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**Probability Theory** and Related Fields



# On the Itô Excursion Process

Thomas S. Salisbury

Department of Mathematics, York University, North York, Ontario M 3J 1 P3, Canada

**Summary.** Necessary and sufficient conditions are given, for a process to be the excursion process of some strong Markov process. These are modifications of necessary conditions of Itô, which are here shown by example not to be sufficient.

## 1. Introduction

In his classic paper "Poisson Point Processes attached to Markov Processes", K. Itô constructs the excursion point process of a given standard process at a recurrent point. He shows that the characteristic measure of the excursion process satisfies certain conditions, and then indicates how to reverse his construction provided those conditions hold. That is, given a PPP whose characteristic measure obeys these conditions, it should be possible to construct a process whose excursion process is that PPP. Itô doesn't state this converse result as a formal theorem, and hence doesn't write out a proof. This paper will attempt to make his argument rigorous.

This is perhaps more interesting in that the converse is not true as stated above. In Sect. 4, examples will be given showing that unless we strengthen two of his conditions, the process constructed from the PPP may fail to be strong Markov or right continuous. This strengthening gives us necessary and sufficient conditions for the process obtained to be a right process.

The argument for sufficiency is presented in Sect. 3, and that for necessity in Sect. 5. The following paper [18] will show how some of these results may be simplified and extended, provided the original process is a Ray process or a right process. Section 6 is an Appendix, giving some other results that may be obtained from the proof of the main lemma (Lemma 7).

## 2. Notation and results

$E$ will be a separable metric space, and $\mathscr{E}^0$ the $\sigma$-field of its Borel subsets. $\mathscr{E}$ will be the universal completion of $\mathscr{E}^0$. $U$ will be the set of right-continuous $E$-

valued paths, and, for $a \in E$, $U^a$ will be the set of all such paths $u$, satisfying $u(0) = a$. $(W_t)_{t \geqq 0}$ will be the coordinate process on $U$; $W_t(u) = u(t)$. For $t \geqq 0$, $\mathscr{U}_t^0$ will be the smallest $\sigma$-field such that for every $s \in [0, t]$, $W_s$ is a measurable function from $(U, \mathscr{U}_t^0)$ to $(E, \mathscr{E}^0)$. $\mathscr{U}_t$ will be the universal completion of $\mathscr{U}_t^0$, and $\mathscr{U}$ that of $\mathscr{U}_\infty^0$.

$(\Pi, \mathscr{P})$ will be the measurable space of $(U, \mathscr{U})$-valued point functions. That is, we adjoin a point $\delta$ to $U$, and let $\Pi$ be the set of functions $p: [0, \infty) \to U \cup \{\delta\}$ such that $p(t) = \delta$ except for countably many $t$. $\mathscr{P}$ is the $\sigma$-field on $\Pi$ generated by the functions $p \mapsto N(A, p)$, where $N(A, p)$ is the number of times $t$ such that $(t, p(t)) \in A \subset [0, \infty) \times U$. Here $A$ belongs to the product $\mathscr{B} \otimes \mathscr{U}$ of the Borel field $\mathscr{B}$ on $[0, \infty)$, and of $\mathscr{U}$. $(y_t)_{t \geqq 0}$ will be the coordinate process on $\Pi$; $y_t(p) = p(t)$.

For $A \in \mathscr{B} \otimes \mathscr{U}$, we define the restriction of $p \in \Pi$ to $A$ to be:

$$p|_A(t) = \begin{cases} p(t), & \text{if } (t, p(t)) \in A \\ \delta, & \text{otherwise.} \end{cases}$$

Special cases of this will be the killing operators:

$$\alpha_s(p) = p|_{[0, s] \times U},$$
$$k_s(p) = p|_{[0, s) \times U}.$$

For $t \geqq 0$, $\mathscr{P}_t$ will be the sub $\sigma$-field $\alpha_t^{-1}(\mathscr{P})$, of $\mathscr{P}$. We define the shift operators of $\Pi$ to be:

$$\Theta_s(p)(t) = p(t+s), \quad \text{for } t \geqq 0,$$
$$\Theta_s^0(p) = \Theta_s(p)|_{(0, \infty) \times U}.$$

The same notation will be used for the corresponding shift operators on $U$.

