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The Incipient Infinite Cluster in Two-Dimensional Percolation

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Summary. Let P_p be the probability measure on the configurations of occupied and vacant vertices of a two-dimensional graph $\mathscr{G}$, under which all vertices are independently occupied (respectively vacant) with probability p (respectively $1-p$). Let p_H be the critical probability for this system and W the occupied cluster of some fixed vertex w_0. We show that for many graphs $\mathscr{G}$, such as $\mathbb{Z}^2$, or its covering graph (which corresponds to bond percolation on $\mathbb{Z}^2$), the following two conditional probability measures converge and have the same limit, ν say:

i) $P_{p_H}\{\cdot \mid w_0$ is connected by an occupied path to the boundary of the square $[-n,n]^2\}$ as $n \to \infty$,

ii) $P_p\{\cdot \mid W$ is infinite$\}$ as $p \downarrow p_H$.

On a set of ν-measure one, w_0 belongs to a unique infinite occupied cluster, $\tilde{W}$ say. We propose that $\tilde{W}$ be used for the "incipient infinite cluster". Some properties of the density of $\tilde{W}$ and its "backbone" are derived.

1. Introduction

The "incipient infinite cluster" or "infinite cluster at criticality" is frequently used in articles on percolation (e.g., in [1, 8, 14]). The concept seems to be as ill defined as "infinitesimals" in Leibniz' time. The difficulty arises because one would like for the incipient infinite cluster an infinite occupied cluster *at* the critical probability, when "infinite clusters just begin to form". Unfortunately with probability one no infinite occupied cluster exists *at* the critical probability (at least in the common percolation models for which this question has been decided); they only exist when p is strictly greater than the critical probability. We therefore propose to force the occurrence of an infinite cluster by taking limits of certain conditional probability measures. J.T. Chayes and L. Chayes

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proposed another definition of the incipient infinite cluster (as the invaded region in invasion percolation, cf. "Percolation and Random Media", p. 136, Lecture Notes for Les Houches Summer School, 1984). It is not clear what the relation is between the different definitions of the incipient infinite cluster.

Unfortunately the method below applies only to certain two-dimensional systems. The important special feature which makes the proof possible is that these systems at or above the critical probability contain infinitely many occupied circuits. Site percolation on $\mathbb{Z}^2$ contains all features of interest and without great loss the reader may take $\mathscr{G}=\mathbb{Z}^2$ throughout. We briefly describe the set-up and notation. We generally adhere to the notation of [6]; more detailed definitions can be found in Chap. 2 and 3 of this reference. Let $(\mathscr{G}, \mathscr{G}^*)$ be a matching pair of periodic graphs imbedded in the plane (in the sense of Sect. 2.1, 2.2) of [6]). Throughout $\Lambda=\Lambda(\mathscr{G})$ is a constant $\geqq 1$ such that

$$\text{length of any edge of } \mathscr{G} \leqq \Lambda$$

and for any vertex $v=(v(1), v(2))$ of $\mathscr{G}$ and $k \in \mathbb{Z}$

$$\begin{aligned}&v \text{ can be connected to } v+(0,k)\ (v+(k,0))\\ &\text{by a path on } \mathscr{G} \text{ in the strip}\\ &(v(1)-\Lambda,\ v(1)+\Lambda)\times \mathbb{R} \text{ (respectively}\\ &\mathbb{R}\times(v(2)-\Lambda,\ v(2)+\Lambda)).\end{aligned}$$

By periodicity it suffices to make the last requirement only for $k=1$. We consider site percolation on $\mathscr{G}$, i.e., each vertex can be occupied or vacant, and we assume that all vertices are independent of each other. The probability of a vertex being occupied is taken to be the same[1] for all vertices and denoted by p. The corresponding probability measure on the configurations of occupied and vacant vertices is denoted by P_p. $W(w)$ is the occupied cluster of w, i.e., the collection of vertices which are connected to w by a path on $\mathscr{G}$ all of whose vertices are occupied. A path all of whose vertices are occupied will henceforth be called *an occupied path.* We write $v \rightsquigarrow w$ if there exist an occupied path from v to w (in particular v and w have to be occupied for this to happen). Similarly $v \rightsquigarrow B (A \rightsquigarrow B)$ means that $v \rightsquigarrow w$ for some $w \in B$ (respectively, for some $v \in A$ and $w \in B$). Occasionally, the paths have to be restricted. We shall write $v \rightsquigarrow w$ in C if there exists an occupied path from v to w, all of whose vertices lie in C. A similar definition applies to $v \rightsquigarrow B$ in C and $A \rightsquigarrow B$ in C. $\# W$ denotes the number of vertices in W and the critical probability is

$$p_H = \inf\{p : P_p\{\# W(w)=\infty\}>0\}$$

(see [6], (3.62); p_H is independent of w). To avoid double subscripts we shall write P_{cr} for the probability measure P_{p_H}, and E_{cr} for expectation with respect to P_{cr}.

[1] The method below can also be used to treat periodic multiparameter problems (as described in [6], Sect. 3.2) in which the probability of being occupied has a finite number of different values. For simplicity we consider here only the one-parameter situation

Our *fundamental assumption* is

(1) *there exists a constant* $\delta > 0$ *such that for all* $n \geqq 3\Lambda$

$$P_{cr}\{[-\Lambda,0]\times[0,n] \rightsquigarrow [3n,3n+\Lambda]\times[0,n] \quad \textit{in} \quad [-\Lambda,3n+\Lambda]\times[0,n]\} \geqq \delta \quad \textit{and}$$

$$P_{cr}\{[0,n]\times[-\Lambda,0] \rightsquigarrow [0,n]\times[3n,3n+\Lambda] \quad \textit{in} \quad [0,n]\times[-\Lambda,3n+\Lambda]\} \geqq \delta.$$

(2) *Remark.* We shall call a path on $\mathscr{G}$ which connects

$$[a-\Lambda,a]\times[c,d] \quad \text{to} \quad [b,b+\Lambda]\times[c,d] \quad \text{in} \quad [a-\Lambda,b+\Lambda]\times[c,d]$$

a horizontal crossing of $[a,b]\times[c,d]$. Vertical crossings are defined similarly. Thus (1) says that at the critical probability (and a fortiori for $p \geqq p_H$) there is a probability of at least δ that there is an occupied crossing in the long direction of the rectangles $[0,3n]\times[0,n]$ and $[0,n]\times[0,3n]$. Because the vertices of $\mathscr{G}$ are not necessarily located on lines of the form x=integer or y=integer, the crossing of $[0,3n]\times[0,n]$ does not necessarily start on the left edge, $\{0\}\times[0,n]$, but somewhere in the "slightly fattened up" left edge, $[-\Lambda,0]\times[0,n]$. The reader should ignore this minor technicality. The important point is that condition (1) is satisfied when $\mathscr{G}=\mathbb{Z}^2$ or the covering graph of $\mathbb{Z}^2$ (the latter corresponds to bond percolation on $\mathbb{Z}^2$, see [6] Sect. 3.1) as well as for the triangular and honeycomb lattices. Proof of these facts can be found in [12, 10, 11, 16]; see also [13] Sect. 3.4. More generally, it follows from [6], Theorems 5.1, 6.1 and the methods of Chap. 3 that (1) holds when the y-axis (or x-axis) is an axis of symmetry for $\mathscr{G}$ and if in addition $\mathscr{G}$ is invariant under a rotation over an angle $\phi \in (0,\pi)$ (compare application (v) of [6] Sect. 3.4).

One final definition before we formulate our main result. A *cylinder event* is an event which depends on the state of finitely many vertices only.

(3) **Theorem.** *Let* $S(n)$ *be the square* $[-n,n]^2$ *and* w_0 *a fixed vertex of* $\mathscr{G}$ *and* $W=W(w_0)$. *If* (1) *holds then for every cylinder event* E *the limits*

$$\lim_{n\to\infty} P_{cr}\{E \mid w_0 \rightsquigarrow \mathbb{R}^2\backslash S(n)\}$$

and

$$\lim_{p\downarrow p_H} P_p\{E \mid \# W=\infty\}$$

exist and are equal. If we denote their common value by $\nu(E)$, *then* ν *extends uniquely to a probability measure on the configurations of occupied and vacant vertices, and*

$$\nu\{\exists \text{ exactly one infinite occupied cluster } \tilde{W}, \text{ and } \tilde{W} \text{ contains } w_0\}=1.$$

(4) *Remark.* ν is not translation invariant. However, the squares $S(n)$ in Theorem 3 may be replaced by any sequence of polygons $P(n)$, provided

$P(n) \subset$ interior of $P(n+1)$, $n=1,2,\ldots$, and for every fixed compact set K, the boundary of $P(n)$ lies outside K eventually. $\square$

We also derive some properties of the cluster $\tilde{W}$, whose existence is guaranteed a.e. [v] by (3). These properties will be used in [7] to show that a random walk on $\tilde{W}$ has subdiffusive behavior. Set

$$\pi(n) = \pi_n = P_{cr}\{w_0 \rightsquigarrow (n, \infty) \times \mathbb{R}\}.$$

This is the probability that w_0 is connected to a halfspace at distance n away from the origin. It is known (cf. [15], Cor. 3.15 and [6] Lemma 8.5) that

$$C_1 n^{-\frac{1}{2}} \leqq \pi_n \leqq C_2 n^{-\eta_1}$$

for some constants $C_i > 0$, and $\eta_1 > 0$. In fact combining the argument of [15], Cor. 3.15 and [6], Lemma 8.2 one can show that even

$$C_1 n^{-\frac{1}{2}+\eta_2} \leqq \pi_n \leqq C_2 n^{-\eta_1} \tag{5}$$

for some $\eta_2 > 0$. It is widely believed that the actual asymptotic behavior of π_n is like $n^{-\eta_3} L(n)$ for some $0 < \eta_3 < 1/2$ and a slowly varying function L (which may be a constant). As an indication that π_n is fairly smooth as a function of n we shall show in Sect. 3 that

$$\pi_n \quad \text{is decreasing, but } \pi_{2n} \geqq C_3 \pi_n, \tag{6}$$

and

$$\pi_n \leqq \frac{1}{n} \sum_{k=1}^{n} \pi_k \leqq C_4 \pi_n, \tag{7}$$

for suitable constants $0 < C_i < \infty$.

Some more notation: $S(n) = [-n, n]^2$, $S^c(n) = \mathbb{R}^2 \setminus S(n)$. E_v denotes the expectation with respect to v. $\# D$ denotes the number of vertices in D. C_i will always be a strictly positive and finite constant whose specific value is without importance for our purposes, and which may change from one appearance to another. Finally, for positive sequences $\{f(n)\}$ and $\{g(n)\}$, $f(n) \asymp g(n)$ means that $f(n)/g(n)$ is bounded away from 0 and ∞ as $n \to \infty$.

