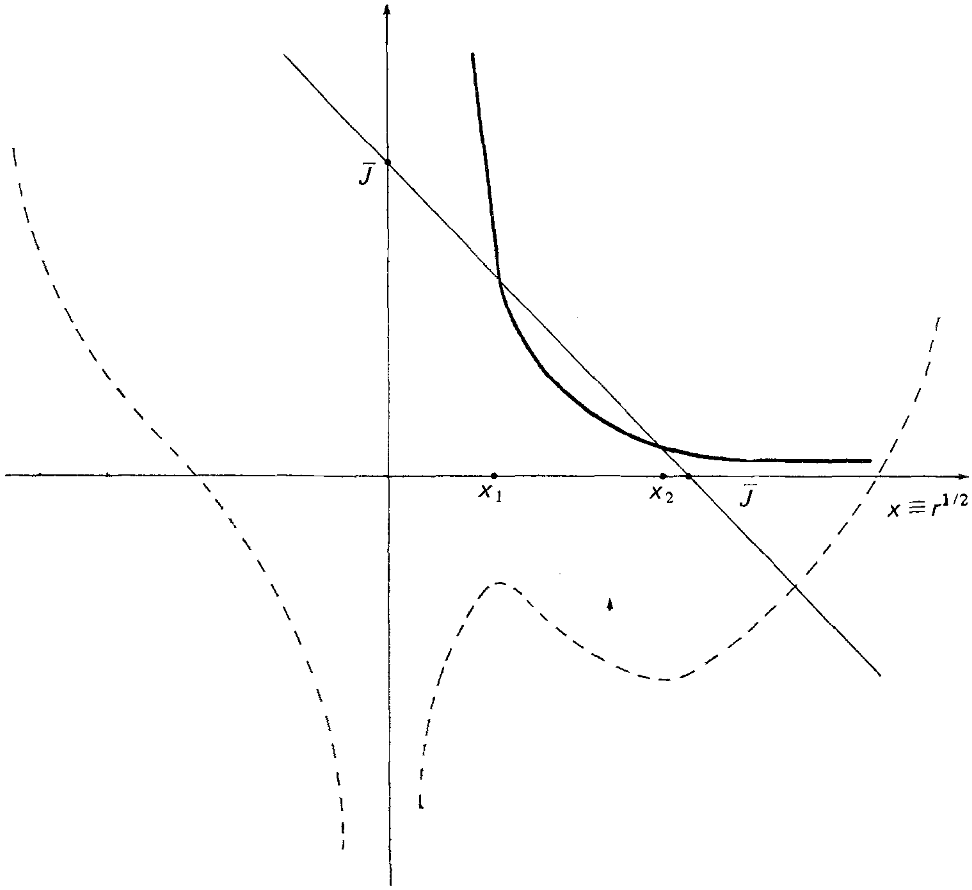

Fig. 3. The heavy curve is the satellite's corotation curve which gives its spin angular velocity as function of the orbital a.m., x,: $\omega_s = 1/x^3$. The light curve is the satellite's spin a.m. as function of x: $\omega_s = \bar{J} - x$. The broken curve is the energy of the system as function of x. This figure corresponds to the case $\bar{J} > 1.75$. It is not on scale.

(α_2) – the actual orbital a.m. is much greater than the spin a.m. of the satellite: i.e.,

$$(\omega_s + x)_{t_a} \simeq (x)_{t_a}.$$

(α_3) – the satellite's corotation time, necessary in order that the separation distance reaches the value d_2 owing to tidal friction on the satellite, is consistent with the age of the system

$$t_c < 10^9 \,\text{yr}$$

we can say that

$$0.87 r_a < d_2 < r_a, \tag{13}$$

where r_a is the actual value of the distance between planet and satellite.