For $u \in U$ we define the hitting time and début of $\{a\} \subset E$ to be

$$\sigma_a(u) = \inf\{t > 0; u(t) = a\},$$
$$\tau_a(u) = \inf\{t \geqq 0; u(t) = a\}.$$

It is well known that both $\sigma_a$ and $\tau_a$ are $(\mathscr{U}_{t+})$ stopping times.

A *Poisson point process* on a probability space $(\Omega, \mathscr{F}, P)$, with values in $(U, \mathscr{U})$ is a measurable function $Y: (\Omega, \mathscr{F}) \to (\Pi, \mathscr{P})$, together with a filtration $(\mathscr{F}_t)_{t \geqq 0}$ of $(\Omega, \mathscr{F})$ $(\mathscr{F}_t \subset \mathscr{F}_s \subset \mathscr{F}$ for $t \leqq s)$, such that:

(a) $\alpha_t(Y) \in \mathscr{F}_t$ for $t \geqq 0$.

(b) $\Theta_t^0(Y)$ is independent of $\mathscr{F}_t$ and has the same law as $Y|_{(0, \infty) \times U}$ for $t \geqq 0$.

(c) There exist sets $A_k \in \mathscr{U}$, $A_k \uparrow U$ such that $N([0, t] \times A_k, Y) < \infty$ a.s. for each $k, t$.

The special role of time $t = 0$ in (b) will be useful when we consider the point process of excursions away from a point $a \in E$, of a Markov process taking values in $E$. Time $t = 0$ corresponds to the first excursion, which is exceptional in that we will want to allow it to start in any initial distribution. In contrast, the other excursions will start in a manner dictated by the transition probabilities of the given Markov process.

Under conditions (a), (b), (c) above, there exists a $\sigma$-finite measure $n$ on $U$ (the *characteristic measure* of $Y$) such that

$$E[N((0,t)\times A, Y)] = t\cdot n(A) \quad \text{for } A\in\mathscr{U},\ t\geqq 0.$$

This measure determines the law of $Y|_{(0,\infty)\times U}$. In fact, if $(s_i,t_i)\times A_i$ are disjoint and $0\leqq s_i\leqq t_i$, for $i=1,\dots,k$, then the $N((s_i,t_i)\times A_i, Y)$ are independent Poisson random variables, of means $(t_i-s_i)\cdot n(A_i)$.

In the theory of right processes, it is customary to equip a right continuous, $E$-valued process $X$ based on $(\Omega,\mathscr{F})$, with laws $\hat{P}^b$ on $(\Omega,\mathscr{F})$, one for each $b\in E$. We will reserve the notation $P^b$ for laws on the canonical space $(U,\mathscr{U})$.

In general, on a fixed measurable space $(\Omega,\mathscr{F})$, let $(X_t)_{t\geqq 0}$ be a right continuous process, based on $\Omega$, with values in $E$. We will say that $(X_t,\mathscr{F}_t,\mu,P^b)$ has the *strong Markov property at* $T$ if the following situation holds; $(\mathscr{F}_t)_{t\geqq 0}$ is a filtration of $(\Omega,\mathscr{F})$, $X^{-1}(\mathscr{U}_t)\subset\mathscr{F}_t$, $\mu$ is a $\sigma$-finite measure on $(\Omega,\mathscr{F})$, $(P^b)_{b\in E}$ is a family of probability measures on $(U,\mathscr{U})$ such that $b\mapsto P^b(A)$ is $\mathscr{E}$-measurable for each $A\in\mathscr{U}$, $T$ is a stopping time for the filtration $(\mathscr{F}_{t+})$, and

$$\mu(X_{T+\cdot}\in A, T<\infty, B) = \int_{B\cap\{T<\infty\}} P^{X_T}(A)\,d\mu \quad \text{for } A\in\mathscr{U},\ B\in\mathscr{F}_{T+}.$$

Except in Lemma 6 below, we will always take $\mu$ to be a probability measure.