(8) **Theorem.** *Assume that* (1) *holds. Then for any* $t \geqq 1$

$$E_v\{[\#(\tilde{W} \cap S(n))]^t\} \asymp [n^2 \pi_n]^t.$$

Moreover

$$v\left\{\varepsilon \leqq \frac{\#(\tilde{W} \cap S(n))}{n^2 \pi_n} \leqq \varepsilon^{-1}\right\} \to 1$$

as $\varepsilon \to 0$, *uniformly in* n.

(9) *Remark.* Note that if indeed $\pi_n \sim n^{-\eta_3} L(n)$, then (8) shows that $\#(\tilde{W} \cap S(n))$ behaves like $n^{2-\eta_3} L(n)$. M. Aizenman (private communication) pointed out to us that in any case the proof of this theorem implies that for suitable $\varepsilon_0 > 0$

and large n

$$P_{cr}\{\#W \geqq \varepsilon_0 n^2 \pi_n\}$$
$$\geqq \pi_n P_{cr}\{\#W \geqq \varepsilon_0 n^2 \pi_n | w_0 \rightsquigarrow S^c(n)\} \geqq \varepsilon_0 \pi_n.$$

Together with $\pi_n \geqq C_1 n^{-\frac{1}{2}}$ this shows that

$$P_{cr}\{\#W \geqq k\} \geqq C_5 k^{-\frac{1}{3}},$$

which by a simple Abelian argument shows that

$$\lim_{h \downarrow 0} h^{-\frac{1}{3}} \sum_k P_{cr}\{\#W = k\}(1-e^{-kh}) \geqq C_6 > 0$$

Thus, if we set

$$m(h) := \sum_k P_{cr}\{\#W = k\}(1-e^{-kh}),$$

then

(10) $$\frac{1}{\delta} := \limsup_{h \downarrow 0} \frac{\log m(h)}{\log h} \leqq \tfrac{1}{3} \quad \text{or } \delta \geqq 3.$$

The so called "mean field value" for δ is 2. Therefore, *in dimension 2, δ does not take its mean field value.* We shall return to this in a forthcoming article "Scaling relations for 2D-percolation." □

For calculations of electrical resistances and the displacement of a random walk on $\tilde{W}$ it is important to consider the "backbone" of $\tilde{W}$. We define this as follows:

(11) $\tilde{B}_n := \{v: \exists$ two occupied paths r_1 and r_2 on $\mathscr{G}$ in $S(n)$ connecting v to w_0 and to $S^c(n)$, respectively, and such that r_1 and r_2 have no other vertex but v in common$\}$.

(by definition $w_0 \in \tilde{B}_n$);

$$\tilde{B} = \liminf \tilde{B}_n = \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \tilde{B}_n.$$

Roughly speaking $\tilde{B}$ consists of all vertices which have disjoint occupied connections to w_0 and ∞. In any case

$$\tilde{B}_n \subset \tilde{W} \quad \text{and} \quad \tilde{B} \subset \tilde{W},$$

since all vertices in $\tilde{B}_n$ are connected to w_0 by occupied paths. We shall also need the following probability

(12) $\rho_n := P_{cr}\{w_0$ is connected to $S^c(n)$ by two occupied paths which have no other vertex than w_0 in common$\}$.

It follows easily from [15] that

(13) $$\rho_n \leqq C_1 \pi_n^2.$$

Moreover (6) and (7) remain valid when π is replaced by ρ everywhere (see Remark 37).

(14) **Theorem.** *For any* $t \geqq 1$

$$E_\nu\{[\#\tilde{B}_n]^t\} \asymp [n^2 \rho_n]^t.$$

(14), (13), (8) and the second inequality of (5) show that in some sense

$$\#\tilde{B}_n \sim n^2 \rho_n \leqq C_1 n^2 \pi_n \cdot \pi_n \leqq C_2 n^{-\eta_1} E_\nu\{\#(\tilde{W} \cap S(n))\}.$$

Thus, the backbone of $\tilde{W}$ is much thinner than $\tilde{W}$ itself. This is the principal reason why the typical displacement in t steps of a random walk on $\tilde{W}$ (the so called ant in the labyrinth) is $\leqq t^{1/2-\eta}$ for some $\eta > 0$. We discuss this in detail in [7].

2. Proof of Theorem 3

To avoid minor technical complications *we shall henceforth assume that $\mathscr{G}$ is planar*, i.e., that two edges can intersect only in a vertex of $\mathscr{G}$. (This covers for instance the cases $\mathscr{G} = \mathbb{Z}^2$, the triangular or the honeycomb lattice). If $\mathscr{G}$ is not planar one has to go over to a planar modification, as explained in [6], Sect. 2.3.

By a *circuit* (on $\mathscr{G}$) we mean a path on $\mathscr{G}$ which has no self intersections when viewed as a curve in $\mathbb{R}^2$, except that its initial point coincides with its endpoint. (Recall that $\mathscr{G}$ is imbedded in $\mathbb{R}^2$). When $\mathscr{C}$ is a circuit we shall use the following notation:

$$\mathring{\mathscr{C}} = \text{interior of } \mathscr{C}, \mathscr{C}^e = \text{exterior of } \mathscr{C}$$

(when $\mathscr{C}$ is viewed as a Jordan curve in $\mathbb{R}^2$),

$$\bar{\mathscr{C}} = \mathscr{C} \cup \mathring{\mathscr{C}}, \qquad \bar{\mathscr{C}}^e = \mathscr{C} \cup \mathscr{C}^e.$$

We say that $\mathscr{C}$ *surrounds* D if $D \subset \mathring{\mathscr{C}}$. In analogy with this notation we write $\mathring{S}(n)$ for the interior of $S(n)$, i.e., for the open square $(-n, n)^2$. As is well known (cf. [12] Lemma 5.4, [13] Sect. 3.4; for the Harris-FKG inequality see [2], [6] Sect. 4.1) (1) and the Harris-FKG inequality imply that

$$\begin{aligned} P_{cr}\{&\exists \text{ occupied circuit surrounding } \mathring{S}(3^k) \\ &\text{in the annulus } S(3^{k+1}) \setminus \mathring{S}(3^k)\} \geqq \delta^4. \end{aligned}$$

Since circuits in disjoint annuli are independent we can find $3\Lambda \leqq k_1 < k_2 < \ldots$ such that

$$\begin{aligned} \alpha_i := P_{cr}\{&\exists \text{ occupied circuit surrounding } S(3^{k_i}) \\ &\text{in the annulus } S(3^{k_i+1}) \setminus S(3^{k_i})\} \to 1, \ i \to \infty. \end{aligned}$$

We fix such k_i for the remainder of this section and write

$$A(i)=A_i=S(3^{k_{i+1}})\setminus S(3^{k_i}).$$

By the method of [2] or [5], Lemma 1, it is not hard to show that among all occupied circuits which surround $S(n)$ in an annulus $S(n)\setminus S(m)$ $(m<n)$ there is a unique innermost one, i.e., a circuit $\mathscr{C}$ with minimal interior $\mathring{\mathscr{C}}$. If $\mathscr{C}\subset A_i$ and $\mathscr{C}$ surrounds $S(3^{k_i})$ then we shall use the abbreviation $F_i(\mathscr{C})$ for the following event

$$F_i(\mathscr{C})=\{\mathscr{C} \text{ is the innermost occupied circuit in } A_i \text{ which surrounds } S(3^{k_i})\}.$$

Also we write

$$F_i=\bigcup F_i(\mathscr{C}),$$

where the union is over all circuits $\mathscr{C}$ in A_i surrounding $S(3^{k_i})$. Note that this is a disjoint union and hence[2]

$$\alpha_i=P_{cr}\{F_i\}=\sum_{\mathscr{C}\subset A_i} P_{cr}\{F_i(\mathscr{C})\}. \tag{15}$$

As observed already by Harris [2], the event $F_i(\mathscr{C})$ depends only on the occupancy of vertices on $\mathscr{C}$ or in $\mathring{\mathscr{C}}\cap A_i$, but not on vertices outside A_i or in $\mathscr{C}^e$. Thus events depending only on the occupancy of vertices in $\mathscr{C}^e\cup(\mathbb{R}^2\setminus A_i)$ are independent of $F_i(\mathscr{C})$. For the vertices on $\mathscr{C}$, the occurrence of $F_i(\mathscr{C})$ of course implies that all of them are occupied. Now let E be any cylinder event depending only on the occupancy of vertices in $S(l)$ and let $l<3^{k_i}<3^{k_{i+1}}<n$, $w_0\in\mathring{S}(3^{k_i})$. Then[3]

$$\begin{aligned} E\cap\{w_0\rightsquigarrow S^c(n)\}&=E\cap F_i^c\cap\{w_0\rightsquigarrow S^c(n)\}\\ &\cup\Big[\bigcup_{\mathscr{C}\subset A_i}(E\cap F_i(\mathscr{C})\cap\{w_0\rightsquigarrow S^c(n)\})\Big]. \end{aligned}$$

Furthermore, since any circuit $\mathscr{C}$ in A_i surrounds w_0 but is contained in $\mathring{S}(n)$, we see that any path from w_0 to $S^c(n)$ must intersect $\mathscr{C}$. Thus, if $\mathscr{C}$ is occupied, then $w_0\rightsquigarrow S^c(n)$ occurs if and only if $w_0\rightsquigarrow\mathscr{C}$ in $\bar{\mathscr{C}}$ and $\mathscr{C}\rightsquigarrow S^c(n)$ in $\bar{\mathscr{C}}^e$. Given that $\mathscr{C}$ is occupied, the latter two events are conditionally independent. Thus

$$\begin{aligned} &P_p\{E\cap F_i(\mathscr{C})\cap\{w_0\rightsquigarrow S^c(n)\}\}\\ &\quad=P_p\{E\cap F_i(\mathscr{C})\cap\{w_0\rightsquigarrow\mathscr{C} \text{ in } \bar{\mathscr{C}}\}\\ &\quad\quad\cdot P_p\{\mathscr{C}\rightsquigarrow S^c(n) \text{ in } \bar{\mathscr{C}}^e\,|\,\mathscr{C} \text{ is occupied}\}. \end{aligned} \tag{16}$$

2 The sum in (15), and later similar sums or unions are over circuits $\mathscr{C}$ in A_i which surround $S(3^{k_i})$. The latter restriction shall usually not be indicated in the formulae