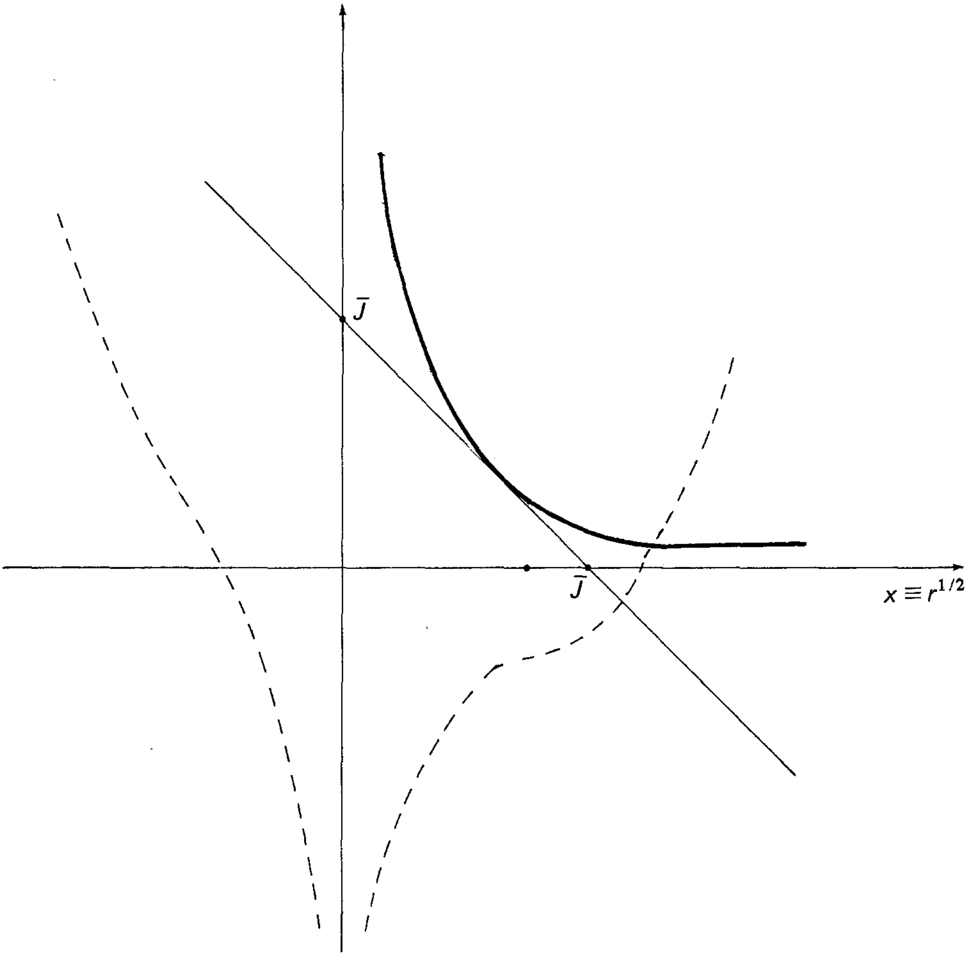

Fig. 4. The heavy curve is the satellite's corotation curve which gives its spin angular velocity as function of the orbital a.m., x,: $\omega_s = 1/x^3$. The light curve is the satellite's spin a.m. as function of x: $\omega_s = \bar{J} - x$. The broken curve is the energy of the system as function of x. This figure corresponds to the case $\bar{J} = 1.75$. It is not on scale.

3. The Consequences of Satellite's Stable Corotation

After the satellite's stable corotation has been established the tidal evolution of the system ensues which has been studied by Darwin (1908) and by Poincaré (1913) under the following hypotheses:

Hypotheses B

(i) circular orbits ($e = 0$)

(ii) mass-point satellite

(iii) planet's spin angular velocity parallel to the orbital angular velocity ($\vec{\omega}_p \| \vec{n}$)