For $(X_t)$ right continuous with values in $E$, we will say that $(X_t,\mathscr{F}_t,\mu,P^b)$ is *strong Markov* if $(X_t,\mathscr{F}_t,\mu,P^b)$ has the strong Markov property at each $(\mathscr{F}_{t+})$ stopping time. For $a\in E$, $(L_t)_{t\geqq 0}$ is a *local time* at $a$, if $L$ is continuous, nondecreasing, and adapted to $(\mathscr{F}_t)$, with set of increase exactly $\{t;\,X_t=a\}$, such that for every $(\mathscr{F}_{t+})$ stopping time $T$ with $X_T=a$, $(X_{\cdot+T}, L_{\cdot+T}-L_T)$ is independent of $\mathscr{F}_{t+}$, with the same law as $(X_{\cdot+\sigma_a(X)}, L_{\cdot+\sigma_a(X)})$.

Itô performed the following construction. (Actually, he considered only the case of a standard process, but as pointed out to me by J. Pitman, his arguments apply without change. Henceforth similar qualifications will be omitted.) Let $P$ be a probability measure on $(\Omega,\mathscr{F})$ under which $\mathscr{F}$ is complete, and suppose the following conditions hold:

(2.1) $(X_t)$ is right continuous with values in $E$, $(X_t,\mathscr{F}_t,P,P^b)$ is strong Markov, and each $\mathscr{F}_t$ contains all the $P$-null sets of $\mathscr{F}$.

(2.2) $X$ is recurrent at a point $a\in E$. ($P^b(\sigma_a<\infty)=1$ for $b\in E$).

(2.3) If $P^a(\sigma_a=0)=1$, then there is a local time $(L_t)$ for $X$ at $a$, which is canonical in the sense that it is normalized to make

$$E[e^{-\sigma_a(X)}] = E\left[\int_0^\infty e^{-t}\,dL_t\right].$$

(Condition (2.3) holds if $X$ is a right process).

If $P^a(\sigma_a=0)=1$, let $S(s)$ be the right continuous inverse local time: $S(s)=\inf\{t\geqq 0;\,L_t>s\}$, $S(0-)=0$. Let

$$Y_s(t)=\begin{cases} X(S(s-)+t), & \text{if } 0\leqq t<S(s)-S(s-)\\ a, & \text{if } t\geqq S(s)-S(s-)>0\end{cases}$$

$$Y_s=\delta, \quad \text{if } S(s)-S(s-)=0.$$

Then Itô shows that $Y$ is a $(U, \mathscr{U})$-valued PPP with respect to $P$ and the filtration of $Y$.

If, on the contrary, $P^a(\sigma_a=0)=0$, then $X$ visits $a$ at a discrete set of times. In this case, let $S(k)$ be the $k$th hitting time of $a$; $S(0)=0$, $S(k+1)=\inf\{t>S(k); X_t=a\}$. Let

$$Y_k(t)=\begin{cases} X(S(k)+t), & \text{if } 0\leqq t<S(k+1)-S(k) \\ a, & \text{if } t\geqq S(k+1)-S(k). \end{cases}$$

Then Itô shows that under $P$, the $Y_k$, $k\geqq 1$ are *IID*, $(U, \mathscr{U})$-valued, $\mathscr{F}$-measurable random variables.

Let $n$ be the characteristic measure of $Y$ in the first case, and the common distribution of the $Y_k$, $k\geqq 1$ in the second. We call $n$ the *excursion measure* of $X$ from $a$.