3 For an event G, G^c will denote its complement

Finally then, for $p \geqq p_H$,

$$(17)\qquad |P_p\{E, w_0 \rightsquigarrow S^c(n)\} - \sum_{\mathscr{C} \subset A_i} P_p\{E \cap F_i(\mathscr{C}) \cap \{w_0 \rightsquigarrow \mathscr{C} \text{ in } \bar{\mathscr{C}}\}\} P_p\{\mathscr{C} \rightsquigarrow S^c(n) \text{ in } \bar{\mathscr{C}}^e | \mathscr{C} \text{ is occupied}\}|$$
$$\leqq P_p\{F_i^c \cap (w_0 \rightsquigarrow S^c(n))\} \leqq P_p\{F_i^c\} P_p\{w_0 \rightsquigarrow S^c(n)\}$$
$$(\text{by Harris-FKG inequality}) \leqq (1-\alpha_i) P_p\{w_0 \rightsquigarrow S^c(n)\}$$

(see (15)). In essentially the same way we obtain for $\mathscr{C} \subset A_i$ and

$$3^{k_i+1} < 3^{k_j} < 3^{k_j+1} < n,$$

$$(18)\qquad |P_p\{\mathscr{C} \rightsquigarrow S^c(n) \text{ in } \bar{\mathscr{C}}^e | \mathscr{C} \text{ is occupied}\} - \sum_{\mathscr{D} \subset A_j} P_p\{F_j(\mathscr{D}),\ \mathscr{C} \rightsquigarrow \mathscr{D} \text{ in } \bar{\mathscr{C}}^e \cap \bar{\mathscr{D}} | \mathscr{C} \text{ is occupied}\}$$
$$\cdot P_p\{\mathscr{D} \rightsquigarrow S^c(n) \text{ in } \bar{\mathscr{D}}^e | \mathscr{D} \text{ is occupied}\}|$$
$$\leqq (1-\alpha_j) P_p\{\mathscr{C} \rightsquigarrow S^c(n) \text{ in } \bar{\mathscr{C}}^e | \mathscr{C} \text{ is occupied}\}.$$

We shall write

$$M(\mathscr{C}, \mathscr{D}, j) = M(\mathscr{C}, \mathscr{D}, j, p) = P_p\{F_j(\mathscr{D}),\ \mathscr{C} \rightsquigarrow \mathscr{D} \text{ in } \bar{\mathscr{C}}^e \cap \bar{\mathscr{D}} | \mathscr{C} \text{ is occupied}\}$$

and

$$\gamma(\mathscr{D}, n) = \gamma(\mathscr{D}, n, p) = P_p\{\mathscr{D} \rightsquigarrow S^c(n) \text{ in } \bar{\mathscr{D}}^e | \mathscr{D} \text{ is occupied}\}.$$

In this notation (18) says

$$(19)\qquad |\gamma(\mathscr{C}, n) - \sum_{\mathscr{D} \subset A_j} M(\mathscr{C}, \mathscr{D}, j)\, \gamma(\mathscr{D}, n)| \leqq (1-\alpha_j)\, \gamma(\mathscr{C}, n).$$

To prove that

$$\lim_{n\to\infty} P_{cr}\{E | w_0 \rightsquigarrow S^c(n)\}$$

exists it suffices to show that

$$(20)\qquad \lim_{n\to\infty} \frac{P_{cr}\{E | w_0 \rightsquigarrow S^c(n)\}}{P_{cr}\{E' | w_0 \rightsquigarrow S^c(n)\}} = \lim_{n\to\infty} \frac{P_{cr}\{E, w_0 \rightsquigarrow S^c(n)\}}{P_{cr}\{E', w_0 \rightsquigarrow S^c(n)\}}$$

exists for any cylinder event E' (in fact it suffices to show this with $E' = E^c$ or E' = the certain event). Since the sum over $\mathscr{C}$ in (17) is a finite sum with range independent of n, and since $1-\alpha_i$ can be made arbitrarily small by choosing i large, (20) will follow once one shows

$$(21)\qquad \lim_{n\to\infty} \frac{\gamma(\mathscr{C}', n, p_H)}{\gamma(\mathscr{C}'', n, p_H)}$$

exists for any circuits $\mathscr{C}'$, $\mathscr{C}'' \subset A_i$. By (19) we can for fixed i and $\varepsilon > 0$ find a j such that

$$e^{-\varepsilon}\gamma(\mathscr{C},n) \leqq \sum_{\mathscr{D}\subset A_j} M(\mathscr{C},\mathscr{D},j)\,\gamma(\mathscr{D},n) \leqq e^{\varepsilon}\gamma(\mathscr{C},n)$$

uniformly in $\mathscr{C}\subset A_i$ and $p \geqq p_H$. By iteration (with ε replaced successively by $\varepsilon/2$, $\varepsilon/4, \ldots$,) we can for fixed $i, \varepsilon > 0$, find $3\Lambda \leqq j_1 < j_2 < \ldots < j_s$ with $j_l \geqq j_{l-1} + 6$, $l = 2, \ldots, s$ (depending on i and ε only) such that

$$\begin{aligned}(22)\qquad & e^{-2\varepsilon}\gamma(\mathscr{C},n)\\ &\leqq \sum_{\mathscr{D}_1\subset A(j_1)} \cdots \sum_{\mathscr{D}_s\subset A(j_s)} M(\mathscr{C},\mathscr{D}_1,j_1)\ldots M(\mathscr{D}_{s-1},\mathscr{D}_s,j_s)\,\gamma(\mathscr{D}_s,n)\\ &\leqq e^{2\varepsilon}\gamma(\mathscr{C},n)\end{aligned}$$

for all $p \geqq p_H$ and $n > 3^{k_s+1}$. We shall think of $M(\mathscr{D}_{l-1}, \mathscr{D}_l)$ as a positive matrix with entries indexed by the $\mathscr{D}$'s.

Towards the end of this section we shall prove the following lemma.

(23) **Lemma.** *There exists a constant* $1 < \kappa < \infty$ (*independent of* ε *and* s *and the* j_l *provided* $j_l \geqq j_{l-1} + 6$) *such that for all* $p \geqq p_H$, $\mathscr{D}', \mathscr{D}'' \subset A_{l-1}$, $\mathscr{E}', \mathscr{E}'' \subset A_l$

$$\frac{M(\mathscr{D}',\mathscr{E}',j_l)\,M(\mathscr{D}'',\mathscr{E}'',j_l)}{M(\mathscr{D}',\mathscr{E}'',j_l)\,M(\mathscr{D}'',\mathscr{E}',j_l)} \leqq \kappa^2.$$

Before proving the lemma we show how it, together with (22) and standard contraction properties of multiplication by positive matrices implies Theorem 3. For any two row vectors $u' = (u'(1), \ldots, u'(\lambda))$ and u'' with strictly positive components and the same dimension set

$$\operatorname{osc}(u', u'') = \max_{i,j} \left| \frac{u'(i)}{u''(i)} - \frac{u'(j)}{u''(j)} \right|$$

Hopf, [3] Theorem 1, showed that if $M = (m_{i,j})_{1 \leqq i \leqq \lambda,\, 1 \leqq j \leqq \rho}$ is a $\lambda \times \rho$-matrix with strictly positive entries which satisfy

$$(24)\qquad \max_{i_1,i_2,j_1,j_2} \frac{m(i_1,j_1)\,m(i_2,j_2)}{m(i_1,j_2)\,m(i_2,j_1)} \leqq \kappa^2,$$

then

$$(25)\qquad \operatorname{osc}(u'M, u''M) \leqq \frac{\kappa - 1}{\kappa + 1}\operatorname{osc}(u', u'').$$

We apply this with $u'(u'')$ equal to the row vector $M(\mathscr{C}', \cdot, j_1)$ (respectively $M(\mathscr{C}'', \cdot, j_1)$ for some fixed $\mathscr{C}', \mathscr{C}'' \subset A_i$. Then

$$\sum_{\mathscr{D}_1\subset A(j_1)} \cdots \sum_{\mathscr{D}_{s-1}\subset A(j_{s-1})} M(\mathscr{C}',\mathscr{D}_1,j_1)\ldots M(\mathscr{D}_{s-1},\mathscr{D}_s,j_s)$$

is the $\mathscr{D}_s$ component of $u'\ M_2 \ldots M_s$, where $M_l(\cdot,\cdot) = M(\cdot,\cdot,j_l)$ satisfies (24). Similarly when u' and $\mathscr{C}'$ are replaced by u'' and $\mathscr{C}''$. Thus by (25) and induction on s

$$\max_{\mathscr{D}_s, \mathscr{D}_s' \subset A(j_s)} \left| \frac{u' M_2 \dots M_s(\mathscr{D}_s')}{u'' M_2 \dots M_s(\mathscr{D}_s')} - \frac{u' M_2 \dots M_s(\mathscr{D}_s'')}{u'' M_2 \dots M_s(\mathscr{D}_s'')} \right| \leqq \left(\frac{\kappa-1}{\kappa+1}\right)^{s-1}.$$

In other words, there exists a number $\xi = \xi(\mathscr{C}', \mathscr{C}'', p, s)$ such that

$$\left| \frac{u' M_2 \dots M_s(\mathscr{D}_s)}{u'' M_2 \dots M_s(\mathscr{D}_s)} - \xi \right| \leqq \left(\frac{\kappa-1}{\kappa+1}\right)^{s-1} \quad \text{for all } \mathscr{D}_s \subset A(j_s).$$

Together with (22) this implies

$$e^{-4\varepsilon}\left\{\xi - \left(\frac{\kappa-1}{\kappa+1}\right)^{s-1}\right\} \leqq \frac{\gamma(\mathscr{C}', n)}{\gamma(\mathscr{C}'', n)} \leqq e^{4\varepsilon}\left\{\xi + \left(\frac{\kappa-1}{\kappa+1}\right)^{s-1}\right\}$$

for all sufficiently large n. Since ε and s are arbitrary, and κ is independent of ε and s it follows that

$$\lim_{n\to\infty} \frac{\gamma(\mathscr{C}', n, p)}{\gamma(\mathscr{C}'', n, p)} \quad \text{exists, uniformly in } p \geqq p_H.$$

In particular (21) holds and the first limit in (3) exists. In fact the same argument shows that

$$\lim_{n\to\infty} P_p\{E \mid w_0 \rightsquigarrow S^c(n)\} \quad \text{exists, uniformly in } p \geqq p_H$$

for any cylinder event E. However, for $p > p_H$ this last limit equals $P_p\{E \mid \# W = \infty\}$, and for each fixed n $P_p\{E \mid w_0 \rightsquigarrow S^c(n)\}$ is a continuous function of p. Thus also $p \to P_p\{E \mid \# W = \infty\}$ is continuous on $[p_H, 1]$ and the second limit in (3) exists and is the same as the first limit in (3).