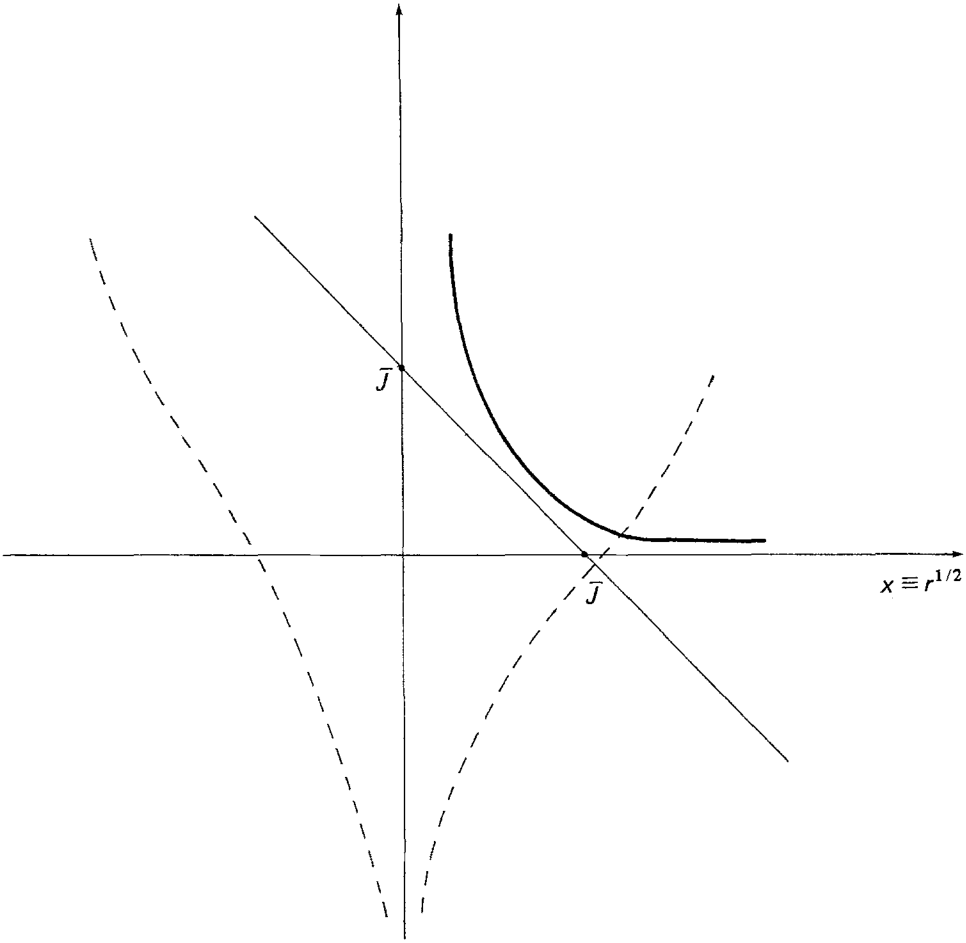

Fig. 5. The heavy curve is the satellite's corotation curve which gives its spin angular velocity as function of the orbital a.m., x: $\omega_s = 1/x^3$ The light curve is the satellite's spin a.m. as function of x: $\omega_s = \bar{J} - x$. The broken curve is the energy of the system as function of x. This figure corresponds to the case $\bar{J} < 1.75$. It is not on scale.

The hypothesis (ii) is correct because, after the satellite has neared its corotation state, under the hypothesis of circular orbits there is no contribution of satellite's tide to the tidal evolution of the system. Moreover, the spin a.m. and the spin energy of the satellite are always negligible in the planet-satellite systems of the solar system.

Under these hypotheses, and using the new system of units given by (2), the total a.m. and the total energy of the system, L and ϵ, satisfy the equations

$$\omega_p(x) = \frac{1}{\lambda}(L - x), \quad 2\epsilon(x) = \frac{1}{\lambda}(L - x)^2 - \frac{1}{x^2}. \tag{14}$$

Owing to the tidal friction excited on the planet, there occurs a dissipation of the energy: the total energy of the system decreases, the total a.m. remaining constant. The total a.m. does not change but there is a transfer between the planet's spin a.m. and the satellite's orbital a.m. The orbital a.m. of the system $r^{1/2} \equiv x$ can be adopted as an independent variable.