Itô is concerned with classifying all processes that agree up till the début of a point $a\in E$. Specifically, suppose that

(2.4) $(P_0^b)$ is a family of probability measures on $(U, \mathscr{U})$ such that for each $c\in E$, the coordinate process $(W_t, \mathscr{U}_t, P_0^c, P_0^b)$ is strong Markov. (Note that here, $c$ is fixed, and $b$ ranges over $E$.)

(2.5) $P_0^b\{u; \tau_a(u)<\infty$, and $u(t)=a$ for $t\geqq\tau_a(u)\}=1$ for each $b\in E$.

The problem is to classify all families $(P^b)$ of probability measures on $(U, \mathscr{U})$ for which there exists $(X_t, \mathscr{F}_t, P, P^b)$ on some probability space, which is a *recurrent extension* of $(P_0^b)$ in the sense that (2.1), (2.2), (2.3) hold, and

(2.6) $$P^b\{u; u(\cdot\wedge\tau_a(u))\in A\}=P_0^b(A) \quad \text{for each } b\in E,\ A\in\mathscr{U}.$$

Itô achieves this classification in terms of the excursion measure $n$ of $X$ from $a$. He shows that the $P_0^b$ and $n$ determine the $P^b$, and then derives the following list of conditions that $n$ obeys

**Theorem 1** (Itô). *Let* $(X_t, \mathscr{F}_t, P, P^b)$ *satisfy* (2.1), (2.2), (2.3). *Let $n$ be the excursion measure of $X$ from $a$, and define $P_0^b(A)$ to be $P^b\{u; u(\cdot\wedge\tau_a(u))\in A\}$. Then the following conditions are satisfied:*

(i) *$n$ is concentrated on* $\{u; 0<\sigma_a(u)<\infty,\ u(t)=a \text{ for } t\geqq\sigma_a(u)\}$.

(ii) $n\{u; u(0)\notin V\}<\infty$ *for every open neighborhood $V$ of $a$.*

(iii) $\int(1-e^{-\sigma_a})\,dn\leqq 1$.

(iv) $n\{u; \sigma_a(u)>t,\ u\in\Lambda,\ \Theta_t(u)\in M\}=\int_{\Lambda\cap\{\sigma_a>t\}} P_0^{u(t)}(M)\,n(du)$ *for* $t>0$, $\Lambda\in\mathscr{U}_t$, $M\in\mathscr{U}$.

(v) $n\{u; u(0)\in B,\ u\in M\}=\int_{\{u: u(0)\in B\}} P_0^{u(0)}(M)\,n(du)$ *for* $M\in\mathscr{U}$, *and* $B\in\mathscr{E}$ *such that* $a\notin B$.

(vi) *Either* (a) *$n$ is a probability measure concentrated on* $U^a=\{u; u(0)=a\}$ *(discrete visiting case)*;

or (b) *$n$ is finite,* $n(U^a)=0$, *and* $\int(1-\exp(-\sigma_a))\,dn<1$ *(exponential holding case)*;

or (c) *$n$ is infinite and* $n(U^a)=0$ *or* $\infty$ *(instantaneous case).*

The main result of this paper is that if conditions (ii) and (vi) are strengthened, we obtain conditions that are necessary and sufficient for a $\sigma$-finite positive measure $n$ to arise as the excursion measure of a recurrent extension of a family $(P_0^b)$ satisfying (2.4) and (2.5).

The conditions are:

(ii′) $n\{u; u \text{ leaves } V\} < \infty$ for every open neighbourhood $V$ of $a$.

(vi′) Either (a) $n$ is a probability measure concentrated on $U^a$. If $n \geqq n' \geqq 0$ and $n'$ satisfies (iv), then $n'$ is a multiple of $n$;
or (b) as in (vi)(b);
or (c) $n$ is infinite. If $n \geqq n' \geqq 0$ and $n'$ satisfies (iv), then $n'(U^a) = 0$ or $\infty$.