Once we know that the common limit in (3), $v(E)$ say, exists, it is immediate from Kolmogorov's extension theorem [9], Sect. III.3, especially Cor. on p. 83, that v extends to a probability measure on the occupancy configurations. Trivially $v\{w_0 \rightsquigarrow S^c(k)\} = 1$ for each k so that $v\{\tilde{W} = W(w_0) \text{ is infinite}\} = 1$. Also, by the Harris-FKG inequality,

$$v\{\exists \text{ occupied circuit in } A\} \geqq P_{cr}\{\exists \text{ occupied circuit in } A\}$$

for any annulus. Therefore, as in [13], Lemma 3.6 and Theorem 3.14 or [6], pp. 178 and 194 there exist infinitely many occupied circuits a.e. $[v]$, and $\tilde{W}$ is unique. Thus also the last part of Theorem 3 will follow and it remains to prove (23).

(23) will be a consequence of a general connectivity argument. Several variants of this argument will be needed. We formulate the most important one as a separate lemma. We remind the reader that an event G is called *increasing* if its indicator function can only increase when any vertex is changed from vacant to occupied (cf. [6], Def. 4.1).

Lemma. *For each* $k>1$ *there exists a* $\delta_k>0$ *such that for all* $p\geqq p_H$, $n\geqq 3\Lambda$

(26) $$P_p\{\exists \text{ occupied horizontal crossing of } [0,kn]\times[0,n]\}\geqq\delta_k,$$

(27) $$P_p\{\exists \text{ occupied vertical crossing of } [0,n]\times[0,kn]\}\geqq\delta_k,$$

and for $k(3\Lambda+1)\leqq m\leqq\frac{k-1}{k}n$

(28) $$P_p\{\exists \text{ occupied circuit surrounding } S(m) \text{ in the annulus } S(n)\backslash S(m)\}\geqq\delta_{2k}^4.$$

There also exists a $\tilde{\delta}_k>0$ *with the following property: If* A_i^* *is an annulus* $S(n_i)\backslash S(m_i)$, *with* $m_i\leqq\frac{k-1}{k}n_i$, $i=1,2$, *and* $k(3\Lambda+1)\leqq m_1<n_1\leqq m_2<n_2$ *with* $m_2\leqq k n_1$, *then for any increasing event* G *and* $p\geqq p_H$

(29) $$\begin{aligned}P_p\{G,\exists &\text{ occupied circuits } \mathscr{C}_i \text{ in } A_i^* \text{ surrounding}\\ &S(m_i) \text{ for } i=1,2, \text{ with } \mathscr{C}_1\rightsquigarrow\mathscr{C}_2\\ &\text{in } S(n_2)\backslash S(m_1)\}\geqq\tilde{\delta}_k P_p\{G\}.\end{aligned}$$

Proof. (26) follows easily from (1) by combining horizontal crossings of $[jn,(j+3)n]\times[0,n]$, $0\leqq j\leqq k-3$ with a number of vertical crossings of $[jn,(j+1)n]\times[0,n]$ (cf. [10] Lemma 4, [12] Lemma 5.3, or [13] Lemma 3.4). Similarly for (27). (28) follows from (26) and (27) by combining two vertical crossings, one each of $[-n,-m)\times[-n,n]$ and $(m,n]\times[-n,n]$, with two horizontal crossings, one each of $[-n,n]\times[-n,-m)$ and $[-n,n]\times(m,n]$ (see Fig. 1 and [12] Lemma 5.4 or [13] Lemma 3.5).

A similar argument works for (29). Let H_i be the event that there exists an occupied circuit surrounding $S(m_i)$ in $\tilde{A}_i$, $i=1,2$, where

$$\tilde{A}_1=S(n_1)\backslash S\left(\frac{k-1}{k}n_1\right)\subset A_1^*,$$

$$\tilde{A}_2=S\left(\frac{k}{k-1}m_2\right)\backslash S(m_2)\subset A_2^*.$$

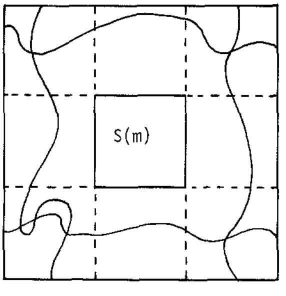

Fig. 1. A circuit can be formed from two vertical and two horizontal crossings

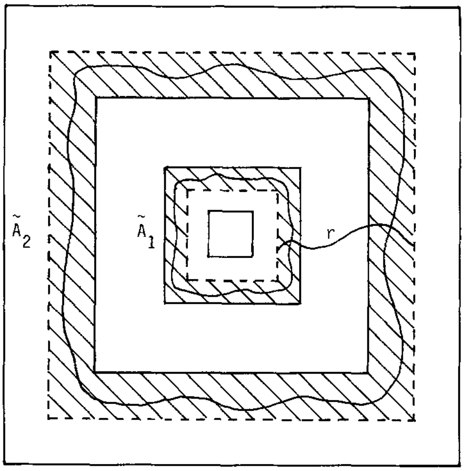

Fig. 2. The *solid squares* are (starting from the *inside*) $S(m_1)$, $S(n_1)$, $S(m_2)$ and $S(n_2)$. The *two dashed squares* are $S(\tilde{m}_1)$ (the smaller one) and $S(\tilde{n}_2)$ (the larger one). The annuli $\tilde{A}_1$ and $\tilde{A}_2$ are *hatched*. The path r connects $S(\tilde{m}_1)$ and $S^c(\tilde{n}_2)$

(see Fig. 2). $\tilde{A}_1 \subset A_1^*$ because $m_1 \leqq \tilde{m}_1 := (k-1)n_1/k$ and $\tilde{A}_2 \subset A_2^*$ because $n_2 \geqq \tilde{n}_2 := k m_2/(k-1)$.

Also denote by K the event $\{S(\tilde{m}_1) \rightsquigarrow S^c(\tilde{n}_2)\}$. Then by the Harris-FKG inequality and (28) the left hand side of (29) is at least

$$\begin{aligned} P_p\{G \cap H_1 \cap H_2 \cap K\} &\geqq P_p\{G\}\, P_p\{H_1\}\, P_p\{H_2\}\, P_p\{K\} \\ &\geqq (\delta_{2k})^8\, P_p\{G\}\, P_p\{K\}. \end{aligned}$$

Moreover (see Fig. 2)

$$P_p\{K\} \geqq P_p\{\exists \text{ occupied horizontal crossing of } [\tilde{m}_1, \tilde{n}_2] \times [-\tilde{m}_1, \tilde{m}_1]\} \geqq \delta_l$$

for any $l \geqq \frac{1}{2} k^3 (k-1)^{-2}$. The last inequality follows from

$$\tilde{n}_2 = \frac{k}{k-1} m_2 \leqq \frac{k^2}{k-1} n_1 \leqq \frac{k^3}{(k-1)^2} \tilde{m}_1 \quad \text{and (26).} \quad \Box$$

(30) *Remark.* We shall want to apply (29) in a case where the occurrence of G forces the existence of occupied paths r_i connecting $\mathbb{R}^2 \setminus S(n_i)$ to $S(m_i)$ in A_i^*, $i = 1, 2$. The circuits $\mathscr{C}_i$ plus a path from $\mathscr{C}_1$ to $\mathscr{C}_2$ in $S(n_2) \setminus S(m_1)$ then connect r_1 to r_2 in $S(n_2) \setminus S(m_1)$ (see Fig. 3). $\Box$

Proof of Lemma 23. We shall prove that there exists a $\kappa \geqq 1$ such that for $\mathscr{D} \subset A(j_{l-1})$, $\mathscr{E} \subset A(j_l)$ one has with

$$s = k_{j_{l-1}+3}, \qquad t = 3^s,$$

(31)
$$\begin{aligned} &\kappa^{-1} \gamma(\mathscr{D}, t, p)\, P_p\{F_{j_l}(\mathscr{E}),\ S(t+3\Lambda) \rightsquigarrow \mathscr{E}\} \\ &\leqq M(\mathscr{D}, \mathscr{E}, j_l) \\ &\leqq \gamma(\mathscr{D}, t, p)\, P_p\{F_{j_l}(\mathscr{E}),\ S(t+3\Lambda) \rightsquigarrow \mathscr{E}\}. \end{aligned}$$

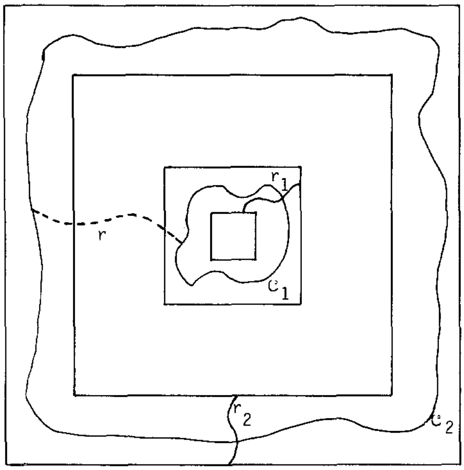

Fig. 3. The *dashed path* r connects $\mathscr{C}_1$ with $\mathscr{C}_2$. The paths r_i which connect the outer and inner boundary of A_i intersect $\mathscr{C}_i$, $i=1,2$

This shows, that within a factor κ, $M(\mathscr{D},\mathscr{E},j)$ is a product of two factors, one depending on $\mathscr{D}$ only and another on $\mathscr{E}$ only. (23) will be immediate from this. The second inequality in (31) is proved in the same way as (16): any occupied path r from $\mathscr{D}$ to $\mathscr{E}$ must cross the boundaries of $S(t)$ and $S(t+3\Lambda)$, since

$$\mathscr{D}\subset A(j_{l-1})\subset \mathring{S}(t)\subset S(t+3\Lambda)\subset \mathring{S}(3^{s+1})\subset \mathscr{E}.$$

Therefore r must contain a piece r_1 connecting $\mathscr{D}$ to $S^c(t)$ and a piece r_2 connecting $S(t+3\Lambda)$ to $\mathscr{E}$. The existence of r_1 and $\{F_{j_l}(\mathscr{E})\cap(r_2 \text{ exists})\}$ are independent events. Indeed the existence of an occupied connection from $\mathscr{D}$ to $S^c(t)$ depends only on vertices in $S(t+\Lambda)$. Similarly $F_{j_l}(\mathscr{E})$ and the existence of an occupied connection from $S(t+3\Lambda)$ to $\mathscr{E}$ depend only on vertices outside $\mathring{S}(t+2\Lambda)$. Therefore, the probability of $F_{j_l}(\mathscr{E})$ and the existence of r_1 and r_2 is given by the last member of (31).