The condition $d\epsilon/dx = 0$ is given by the equation

$$x^4 - Lx^3 + \lambda = 0, \tag{15}$$

which represents the planet's corotation. In fact, if we want the planet's corotation, $\omega_p = n$, and the conservation of total a.m., using (6), we have the system of equations

$$\omega_p = 1/x^3, \quad L = \lambda\omega_p + x, \tag{16}$$

which give directly (15).

Depending on the values of L and λ, there are three cases for the solutions of (15):

Case A – $L > L_{\min} = 1.75\lambda^{1/4}$. (15) has two real roots X_1 and X_2.

X_1 and X_2 correspond, respectively, to unstable and stable planetary corotation.

Case B – $L = L_{\min} = 1.75\lambda^{1/4}$. (15) has only one real root $X^* = \frac{3}{4}L$.

The planet's corotation distance at $X = X^*$ is clearly unstable.

Case C – $L < L_{\min} = 1.75\lambda^{1/4}$. (15) has no real root.

In the case A, from the values X_1 and X_2, knowing M, m and L it is possible to calculate the numerical values, in cm, of the distances D_1 and D_2 of unstable and stable corotation of the planet.

When the planet-satellite distance has reached the value d_2 of satellite's stable corotation, if $L > L_{\min}$, the planet-satellite system can have three different evolutions: (α), (β), (γ). (Figure 6)

(α) – At satellite's stable corotation it holds $X_1 < x_2 < X_2$.

Owing to tidal friction on the planet the system evolves towards planet's stable corotation with $\dot{\omega}_p < 0$ and $\dot{r} > 0$.

(β) – For the satellite's stable corotation, $0 < x_2 < X_1$.

Owing to the tidal friction on the planet, the planet-satellite distance decreases while the planet's spin a.m. increases – i.e., $\dot{r} < 0$, $\dot{\omega}_p > 0$.

(γ) – For the satellite's stable corotation, $x_2 < 0$.

This is the case of a corotating satellite whose spin angular velocity is parallel to the orbital angular velocity but antiparallel to the planet's spin angular velocity. Owing to tidal friction on the planet the planet-satellite distance decreases and the planet's spin angular velocity decreases, i.e.

$$\frac{d|r|}{dt} < 0, \quad \dot{\omega}_p < 0.$$

We do not study the case $x_2 > X_2$ because $x_2 > X_2$ means that the satellite's spin a.m. is greater than the planet's spin a.m. and this relation never holds good in the solar system.