The statement of necessity is:

**Proposition 1.** *Under the conditions of Theorem* 1, *condition* (ii′) *and* (vi′) *also hold.*

A strong form of sufficiency is:

**Theorem 2.** *Assume that* $(\Omega, \mathscr{F}, P)$ *is complete, and that* $(P_0^b)$ *satisfies* (2.4) *and* (2.5).

(a) *If* $(Y_t, \mathscr{F}_t)$ *is a PPP with values in* $(U, \mathscr{U})$ *and with characteristic measure* $n$, *such that:*

$(\mathscr{F}_t)$ *is right continuous, and each* $\mathscr{F}_t$ *contains all the P-null sets of* $\mathscr{F}$;

$$P(Y_0 \in M) = \int P_0^{Y_0(0)}(M)\, dP \qquad \text{for } M \in \mathscr{U};$$

(i), (ii′), (iii), (iv), (v) *and either* (b) *or* (c) *of* (vi′) *hold.*

*Then there is a right continuous strong Markov process* $(X_t, \mathscr{G}_t, P, P^b)$ *such that:*

*Y is the PPP constructed from X as above, P-a.s.;*

$(X_t, \mathscr{G}_t, P, P^b)$ *is a recurrent extension of* $(P_0^b)$;

$\mathscr{G}_{S(t+)} = \mathscr{F}_t$;

$(\mathscr{G}_t)$ *is right continuous, and each* $\mathscr{G}_t$ *contains all the P-null sets of* $\mathscr{F}$.

(b) *If* $(\mathscr{F}_k)_{k \geqq 0}$ *is an increasing family of sub σ-fields of* $\mathscr{F}$, *each containing all the P-null sets of* $\mathscr{F}$, *and for each* $k \geqq 0$, $Y_k$ *is a measurable function from* $(\Omega, \mathscr{F}_k)$ *to* $(U, \mathscr{U})$ *such that:*

*For* $k \geqq 1$ *the* $Y_k$ *have a common distribution* $n$;

$$P(Y_0 \in M) = \int_{U \setminus U^a} P_0^{u(0)}(M)\, P(Y_0 \in du) + n(M)\, P(Y_0 \in U^a) \qquad \text{for } M \in \mathscr{U};$$

$\sigma(Y_i; i > k)$ *is independent of* $\mathscr{F}_k$ *for* $k \geqq 0$;

(i), (ii′), (iv) *and* (a) *of* (vi′) *hold.*

*Then there is a strong Markov right continuous process* $(X_t, \mathscr{G}_t, P, P^b)$ *such that:*

*The* $Y_k$ *are the excursion random variables constructed from X as above (discrete visiting case), P-a.s.;*

$(X_t, \mathscr{G}_t, P, P^b)$ *is a recurrent extension of* $(P_0^b)$;

$\mathscr{G}_{S(k+1)-} \subset \mathscr{F}_k \subset \mathscr{G}_{S(k+1)}$, *for* $k \geqq 0$;

$(\mathscr{G}_t)$ *is right continuous, and each* $\mathscr{G}_t$ *contains all the P-null sets of* $\mathscr{F}$.

This result can be used to give a rigorous construction of processes such as skew Brownian motion (see [18]).

The key to the proof of Theorem 2 is to find an expression for conditioning $Y_T$ on the strict past $\mathscr{F}_{T-}$, for $T$ an $(\mathscr{F}_t)$ stopping time. This is done in Lemma 7; the relationship with similar results of M. Weil is spelled out in Sect. 6.

In the following paper [18], we will see how the Ray property or the 'hypothéses droites' may be obtained for $(P^b)$, assuming they hold for $(P_0^b)$. In fact, under these additional conditions the proofs of Theorem 2 and Proposition 1 may be shortened. Since under mild conditions ($E$ locally compact, Borel measurable semigroup) every right continuous strong Markov process is a 'right process' (see Getoor [6]), this streamlined proof may meet the needs of nonspecialists. It will be given in [18]. Nonetheless, the present approach has its own merits. First of all, it is elementary, requiring none of the analytical apparatus of resolvents, or the deep results of the theory of right processes (for example, it applies in a general separable metric space, without assuming completeness or Borel embedding in a compact space). Secondly, it is the approach that generalizes to considerations of excursions away from a set (rather than a single point).