For the first inequality in (31) we shall condition on the occupancy configuration, Ξ say, in $A(j_l)$ and on $\mathscr{D}$ being occupied. Fix such a configuration Ξ in $A(j_l)$ for which $F_{j_l}(\mathscr{E})$ occurs. Note that this last event depends on the configuration in $A(j_l)$ only. Set

$$m_1=3^{k_{j_{l-1}}+2},\qquad n_1=3^{k_{j_{l-1}}+3}=t,\qquad m_2=t+3\Lambda$$
$$n_2=3m_2,\qquad A_i^*=S(n_i)\setminus S(m_i),\qquad i=1,2.$$

Define the increasing event G as

$$\{\mathscr{D}\rightsquigarrow S^c(t)\ \text{in}\ \bar{\mathscr{D}}^e\ \text{and}\ S(t+3\Lambda)\rightsquigarrow\mathscr{E}\}.$$

Since we already fixed all vertices on $\mathscr{D}$ as occupied, as well as the configuration in $A(j_l)$, we can view G as depending only on the vertices in $S(3^{j_l})\cap\mathscr{D}^e$. These sites are independent of those in $\bar{\mathscr{D}}\cup A(j_l)$ and we can therefore still apply (29). Note that as in Remark (30), if the event in (29) occurs for the present G, then the occupied paths from $\mathscr{D} \rightsquigarrow S^c(t)$ and from $S(t+3\Lambda)$ to $\mathscr{E}$ (which exist when G occurs) are connected by the occupied circuits $\mathscr{C}_1$ and $\mathscr{C}_2$ and an occupied path between $\mathscr{C}_1$ and $\mathscr{C}_2$. Consequently in this case $\mathscr{D}$ is actually connected to $\mathscr{E}$ by all these pieces. Thus, conditionally on $\mathscr{D}$ being occupied and on the configuration in $A(j_l)$, the probability of $\mathscr{D} \rightsquigarrow \mathscr{E}$ in $\bar{\mathscr{D}}^e \cap \bar{\mathscr{E}}$ is at least (by (29))

$$\begin{aligned}\tilde{\delta}_2 P_p\{G \mid \mathscr{D} \text{ occupied, } \Xi \text{ in } A(j_l)\}\\ &= \tilde{\delta}_2 P_p\{\mathscr{D} \rightsquigarrow S^c(t) \text{ in } \bar{\mathscr{D}}^e \mid \mathscr{D} \text{ occupied}\}\\ &\quad \cdot P_p\{S(t+3\Lambda) \rightsquigarrow \mathscr{E} \mid \Xi \text{ in } A(j_l)\}\\ &= \tilde{\delta}_2\, \gamma(\mathscr{D}, t, p)\, P_p\{S(t+3\Lambda) \rightsquigarrow \mathscr{E} \mid \Xi \text{ in } A(j_l)\}.\end{aligned}$$

Averaging with respect to all Ξ in $A(j_l)$ for which $F_{j_l}(\mathscr{E})$ occurs we obtain the first inequality of (31) with $\kappa = (\tilde{\delta}_2)^{-1}$.

3. Proofs of Theorems 8 and 14

We begin with the Proof of (6) and (7). It is obvious from the definition that π_n is decreasing. Also, any path from w_0 to $(n, \infty) \times \mathbb{R}$ must leave $S(n)$ so that

$$\begin{aligned}\pi_n &\leqq P_{cr}\{w_0 \rightsquigarrow S^c(n)\}\\ &\leqq 4 P_{cr}\{w_0 \rightsquigarrow \sigma(n) \text{ in } S(n+\Lambda)\} \leqq C_1 \pi_n,\end{aligned} \tag{32}$$

for $\sigma(n)$ one of the four rectangles which make up $S(n+\Lambda)\backslash S(n)$. For the sake of argument assume that

$$P_{cr}\{w_0 \rightsquigarrow S^c(n)\} \leqq 4 P_{cr}\{w_0 \rightsquigarrow [n, n+\Lambda] \times [-n, n] \text{ in } S(n+\Lambda)\}.$$

Take for G the increasing event

$$\{w_0 \rightsquigarrow \sigma(n) \text{ in } S(n+\Lambda) \text{ and } \sigma(n) \rightsquigarrow S^c(2n)\}.$$

Now apply (29) for this G and $A_1^* = S(n)\backslash S(n/2)$, $A_2^* = S(2n)\backslash S(n+\Lambda)$. Just as in Remark (30), if the event in (29) occurs then there exists an occupied path from w_0 to $\sigma(n)$ and another occupied path from $\sigma(n)$ to $S^c(2n)$, and these two paths are connected by pieces of two occupied circuits $\mathscr{C}_1$ and $\mathscr{C}_2$ and an occupied path between the circuits. Thus $w_0 \rightsquigarrow S^c(2n)$ in this situation. Consequently by (32) and (29) and the Harris-FKG inequality

$$\begin{aligned}\pi_{2n} &\geqq C_2 P_{cr}\{w_0 \rightsquigarrow S^c(2n)\}\\ &\geqq C_2 \tilde{\delta}_3 P_{cr}\{G\} \geqq C_3 \tilde{\delta}_3 \pi_n P_{cr}\{\sigma(n) \rightsquigarrow S^c(2n)\}.\end{aligned}$$

The last probability is – by virtue of (1) – for $n \geqq 3\Lambda$ at least

$$P_{cr}\{[n, n+\Lambda] \times [-n, n] \rightsquigarrow (2n, 2n+\Lambda] \times [-n, n]\} \geqq \delta,$$

so that (6) follows.

The first inequality in (7) is immediate from the fact that π_n is decreasing. For the second inequality we consider

$$V_n := \{\text{number of vertices of the form } w_0+(0,k) \text{ with } 0 \leq k \leq 2n \text{ which are connected by an occupied path to the half space } (n,\infty)\times\mathbb{R}\}.$$

Clearly, by periodicity

$$E_{cr} V_n = \sum_{k=0}^{2n} \pi_n = (2n+1)\pi_n. \tag{33}$$

Next we find a lower bound for V_n by considering the "lowest occupied crossing" of a figure which is very close to the rectangle $[-n,n]\times[0,n]$. Because we want the crossing to begin and end on the boundary of our figure, we choose four selfavoiding paths $J_1 - J_4$ on $\mathscr{G}$ such that their concatenation is a Jordan curve and such that

$$\begin{aligned} &J_1 \subset [-n-3\Lambda, -n)\times[-3\Lambda, n+3\Lambda],\\ &J_2 \subset [-n-3\Lambda, n+3\Lambda]\times(n, n+3\Lambda],\\ &J_3 \subset (n, n+3\Lambda]\times[-3\Lambda, n+3\Lambda],\\ &J_4 \subset [-n-3\Lambda, n+3\Lambda]\times[-3\Lambda, 0)\end{aligned}$$

(see Fig. 4).

Once again the reader is advised to think of the case $\mathscr{G}=\mathbb{Z}^2$ in which case we can take for $J_1 - J_4$ simply the four sides of the rectangle $[-n,n]\times[0,n]$. Write J for the interior of the Jordan curve made up of $J_1 - J_4$. If r is a selfavoiding path on $\mathscr{G}$ which has its initial point (endpoint) on J_1 (respectively on J_3) and lies otherwise in J, then denote by $J^-(r)(J^+(r))$ the component of $J\backslash r$ with J_4 (respectively J_2) in its boundary (see Fig. 4). The lowest occupied (horizontal) crossing of J is now defined as that occupied path R on $\mathscr{G}$, connecting J_1 to J_3 and lying in J (except for its endpoints) for which $J^-(R)$ is minimal. As in [5], Lemma 1 or [6], Prop. 2.3 one sees that there exists a

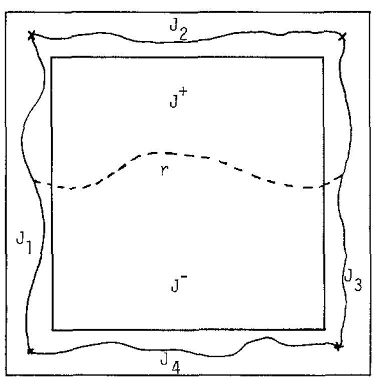

Fig. 4. The *inner rectangle* is $[-n,n]\times[0,n]$ and the *outer rectangle* is $[-n-3\Lambda, n+3\Lambda]\times[-3\Lambda, n+3\Lambda]$

unique lowest occupied crossing R whenever $J_1 \rightsquigarrow J_3$ in $J \cup J_1 \cup J_3$. It follows that R exists under P_{cr} with probability at least

$$P_{cr}\{\exists \text{ occupied horizontal crossing of } [-n-4\Lambda, n+4\Lambda]+[0,n]\} \geqq \delta_3$$

by (26). Moreover, if r_0 is any fixed self avoiding crossing of J from J_1 to J_3 as above, then the event $\{R=r_0\}$ is independent of the vertices in $J^+(r_0)$ (cf. [5], Lemma 1 or [6], Prop. 2.3 and Fig. 4).

Now we give a lower bound for

$$P_{cr}\{(w_0(1), w_0(2)+k) \rightsquigarrow (n,\infty)\times\mathbb{R} \mid R=r_0\}. \tag{34}$$

Denote the highest intersection of r_0 with the line $x=w_0(1)$ by u. Since $r_0 \in J \cup J_1 \cup J_3$ we have $u(2) \leqq n+3\Lambda$. We restrict ourselves to k with

$$u(2)+24\Lambda+8|w_0(1)| < w_0(2)+k \leqq 2n. \tag{35}$$

For such k, $w_0+(0,k)$ lies "above r_0", i.e., it lies in J^+ or in $(\mathbb{R}^2 \setminus J)$. On the event $\{R=r_0\}$, r_0 itself is occupied and has its endpoint in $(n,\infty)\times\mathbb{R}$, so that $w_0+(0,k) \rightsquigarrow (n,\infty)\times\mathbb{R}$ will occur whenever $w_0+(0,k)$ is connected to r_0 by an occupied path in $(-n,n)\times\mathbb{R}$. The piece of such a path from $w_0+(0,k)$ to its first intersection with r_0 lies outside $J^- \cup r_0$ and therefore (just as in [5], step (i) of Prop. 1 or [6], Lemma 8.2) (34) is at least as large as

$$\begin{aligned} &P_{cr}\{w_0+(0,k) \rightsquigarrow r_0 \text{ in } (-n,n)\times\mathbb{R} \mid R=r_0\} \\ &\geqq P_{cr}\{\exists \text{ occupied circuit surrounding } u \text{ in the} \\ &\quad \text{annulus } A \text{ and } w_0+(0,k) \rightsquigarrow T^c\}, \end{aligned} \tag{36}$$

where $l=w_0(2)+k-u(2)$ and

$$A=\left[-\frac{l}{4},\frac{l}{4}\right]+[u(2)-3l, u(2)+3l]\setminus$$
$$\left(-\frac{l}{8},\frac{l}{8}\right)\times(u(2)+2l, u(2)-2l),$$

$$T \text{ is the rectangle } \left[-\frac{l}{4},\frac{l}{4}\right]\times[u(2)-3l, u(2)+3l]$$

and T^c its complement (see Fig. 5). Note that $A \subset T$ and that $w_0+(0,k)$ lies inside the inner rectangular boundary of A, so that a circuit in A surrounding u, also surrounds $w_0+(0,k)$. We leave it to the reader to show that the last probability in (36) is $\geqq C_1\pi_l$ (use (26), (27) and the fact that the dimensions of T and the inner and outer boundary of A are all of order l).