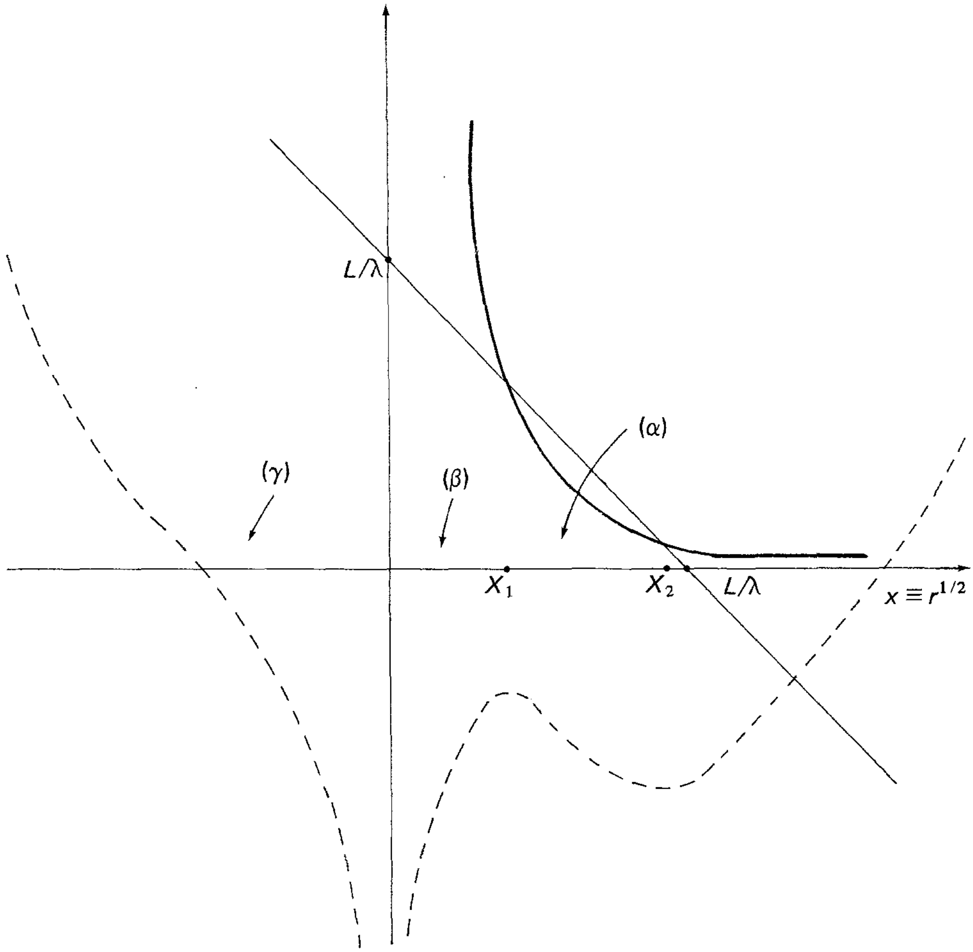

Fig. 6. The heavy curve is the planet's corotation curve which gives its spin angular velocity as function of the orbital a.m., x,: $\omega_p = 1/x^3$. The light curve is the planet's spin angular momentum as function of x: $\omega_p = (L - x)/\lambda$. The broken curve is the energy of the system as function of x. This figure corresponds to the case $L > L_{min}$. It is not on scale.

4. The Planet-Satellite Systems of the Solar System

All the planet-satellite systems of the solar system satisfy the relation $\bar{J} > 1.75$ and it means that Equation (9) has two real roots, x_1 and x_2 as in Figure 3. Clearly there is no satellite with $\bar{J} \leqslant 1.75$, because tidal friction would have destroyed such satellites (Figures 4 and 5).

Further the existing satellites cannot have had an initial orbital a.m. $x(0)$ such that $x(0) < 0$ or $0 < x(0) < x_1$ because in these cases tidal friction would have destroyed them (Figure 3). On the other hand, it is not probable that they have had $x(0) > x_2$ because it means (apart from the small interval $x_2 < x < \bar{J}$) that the satellite's spin angular velocity and the orbital angular velocity are antiparallel. Probably they have had $x_1 < x(0) < x_2$. Therefore, during phase 1, they evolved towards the satellite's stable corotation.

To assert that the existing satellites of the solar system corotate at distance d_2 satisfying (13) we must see if the conditions (α_1), (α_2) and (α_3) of Section 1 are fulfilled. With regard to (α_1), it is verified for all satellites with the exception of the Moon and, probably, Triton. To convince ourselves of this fact, it suffices to evaluate the order of magnitude of the planet's corotation times. The condition (α_2), is verified for all planet-satellite systems of the solar system. For the satellites whose spin angular velocity is not known we compared the orbital a.m. of the system with the spin a.m. of the satellite assuming that it is rotating with the greatest angular velocity consistent with its gravitational stability. For the satellites whose mass is not known, we calculated its order of magnitude assuming reasonable values for the mass density.