This generalization will be carried out in [19]; though the same 'elementary' techniques are used, the powerful machinery comes into play as well (Maisonneuve's 'Exit systems'; [12]). Needless to say, in the present situation, the answers we obtain are simpler and more complete.

As to related work, the idea of constructing processes via excursions has a long history, in the Markov chain setting (e.g., Lamb [11]). (See Rogers [16] for other references), for diffusion (Motoo [14], Sato-Ueno [20]) and for symmetric processes (Fukushima [5], Silverstein [21], [22]). Closer to the present work are Blumenthal [1], and results of S. Watanabe [24] (see also [8]).

It has been pointed out to me that, in addition to the above, parts of the present work have appeared elsewhere; Rogers [16] obtains results similar to some of those of Salisbury [18]; see the latter for a comparison.

In his thesis [10], S. Kabbaj obtained a result similar to Theorem 2. He shows that under Itô's conditions, and with $(\mathscr{G}_t^0)$ the minimal filtration with respect to which the reconstructed process $(X_t)$ is adapted, $(X_t)$ is strong Markov at all $(\mathscr{G}_t^0)$ stopping times. His proof uses a weaker form of Lemma 7 and relies heavily on the theory of right processes. The proof presented here works for $T$ a $(\mathscr{G}_{t+}^0)$ stopping time, is more elementary, and applies in greater generality (we assume no compactness conditions on $E$).

Finally, when we are given a right process $(X_t)$, and let $(Y_t)$ be its excursion process, Lemma 7 is still of interest. In this context, it is closely related to Getoor and Sharpe's last exit decompositions, and a very similar result has appeared in Getoor and Sharpe [7] (see also Pitman [15]). In fact, their methods will be used in Salisbury [19] to give a different proof of a generalization of Lemma 7.

This work forms part of the author's Ph.D. thesis, Salisbury [17]. I am grateful to John Walsh for posing me the problem in the first place, and for his support and helpful comments throughout. Also I would like to thank Yves LeJan for bringing [10] to my attention.

## 3. Proof of Theorem 2

The arguments used in parts (a) and (b) are similar. The author feels that it is important to give the more complicated arguments of part (a) in detail. Thus those of part (b) are only sketched, with a description of the modifications needed to obtain them from those of part (a). Thus we deal with part (a) first. On a first reading however, the reader is urged to reverse this order, first absorbing the simpler, less detailed version. The reader may even wish to first look at Sect. 6, in which the arguments of the key lemma are presented in an even simpler form (Proposition 3, or the even shorter discrete time version; Proposition 4).

In part (a), conditions (i) and (iii) are used in the construction; the latter so that the normalization of local time agrees with (2.3). Condition (ii′) appears in Lemma 3, in the proof of the right continuity of paths. Condition (vi′)(b) also appears in this lemma, and is used to make the "inverse local time" strictly increasing, so that local time will be continuous. Conditions (iv) and (v) are put in a more convenient form in Lemma 6, which, together with Lemma 7, yields Corollary 2. Lemma 7 is also used with conditions (vi′)(b) and (vi′)(c) to give Corollary 1. Note that these two corollaries essentially show the strong Markov property. In part (b), the conditions are put to the same uses, except that we use condition (vi′)(a) instead of conditions (vi′)(b) and (vi′)(c).

We start the proof of part (a), by constructing $X$ as an explicit measurable function of $Y$.