The above estimate for (36) is independent of r_0 and holds for all k which satisfy (35) and a fortiori for $24\Lambda+8|w_0(1)|<l<n-3\Lambda$

$$E_{cr}\{V_n \mid R=r_0\} \geqq C_1 \sum_{24\Lambda<+8|w_0(1)|<l<n-3\Lambda} \pi_l$$

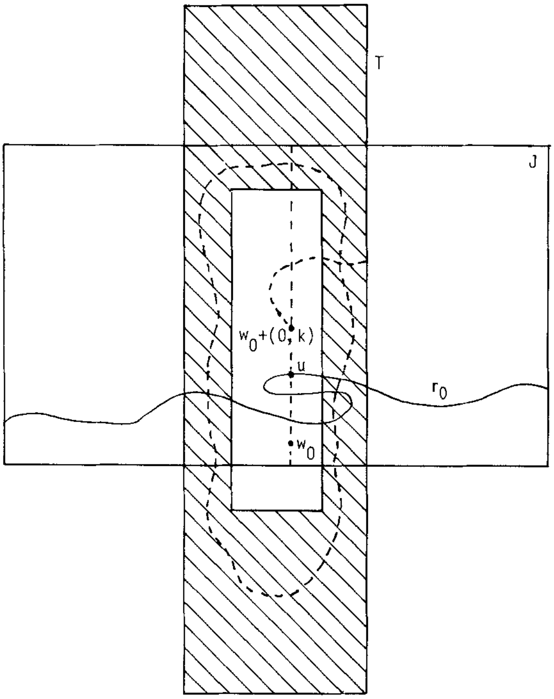

Fig. 5. A is the *hatched region*. $w_0+(0,k)$ is connected to r_0 by pieces of the *dashed path* and *circuit*

and

$$E_{cr}\{V_n\} \geqq C_1 \sum \pi_l P_{cr}\{R \text{ exists}\}$$
$$\geqq C_2 \delta_3 \sum_{l=1}^{n} \pi_l.$$

Combined with (33) this yields (7). □

(37) *Remark*. The proof of (6) and (7) with π replaced by ρ everywhere is similar. The details are only slightly more complicated and will not be given here.

Proof of Theorem 8. We begin with a lower bound for $E_v\{Z(n)\}$, where

$$Z(n) = \#(\tilde{W} \cap S(n)).$$

This is very similar to the proof of the second inequality in (7). Let $v = w_0 + (k, l)$ with $0 \leqq l \leqq k < n$ for the sake of argument, and let r be a path from w_0 to $S^c(m)$, $m > 3n$. Then v will be connected to r (and hence will belong to $\tilde{W}$ if r is occupied) if there exists an occupied circuit in the annulus $A := S(3k) \backslash S(2k)$ and if $v \rightsquigarrow S^c(3k)$ (note that v, as well as w_0, lies in $S(2k)$ if k is large enough). Thus, again by the Harris-FKG inequality and (28) we have for large enough

k, say $k \geqq k_0$,

$$\begin{aligned}
&P_{cr}\{w_0 \rightsquigarrow S^c(m) \text{ and } v \rightsquigarrow w_0\} \\
&\quad \geqq P_{cr}\{w_0 \rightsquigarrow S^c(m)\}\, P_{cr}\{\exists \text{ occupied circuit in } A\} \\
&\qquad \cdot P_{cr}\{v \rightsquigarrow S^c(3k)\} \\
&\quad \geqq C_1\, \pi_{3k}\, P_{cr}\{w_0 \rightsquigarrow S^c(m)\}.
\end{aligned}$$

If we divide both sides by $P_{cr}\{w_0 \rightsquigarrow S^c(m)\}$ and let $m \to \infty$ we obtain

$$\nu\{v \in W\} \geqq C_1\, \pi_{3k} \geqq C_2\, \pi_k \quad \text{(by (6))}.$$

Since there are $(k+1)$ choices for l with $0 \leqq l \leqq k$ we find

$$\begin{aligned}
E_\nu\{Z(n)\} &\geqq \sum_{k=k_0}^{n} \sum_{l=0}^{k} C_2\, \pi_k \\
&\geqq C_2\, \pi_n \sum_{k=k_0}^{n} (k+1) \geqq C_3\, n^2\, \pi_n.
\end{aligned}$$

For any positive random variable X, Jensen's inequality gives

$$E\{X^t\} \geqq [E\{X\}]^t, \qquad t \geqq 1,$$

so that the above proves

$$E_\nu\{Z^t(n)\} \geqq C(t)\,[n^2\, \pi_n]^t, \qquad t \geqq 1.$$

For an upper bound we begin with some remarks. Firstly, for any vertex $v \in [0,1]^2$ we have by the Harris-FKG inequality for any set T

$$P_{cr}\{v \rightsquigarrow T\} \geqq P_{cr}\{v \rightsquigarrow w_0\}\, P_{cr}\{w_0 \rightsquigarrow T\}$$

and

$$P_{cr}\{w_0 \rightsquigarrow T\} \geqq P_{cr}\{w_0 \rightsquigarrow v\}\, P_{cr}\{v \rightsquigarrow T\}.$$

In particular, if $S(n, v)$ denotes the square $[v(1)-n, v(1)+n] \times [v(2)-n, v(2)+n]$, and $S^c(v, n)$ its complement, then we obtain uniformly in v

$$C_1\, \pi_n \leqq P_{cr}\{v \rightsquigarrow S^c(v, n)\} \leqq C_2\, \pi_n. \tag{38}$$

(Use (32) if $v \in [0,1]^2$; the general v reduces to the case $v \in [0,1]^2$ by periodicity). Secondly we need a somewhat less trivial inequality. Let $S_1, \ldots, S_t$ be t squares of the form $S_i = S(v_i, n_i)$, $n_i \geqq 9\Lambda$, and let $m \geqq n$ be so large that

$$\bigcup_{i=1}^{t} S(v_i, 2n_i) \subset S(m). \tag{39}$$

Assume further that

$$w_0 \notin \bigcup_{i=1}^{t} S(v_i, 2n_i). \tag{40}$$

We claim that if G is any increasing cylinder event depending only on the occupancies of vertices in $\bigcup \tilde{S}_i$, where $\tilde{S}_i := S(v_i, n_i + \Lambda)$, then

$$P_{cr}\{G, w_0 \rightsquigarrow S^c(m)\} \leqq C_1 P_{cr}\{G\} P_{cr}\{w_0 \rightsquigarrow S^c(m)\} \tag{41}$$

for some constant $C_1 < \infty$ independent of the S_i, G and m (but dependent on t), as long as (39) and (40) hold. To prove (41), let $T = \bigcup \tilde{S}_i$. Then G and $\{w_0 \rightsquigarrow S^c(m)$ in $S(m) \setminus T\}$ are independent events since they depend on different sets of vertices. Thus

$$\begin{aligned} &P_{cr}\{G \text{ and } w_0 \rightsquigarrow S^c(m) \text{ in } S(m)\setminus T\} \\ &\quad \leqq P_{cr}\{G\} P_{cr}\{w_0 \rightsquigarrow S^c(m)\}. \end{aligned} \tag{42}$$

One therefore merely has to show that the left hand side of (42) is at least C_1^{-1} times the left hand side of (41). However, if w_0 is connected by an occupied path r to $S^c(m)$ and if there exists an occupied circuit $\mathscr{C}_i$ in $S(v_i, 2n_i) \setminus \tilde{S}_i$ for $1 \leqq i \leqq t$, then we can replace r by an occupied path $\tilde{r}$ from w_0 to $S^c(m)$ which does not enter $\bigcup \mathring{\mathscr{C}}_i \supset T$. Indeed r starts and ends outside $\mathscr{C}_i$ by (39) and (40). If r enters $\mathring{\mathscr{C}}_i$, replace the piece of r between its first and last intersection with $\mathscr{C}_i$ by an arc of $\mathscr{C}_i$ (see Fig. 6). If the $S(v_i, 2n_i)$, $1 \leqq i \leqq t$, are disjoint, then we can do this successively for $i = 1, \ldots, t$ to obtain the desired path $\tilde{r}$. If the $S(v_i, 2n_i)$ are not disjoint then we can find a number of disjoint curves $\mathscr{C}'_j$, each $\mathscr{C}'_j$ made up of pieces of the $\mathscr{C}_i$, such that each $\tilde{S}_i$ belongs to the interior of some $\mathscr{C}'_j$, $\mathscr{C}'_j$ and $\mathscr{C}'_k$ lie in each other's exterior for $j \neq k$, and such that

$$\left(\bigcup \mathscr{C}'_j\right) \cap \left(\bigcup \tilde{S}_i\right) = \phi.$$

Thus the $\mathscr{C}'_j$ curves surround all the $\tilde{S}_i$ and lie in $\bigcup S(v_i, 2n_i)$. We can then use the preceding construction of $\tilde{r}$ with the $\mathscr{C}'_j$ replacing the $\mathscr{C}_j$. We skip the details since in our application the $S(v_i, 2n_i)$ will be disjoint.