To ensure that (α_3) is verified, it is necessary to calculate t_c. The magnitude of the tidal torque on the planet due to the satellite is

$$N_{sp}(r) = \frac{9Gm^2A^5}{4Q_p'} r^{-6}, \tag{17}$$

where A is the radius of the planet;

$$Q_p' = Q_p\left(1 + \frac{38\pi A^4 \mu_p}{3GM^2}\right); \tag{18}$$

μ_p is the rigidity of the planet; and Q_p, its dissipation function (Goldreich and Soter, 1966). If the planet's spin a.m. is much greater than the satellite's spin a.m. it is possible to neglect $\dot{\omega}_p$ with respect to $\dot{\omega}_s$ but it is not possible to neglect the variation of the orbital a.m. of the satellite which occurs in order that the total a.m. is conserved, so that

$$N_{sp} = \frac{dL_s}{dt}, \tag{19}$$

where L_s is the orbital a.m. of the satellite, linked to the orbital a.m. of the system, L_0, by the relation

$$L_s = \frac{M}{M+m} L_0, \tag{20}$$

$$L_0 = \frac{G^{1/2}Mm}{(M+m)^{1/2}} r^{1/2}. \tag{21}$$

Furthermore, there is a torque on the satellite due to the planet, whose magnitude is given by

$$N_{ps}(r) = \frac{9GM^2a^5}{4Q_s'} r^{-6} \tag{22}$$

where a is the radius of the satellite,

$$Q_s' = Q_s\left(1 + \frac{38\pi a^4 \mu_s}{3Gm^2}\right); \tag{23}$$

μ_s being the rigidity of the satellite, and Q_s its dissipation function (Goldreich and Soter, 1966). The torque N_{ps} tends to decrease (increase) the satellite's spin a.m. and the reation on the planet tends to increase (decrease) its orbital a.m.: i.e.,

$$N_{ps} = \frac{dL_p}{dt}, \tag{24}$$

where L_p is the planet's orbital a.m. connected with the total a.m. L_0 from the relation

$$L_p = \frac{m}{M+m} L_0. \tag{25}$$

From (19), (20) and (21) it follows that

$$(\dot{r})_{sp} = N_{sp}\,\frac{2(M+m)^{3/2} r^{1/2}}{G^{1/2} M^2 m}, \tag{26}$$

where the subscript "*sp*" signifies that $\dot{r}$ given by (26) is due to the torque N_{sp}. From (21), (24) and (25) it follows that

$$(\dot{r})_{ps} = N_{ps}\,\frac{2(M+m)^{3/2} r^{1/2}}{G^{1/2} M^2 m}, \tag{27}$$

where the subscript "*ps*" denotes that $\dot{r}$ given by (27) is due to the torque N_{ps}. From (26) and (27) we have

$$\frac{(\dot{r})_{ps}}{(\dot{r})_{sp}} = \frac{N_{ps}}{N_{sp}}\frac{M}{m}. \tag{28}$$

As it is $M \gg m$ and never results $N_{sp} \gg N_{ps}$ we see that

$$(\dot{r})_{ps} \gg (\dot{r})_{sp}, \tag{29}$$

which holds during phase 1. During phase 2, as we assume a mass-point satellite, it will be $(\dot{r})_{ps} = 0$. From (29) it follows that

$$\dot{r} \simeq (\dot{r})_{ps}; \tag{30}$$

and, by use of (27), (22),

$$\dot{r} = D_s r^{-11/2}, \tag{31}$$

where

$$D_s = \frac{9G^{1/2} M(M+m)^{3/2} a^5}{2m^2 Q_s'}. \tag{32}$$

Therefore, the satellite's corotation time t_c is given by

$$t_c = \frac{2}{13 D_s}\left(r_a^{13/2} - r_i^{13/2}\right), \tag{33}$$

where r_i is the initial planet-satellite distance. As $r_i < r_a$, Equation (33) becomes

$$t_c \simeq \frac{2}{13D_s} r_a^{13/2}. \tag{34}$$

If we want t_c to be smaller than the age of the system

$$t_c < 10^9 \mathrm{y}, \tag{35}$$

assuming $\mu_s \simeq 10^{11}$ dyne/cm^2 we calculated the maximum value of the satellite's Q such that (35) is verified – i.e.,

$$Q < Q_{\mathrm{max}} \Rightarrow t_c < 10^9 \mathrm{y}.$$

From Table I, the following satellites seem to corotate:

Phobos, Deimos;

Amalthea, Io, Europa, Ganymede, Callisto;

Miranda, Ariel, Umbriel, Titania, Oberon.