Put

$$m=1-\int(1-e^{-\sigma_a})\,dn,$$

$$S^-(s,p)=ms+\sum_{r<s}\sigma_a(p(r)), \qquad \text{for } s\geqq 0,\ p\in\Pi$$

(with the convention that $\sigma_a(\delta)=0$), and

$$S^+(s,p)=\lim_{r\downarrow s}S^-(r,p).$$

Then $S^-(\cdot,p)$, $S^+(\cdot,p)$ are nondecreasing, and respectively left and right continuous, with values in $[0,\infty]$. If $S^+(s,p)<\infty$, then $S^+(s,p)=S^-(s,p)+\sigma_a(p(s))$. $S^-(s,\cdot)\in\mathscr{P}_{s-}$ since it is left continuous, and $S^-(s,p)=S^-(s,\alpha_s(p))$. Thus also $S^+(s,\cdot)\in\mathscr{P}_{s+}$.

Put

$$l_t(p)=\inf\{s\geqq 0;\ \infty>S^+(s)\geqq t\},$$

with the usual convention that $\inf(\emptyset)=+\infty$. Then $l_t$ is a $(\mathscr{P}_{s+})$ stopping time, and

$$S^-(l_t(\cdot),\cdot)\leqq t\leqq S^+(l_t(\cdot),\cdot)$$

(with the convention that $S^+(\infty,p)=\infty$). Put

$$x_t=\begin{cases} y_{l_t}(t-S^-(l_t,\cdot)), & \text{if } y_{l_t}\neq\delta\\ a, & \text{otherwise.}\end{cases}$$

We will show next that $x_t$ is measurable from $\mathscr{P}_{l_{t+}}$ to $\mathscr{E}^0$. Since $(u,r)\mapsto u(r)$ is measurable from $\mathscr{U}\otimes\mathscr{B}$ to $\mathscr{E}^0$, and $S^-(l_t(\cdot),\cdot)\in\mathscr{P}_{l_t}$ since $S^-$ is predictable, we need only show that $y_{l_t} \in \mathscr{P}_{l_t+}$. We state this in a more general form, to be useful later, as;

**Lemma 1.** *Let $(\mathscr{F}_t)$ be a filtration of a measurable space $(\Omega, \mathscr{F})$, and let $Y$ be a function which is measurable from $(\Omega, \mathscr{F}_t)$ to $(\Pi, \mathscr{P}_t)$, for every $t \geqq 0$. Let $R$ be an $(\mathscr{F}_t)$ stopping time such that for each $\varepsilon > 0$,*

$$\{R < \infty\} \subset \{\sigma_a(Y_t) > \varepsilon \text{ for only finitely many times } t \text{ in any compact time set}\}.$$

*Then $Y_R \in \mathscr{F}_R$.*

*Remark.* In the present situation, we apply the lemma with $Y$ the identity map, and with $\mathscr{F}_t = \mathscr{P}_{t+}$.

*Proof.* Let $A \in \mathscr{U}$, $s \geqq 0$. Then

$$\{Y_R \in A, R \leqq s\} = \{Y_R \in A, R \leqq s, \sigma_a(Y_R) = 0\} \cup \bigcup_{j \geqq 1} \left\{Y_R \in A, R \leqq s, \sigma_a(Y_R) \geqq \frac{1}{j}\right\}.$$

Let $B_i$, $i = 1, 2, \ldots$ be an open base in the space $[0, s]$. We can write

$$\{\sigma_a(Y_R) = 0, R \leqq s\} = \bigcap_{j \geqq 1} \bigcup_{i \geqq 1} \left[\{R \in B_i\} \cap \left\{N\left(B_i \times \left\{\sigma_a \geqq \frac{1}{j}\right\}, Y\right) = 0\right\}\right]$$

and

$$\left\{Y_R \in A, R \leqq s, \sigma_a(Y_R) \geqq \frac{1}{j}\right\}$$
$$= \bigcap_{i \geqq 1} \left[\{R \leqq s\} \cap \left(\{R \notin B_i\} \cup \left\{N\left(B_i \times \left[A \cap \left\{\sigma_a \geqq \frac{1}{j}\right\}\right], Y\right) \geqq 1\right\}\right)\right].$$

Thus $\{Y_R \in A, R \leqq s\} \in \mathscr{F}_s$, as required. □

Put $O = \{p; S^-(s, p) < \infty$ for each $s \geqq 0$,
$S^-(s, p) \to \infty$ as $s \to \infty$,
$S^-(\cdot, p)$ is strictly increasing, and for each open neighbourhood $V$ of $a$,
$p(s)$ leaves $V$ for only finitely many times $s$ in any compact set of times$\}$.