The existence of $\tilde{r}$ shows that the left hand side of (42) is at least as large as

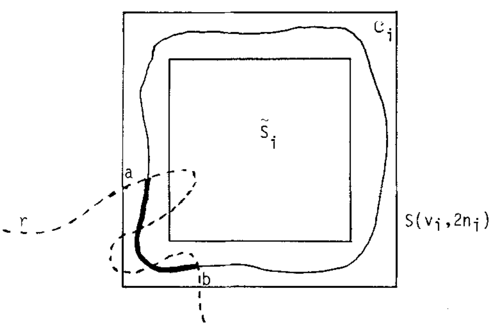

Fig. 6. r is *the dashed path*. To obtain $\tilde{r}$ replace the piece of r from a to b by the boldly drawn arc of $\mathscr{C}_i$

$$\begin{aligned}&P_{cr}\{G \text{ and } w_0 \leadsto S^c(m) \text{ and } \exists \text{ occupied circuit}\\&\quad \mathscr{C}_i \text{ in } S(v_i, 2n_i)\setminus\tilde{S}_i,\ 1 \leqq i \leqq t\}\\&\geqq P_{cr}\{G \text{ and } w_0 \leadsto S^c(m)\}\\&\prod_{i=1}^{t} P_{cr}\{\exists \text{ occupied circuit } \mathscr{C}_i \text{ in } S(v_i, 2n_i)\setminus\tilde{S}_i\}\\&\text{(by the Harris-FKG inequality)}\\&\geqq C_2 P_{cr}\{G \text{ and } w_0 \leadsto S^c(m)\} \quad \text{(by (28))}.\end{aligned}$$

This proves (41).

We turn to the upper bound for $E_v\{Z^t(n)\}$. Again by Jensen's inequality we may restrict ourselves to integer $t \geqq 1$. Then

$$E_v\{Z^t(n)\} = \lim_{m\to\infty} [P_{cr}\{w_0 \leadsto S^c(m)\}]^{-1} \cdot \sum_{v_1,\dots,v_t \in S(n)} P_{cr}\{w_0 \leadsto S^c(m) \text{ and } v_i \leadsto w_0,\ 1 \leqq i \leqq t\}. \tag{43}$$

Next we choose n_i. Take $v_0 = w_0$ and define for $\bar{v} = (v_1, \dots, v_t)$[4]

$$|u|_\infty = \max(|u(1)|, |u(2)|) \qquad (\text{for } u = (u(1), u(2)) \in \mathbb{R}^2),$$

$$n_i = n_i(\bar{v}) = \lfloor \tfrac{1}{4} \min\{|v_i - v_j|_\infty : j \neq i, 0 \leqq j \leqq t\} \rfloor, \tag{44}$$

$S_i = S(v_i, n_i)$ and $\tilde{S}_i = S(v_i, n_i + \Lambda)$ as before,

$$G_i = G_i(\bar{v}) = \{v_i \leadsto S_i^c \text{ in } \tilde{S}_i\}, \qquad G = G(\bar{v}) = \bigcap_{i=1}^{t} G(v_i).$$

First we estimate the contribution to (43) for a $\bar{v}$ for which $n_i \geqq 9\Lambda$, $1 \leqq i \leqq t$. For such a $\bar{v}$,

$$\max(|v_i(1) - w_0(1)|, |v_i(2) - w_0(2)|) \geqq 4n_i > 0, \quad 1 \leqq i \leqq t,$$

and hence $w_0 \notin S(v_i, 2n_i)$. Thus (40) holds, and so does (39) as soon as $m > 3n$. Moreover, by definition of the n_i

$$\tilde{S}_i \cap \tilde{S}_j = \emptyset \qquad \text{for } i \neq j.$$

Finally, under (40), if $v_i \leadsto w_0$, then there must exist an occupied path from v_i to w_0 and a fortiori G_i must occur. Thus, for a $\bar{v}$ with $n_i \geqq 9\Lambda$, $1 \leqq i \leqq t$, the contribution to (43) is at most

$$\begin{aligned}&[P_{cr}\{w_0 \leadsto S^c(m)\}]^{-1} P_{cr}\{G, w_0 \leadsto S^c(m)\}\\&\leqq C_1 P_{cr}\{G\} \text{ (by (41))} = C_1 \prod_{i=1}^{t} P_{cr}\{G_i\} \text{ (the } G_i \text{ are}\\&\text{independent when the } \tilde{S}_i \text{ are disjoint)}\\&\leqq C_2 \prod_{i=1}^{t} \pi(n_i) \text{ (by (38))}.\end{aligned} \tag{45}$$

[4] $\lfloor a \rfloor$ is the largest integer $\leqq a$

We claim that the inequality between the first and last members of (45) remains valid even without the condition $n_i \geqq 9\Lambda$, $1 \leqq i \leqq t$. This is seen by simply replacing G by the intersection of only those G_i for which $n_i \geqq 9\Lambda$. The extra factors $\pi(n_i)$ with $n_i < 9\Lambda$ in the right hand side are harmless. They can be incorporated in C_2 since for $n \leqq 9\Lambda$ $\pi_n \geqq \pi_{9\Lambda} > 0$.

The above shows that (43) is bounded by

$$C_2 \sum_{v_1, \ldots, v_t \in S(n)} \prod_{i=1}^{t} \pi(n_i), \tag{46}$$

and it remains to show that this expression is at most

$$\begin{aligned} C_3 \left[\sum_{k=1}^{n} k\pi_k\right]^t &\leqq C_3 \left[n \sum_{k=1}^{n} \pi_k\right]^t \\ &\leqq C_4 [n^2 \pi_n]^t \leqq C_5 \left[\sum_{k=n/2}^{n} k\pi_k\right]^t \quad \text{(by (6) and (7)).} \end{aligned} \tag{47}$$

For clarity we treat the simplest case, namely $t=1$, separately. For $t=1$

$$n_1 = \lfloor \tfrac{1}{4} \max(|v_1(1) - w_0(1)|, \quad |v_1(2) - w_0(2)|) \rfloor,$$

and the number of vertices v with $n_1 = k$ is at most $C_5 k$. Since n_1 can be at most n for v in $S(n)$, (47) clearly is an upper bound for (46) when $t=1$.

For general t[5] we have to divide the v_i into groups, and apply more or less the same argument as just given to each group separately. For the moment fix $\bar{v}$ and let i_0 be an index for which n_i is minimal, i.e.,

$$n_{i_0} = \min\{n_j : 0 \leqq j \leqq t\}. \tag{48}$$

Set $I_0 = \{i_0\}$. Define successively

$$I_l = \{j : \exists\, i \in I_{l-1} \text{ such that } n_j = \lfloor \tfrac{1}{4} |v_j - v_i|_\infty \rfloor\}.$$

Finally set

$$J_1 = \bigcup_{l \geqq 0} I_l.$$

Note that there must exist an index j_0 such that

$$n_{i_0} = \lfloor \tfrac{1}{4} |v_{i_0} - v_{j_0}|_\infty \rfloor$$

and that this implies $n_{j_0} \leqq n_{i_0}$, hence $n_{j_0} = n_{i_0}$ (since n_{i_0} is minimal) and $j_0 \in I_1$. Note also that if we order J_1 in such a way that all indices in I_l precede all indices in I_k if $l < k$ (but the order within one I_l arbitrary) then for any $i \in J_1 \setminus \{i_0\}$

$$n_i = \min\{\lfloor \tfrac{1}{4} |v_i - v_j|_\infty \rfloor : j \text{ precedes } i \text{ in } J_1\}. \tag{49}$$

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[5] Dr. Bao G. Nguyen has shown me that the upper bound for $E_v\{Z^t(n)\}$ for $t>1$ can be obtained much simpler by induction on t

Thus, the minimum in (44) is taken on for some j in J_1 and even an earlier j. Of course (48) also holds. In addition for $l \notin J_1$, $n_l(\bar{v}) = n_l(v_i : i_l \notin J_1)$, i.e.,

$$n_l(\bar{v}) = \min\{\lfloor \tfrac{1}{4} |v_l - v_j|_\infty \rfloor : j \neq l, j \in \{0, \dots, t\} \setminus J_1\}. \tag{50}$$

Indeed for $l \notin J_1$ the minimum in (44) cannot be taken on at some $j \in J_1$ or l itself would also belong to J_1. We may thus replace $\{0, \dots, t\}$ by $\{0, \dots, t\} \setminus J_1$ and (if this set is not empty) find an ordered set J_2 of indices in $\{0, \dots, t\} \setminus J_1$ with a first index k_0 such that

$$n_{k_0} = \min\{n_j : n \notin J_1\}$$

(this is the analogue of (48)) and such that for $i \in J_2 \setminus \{k_0\}$ (49) holds with J_1 replaced by J_2, and for $l \notin J_1 \cup J_2$ (50) holds when J_1 is replaced by $J_1 \cup J_2$. If $\{0, \dots, t\} \setminus J_1 \cup J_2$ is still not empty we proceed in the same manner, until $\{0, \dots, t\}$ has been partitioned into a number of ordered sets $J_1, \dots, J_\lambda$ with the above properties. Note that (50) and its analogues imply that each J_i has at least two elements. To each $\bar{v}$ there corresponds such a selection of $J_1, \dots, J_\lambda$ (with varying λ) and (46) may be bounded by

$$\sum_{J_1, \dots, J_\lambda} \Big(\sum^{J_1} \prod_{l \in J_1}^{*} \pi(m_l^1)\Big) \dots \Big(\sum^{J_\lambda} \prod_{l \in J_\lambda}^{*} \pi(m_l^\lambda)\Big). \tag{51}$$

Here the outer sum stands for the sum over all choices of the J's and if $J_\tau = \{l_0, \dots, l_{r-1}\}$ with $r = |J_\tau|$, the cardinality of J_τ, then $\sum^{J_\tau}$, $\prod^*$ and m_l^τ stand for the following:

$$m_l^\tau = \min\{\lfloor \tfrac{1}{4} |v_l - v_j|_\infty \rfloor : j \neq l, j \in J_\tau\},$$

$\sum^{J_\tau}$ is the sum over all $v_{l_0}, \dots, v_{l_{r-1}} \in S(n)$ for which

$$m_{l_0}^\tau = \min\{m_l^\tau : l \in J\}, \qquad \text{and for } l \in J_\tau \setminus \{i_0\}$$
$$m_l^\tau = \min\{\lfloor \tfrac{1}{4} |v_l - v_j|_\infty \rfloor : j \text{ precedes } l \text{ in } J_\tau\};$$

finally $\prod^*$ stands for the product over all $l \in J_\tau \setminus \{0\}$ (the factor $\pi(m_0^\tau)$ is excluded because (46) does not contain a factor $\pi(n_0)$). As above we must have $m_{l_1}^\tau = m_{l_0}^\tau$.

We now change our point of view. Instead of fixing $\bar{v}$ and finding the J's we now estimate (51) by fixing the J's and carrying out the sums over the $\bar{v}$'s which yield these J's. We shall prove

$$\sum^J \prod_{l \in J}^{*} \pi(m_l) \leqq C_1 (n^2 \pi_n)^{|J \setminus \{0\}|}. \tag{52}$$

Since there are only $C_2(t)$ ways of choosing the J's, substitution of (52) into (51) will yield the bound (47) for (46).