In the case of Japetus the value of Q_{max} is small to be sure that it is corotating (great value of Q means small energy dissipation and vice versa), but there are reasons to think that it corotates (Cook and Franklin, 1970).

As the eccentricities of these satellites are very small, it is reasonable to assert that their spin period equals the orbital period. The only exception could be Hyperion ($e \simeq 0.1042$) whose spin period is probably smaller than the orbital period, like Mercury in its motion about the Sun.

With regard to the Moon, which does not satisfy (α_1) it began to corotate in a time interval much smaller than its age (Goldreich, 1972), we do not know the value of the Earth-Moon distance when the Moon began to corotate. The value of such distance is known only as function of the initial spin angular velocity of the Earth, which is unknown. So the present work does not give any new result in this case, which is still more complicated for the eccentricity of the orbit and its inclination on the ecliptic.

Triton too, probably, does not satisfy (α_1) and began to corotate in a time interval much smaller than its age. Furthermore it has an orbital a.m. antiparallel to Neptune's spin a.m. If we assume as positive the direction of Neptune spin a.m. it results $\vec{L} > 0$, $L = 1.18 \times 10^{43}\,\mathrm{g\,cm^2\,s^{-1}}$, and $\bar{J} < 0$ also if we do not know its value exactly. So during phase 2 the Neptune-Triton system will have (γ) evolution (Figure 6).

When the planet-satellite distance has reached the value of satellite stable corotation in a time interval smaller than 10^9 y, being $L > L_{\mathrm{min}}$, the planet-satellite system will have evolution (α), (β) or (γ) as shown in the following scheme:

evolution (α)

Moon;

Io, Europa, Ganymede, Callisto;

Janus, Mimas, Enceladus, Tethys, Dione, Rhea, Titanus, Hyperion, Japetus;

Miranda, Ariel, Umbriel, Titania, Oberon.

evolution (β)

TABLE I

	Q_{max}
Phobos	2.09×10^{9}
Deimos	3.31×10^{6}
J V (Amalthea)	3.37×10^{10}
Io	2.66×10^{9}
Europa	1.14×10^{8}
Ganymede	9.36×10^{6}
Callisto	2.1×10^{5}
J VI	4.81×10^{-2}
J VII	1.40×10^{-2}
J X	7.42×10^{-1}
J XII	2.34×10^{-2}
J XI	1.31×10^{-2}
J VIII	1.65×10^{-2}
J IX	8.58×10^{-3}
Janus	4.65×10^{9}
Mimas	4.52×10^{9}
Enceladus	1.03×10^{9}
Tothys	5.16×10^{8}
Dione	1.11×10^{8}
Rhea	1.74×10^{7}
Titanus	1.79×10^{5}
Hyperion	4.75×10^{3}
Japetus	48.4
Phoebe	2.73×10^{-3}
Miranda	2.42×10^{8}
Ariel	3.87×10^{7}
Umbriel	2.99×10^{6}
Titania	2.97×10^{5}
Oberon	3.60×10^{4}
Nereide	9.29×10^{-3}

Phobos.

evolution (γ)

Triton.

In the cases of Deimos and Amalthea, since they have a little eccentricity (0.0023 and 0.003 respectively), it will act the component along the line Mm of the tidal forces on M. Since their spin angular velocity is greater than n but not greater than $n + 50\%n$ they will probably follow the evolution (β).

Observational data confirm evolution (α) for the Moon (Van Flandern, 1970; Morrison, 1972) and evolution (β) for Phobos (Sharpless, 1945). The situation is rather uncertain for Deimos and Amalthea (Kerr and Whipple, 1954). There is no such information about the other satellites.

In the planet-satellite systems having evolution (α) rough estimates of the planets' corotation times say that they are much greater than the age of the universe. Also if we assume very little values for planets' Q.

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