To prove (52) fix $J = \{l_0, l_1, \dots, l_{r-1}\}$ and for the moment also fix $m(l_0), \dots, m(l_{r-1})$. We wish to estimate the number of choices for $v_{l_0}, \dots, v_{l_{r-1}}$ which are consistent with these data. First we consider the case where $0 \notin J$. v_{l_0} can be chosen as any vertex in $S(n)$, i.e., in at most $C_3 n^2$ ways. Then, for any $l_k \in J \setminus \{0\}$ there must be a j preceding l_k for which $m_{l_k} = \lfloor \frac{1}{4} |v_{l_k} - v_j|_\infty \rfloor$. If $v_{l_0}, \dots, v_{l_{k-1}}$ have been picked already then there are at most $k \leqq t+1$ choices for this j, and if v_j is fixed, and v_{l_k} has to satisfy $\lfloor \frac{1}{4} |v_{l_k} - v_j|_\infty \rfloor = m_{l_k}$, then there are at most $C_4 m_{l_k}$ choices for v_{l_k}. In total we have at most

$$C_5 n^2 \prod_{l \in J \setminus \{l_0\}} m_l$$

choices for the v's corresponding to J. Next recall that we also have the restriction $m_{l_0} = m_{l_1}$. If we now carry out the sum over the m_l with this restriction then we see that the left hand side of (52) is at most

$$(53) \qquad C_5 n^2 \sum_{0 \leqq m(l_1), \dots, m(l_{r-1}) \leqq n} m_{l_1} \pi^2(m_{l_1}) m_{l_2} \pi(m_{l_2}) \dots \pi_{l_{r-1}} m(l_{r-1}).$$

Note that by (7)

$$k \pi_k \leqq \sum_{j=1}^{k} \pi_j \leqq \sum_{j=1}^{n} \pi_j \leqq C_4 n \pi_n, \qquad k \leqq n,$$

so that

$$\sum_{0 \leqq m \leqq n} m \pi^2(m) \leqq C_4 n \pi_n \sum_{m=0}^{n} \pi(m) \leqq C_4^2 (n \pi_n)^2.$$

Thus (53) is at most

$$C_6 (n^2 \pi_n)^r,$$

which establishes (52) if $\{0\} \notin J$. A similar argument applies if $0 \in J$. Of course $v_0 = w_0$ is fixed, so that if $0 = l_k$, then we don't get a factor $m(l_k)$ for the number of choices of v_0 (or if $l_0 = 0$, then we don't get the initial factor n^2). However, we don't get a factor $\pi(m_{l_k})$ either in $\prod^*$. It is now easy to verify that (52) again holds in this case. This completes the proof of (52) and of (47) as an upper bound for (46). The stated behavior for the moments of $Z(n)$ has therefore been proved.

We turn to the final statement of Theorem 8 about the distribution of $Z(n)$. By Markov's inequality

$$\nu\{Z(n) \geqq \varepsilon^{-1} n^2 \pi_n\} \leqq \varepsilon \frac{E_\nu\{Z_n\}}{n^2 \pi_n},$$

so that we only have to estimate (for suitable $C_1 > 0$)

$$(54) \qquad \nu\{Z(n) \leqq \varepsilon C_1 n^2 \pi_n\}.$$

To do this consider a triple of annuli $B'(m) := S(3m) \setminus S(m)$, $B(m) := S(9m) \setminus S(3m)$, and $B''(m) = S(27m) \setminus S(9m)$. Assume that there exist occupied circuits $\mathscr{C}'$ in $B'(m)$ and $\mathscr{C}''$ in $B''(m)$. A.e. $[\nu]\, \mathscr{C}'$ and $\mathscr{C}''$ belong to $\tilde{W}$ (see Fig. 7). If $v \in B(m)$ and $v \rightsquigarrow S(m) \cup S^c(27m)$ then some occupied path from v to $S(m)$ or $S^c(27m)$ intersects $\mathscr{C}'$ or $\mathscr{C}''$, and therefore belongs to $\tilde{W}$. Thus, if we define

$$\hat{B}(m) = S(27m + \Lambda) \setminus \mathring{S}(m - \Lambda)$$

and

$$Y(m) = \#\{v \in B(m) : v \rightsquigarrow S(m) \cup S^c(27m) \text{ in } \hat{B}(m)\},$$

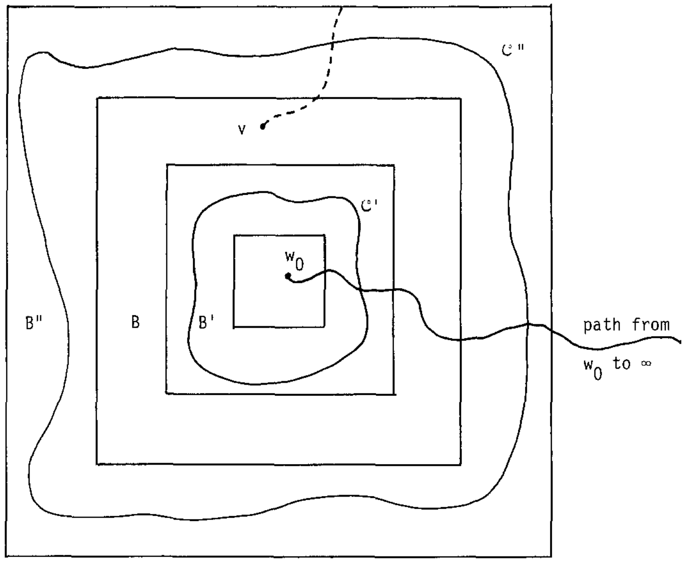

Fig. 7. The four *squares* are (starting from the inside) $S(m)$, $S(3m)$, $S(9m)$ and $S(27m)$

then a.e. $[v]$ on the event

$$F(m) := \{\exists \text{ occupied circuits } \mathscr{C}' \text{ and } \mathscr{C}'' \text{ in } B'(m) \text{ and } B''(m) \text{ respectively}\}$$

we have $Z(n) \geqq Y(m)$ for $n \geqq 27m$. It is also easy to see (use (6)) that

$$E_{cr}\{Y(m)\} \geqq C_1 m^2 \pi_m. \tag{55}$$

Thus if

$$3^k \leqq n < 3^{k+1} \quad \text{and} \quad \varepsilon \leqq \tfrac{1}{2} 3^{-8j-2}, \tag{56}$$

then for $3^{k-4j} \leqq m \leqq 3^{k-3}$

$$\begin{aligned} \tfrac{1}{2} E_{cr}\{Y(m)\} &\geqq \tfrac{1}{2} C_1 3^{2k-8j} \pi_m \\ &\geqq \tfrac{1}{2} C_1 n^2 \pi_n 3^{-8j-2} \geqq \varepsilon C_1 n^2 \pi_n. \end{aligned}$$

In particular for these choices of k, j, ε we see that

$$\begin{aligned} v\{Z(n) \geqq \varepsilon C_1 n^2 \pi_n\} \geqq v\{&Y(3^{k-4l}) \geqq \tfrac{1}{2} E_{cr}\{Y(3^{k-4l})\} \\ &\text{and } F(3^{k-4l}) \text{ occurs for some } 1 \leqq l \leqq j\}. \end{aligned} \tag{57}$$

Moreover, the event in the right hand side of (57) is increasing, so that an application of the Harris-FKG inequality shows that the right hand side of (57) is at least equal to

$$P_{cr}\{Y(3^{k-4l}) \geqq \tfrac{1}{2} E_{cr}\{Y(3^{k-4l})\} \quad \text{and} \quad F(3^{k-4l}) \text{ occurs for some } 1 \leqq l \leqq j\}.$$

The event

$$\{Y(3^{k-4l}) \geqq \tfrac{1}{2} E_{cr}\{Y(3^{k-4l})\} \text{ and } F(3^{k-4l})\}$$

depends only on vertices in $S(3^{k-4l+3}+\Lambda) \setminus \mathring{S}(3^{k-4l}-\Lambda)$, and therefore these events for different l are independent (when $k-l$ is large). Also

$$\{Y(3^{k-4l}) \geqq \tfrac{1}{2} E_{cr}\{Y(3^{k-4l})\} \quad \text{and} \quad F(3^{k-4l})$$

are both increasing events. These observations and another application of the Harris-FKG inequality show that the expression in (54) is at most

$$\prod_{l=1}^{j} [1 - P_{cr}\{Y(3^{k-4l}) \geqq \tfrac{1}{2} E_{cr}\{Y(3^{k-4l})\}\} P_{cr}\{F(3^{k-4l})\}].$$

Finally, by (28)

$$P_{cr}\{F(3^{k-4l})\} \geqq C_2 > 0,$$

and for small ε we can take j large (see (56)). Therefore, it will follow that (54) is small, uniformly in n, when ε is small, as soon as we show

$$P_{cr}\{Y(m) \geqq \tfrac{1}{2} E_{cr}\{Y(m)\}\} \geqq C_3 > 0 \quad \text{for all } m. \tag{58}$$

The one-sided analogue of Chebyshev's inequality ([4], p. 476) shows that the left hand side of (58) is at least

$$\frac{\frac{1}{4}[E_{cr}\{Y(m)\}]^2}{\frac{1}{4}[E_{cr}\{Y(m)\}]^2 + \operatorname{var}_{cr}\{Y(m)\}},$$

where $\operatorname{var}_{cr}\{Y\}$ is the variance of Y under P_{cr}. The proof of (54) has therefore been reduced to the estimate

$$E_{cr}\{Y^2(m)\} \leqq C_4 (m^2 \pi_m)^2 \tag{59}$$

(see (55)). We do not prove (59) except to remark that the same argument as used to go from (43) to (46) shows that[6]

$$E_{cr}\{Y^2(m)\} \leqq \sum_{v, w \in B(m)} \{\pi(\lfloor \tfrac{1}{4} |v-w|_\infty \rfloor \wedge m)\}^2,$$

and the last sum is indeed $O(m^2 \pi_m)^2$ by (52) applied to a J consisting of two indices only. □

The proof of Theorem 14 will not be spelled out. It is essentially the same as that of the first part of Theorem 8. We merely have to replace at various places the events $\{v \rightsquigarrow S^c(v,n)\}$ by $\{v$ is connected to $S^c(v,n)$ by two occupied paths, which only have the vertex v in common$\}$, and correspondingly π_n by ρ_n. (See also Remark (37).)

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[6] $a \wedge b = \min(a,b)$

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