Thus on $O$, $l_t < \infty$ for every $t$, $t \mapsto l_t$ is continuous, and

$$x_t = \begin{cases} y_s(t - S^-(s, \cdot)) & \text{if } S^-(s, \cdot) \leqq t < S^+(s, \cdot) \\ a & \text{if } t = S^+(s, \cdot). \end{cases}$$

**Lemma 2.** *$t \mapsto x_t$ is right continuous on $O$.*

*Proof.* Fix some element $p$ of $O$. By definition of $U$, $x_{\cdot}(p)$ is right continuous on each interval $[S^-(s, p), S^+(s, p))$. If $t$ lies in no such interval, then it must equal $S^+(s, p)$ for some $s$, and hence $x_t(p) = a$. Let $V$ be an open neighbourhood of $a$. Since $p \in O$, there is an $s' > s$ such that $y_r(p)$ remains in $V$ for any $r \in (s, s')$. Thus $x_{r'}(p) \in V$ for any $r' \in (S^+(s, p), S^-(s', p))$. But by definition of $O$, $S^-(s', p)$ is strictly greater than $S^+(s, p)$, so that as $V$ was arbitrary, $x_.(p)$ must be right continuous at $t$. $\square$

(The same argument would show that if we had taken $U$ to be the space of càdlàg paths, then on $O$, $x_.$ would be càdlàg as well.)

For convenience, we separate out the contribution of $p(0)$; we have shown that there is a measurable function

$$F: (U \times \Pi, \mathscr{U} \otimes \mathscr{P}) \to (U, \mathscr{U})$$

such that $x_t = F(y_0, y|_{(0,\infty)})(t)$ on $O$.

Put

$$S^-(s) = S^-(s, Y)$$
$$S^+(s) = S^+(s, Y)$$
$$L_t = l_t(Y)$$
$$X_t = x_t(Y).$$

Since $Y$ is, by the hypotheses of Theorem 2, a measurable function from $(\Omega, \mathscr{F}_s)$ to $(\Pi, \mathscr{P}_{s+})$, for each $s$, the above results imply that $S^-$ and $S^+$ are adapted to $(\mathscr{F}_s)$, and that for each $t \geqq 0$, $L_t$ is an $(\mathscr{F}_s)$ stopping time and $X_t$ is measurable from $\mathscr{F}_{L_t}$ to $\mathscr{E}$ (as $\mathscr{F}_{L_t}$ is complete).

**Lemma 3.** $P(Y \in O) = 1$.

*Proof.* For $f \in \mathscr{B}$ with $f(0) = 0$,

$$\sum_{0<t<s} f(\sigma_a(Y_t)) = \int_{(0,\infty)} f(r)\, N((0, s) \times \{\sigma_a \in dr\}, Y)$$

is $\mathscr{F}$-measurable, and has expectation

$$s \int_{(0,\infty)} f(r)\, n(\sigma_a \in dr) = s \int f \circ \sigma_a\, dn.$$

In particular,

$$E\Big[\sum_{\substack{0<t<s \\ \sigma_a(Y_t) \leqq 1}} \sigma_a(Y_t)\Big] = s \int_{\{\sigma_a \leqq 1\}} \sigma_a\, dn$$
$$\leqq \frac{s}{1-e^{-1}} \int (1 - e^{-\sigma_a})\, dn$$
$$= \frac{s}{1-e^{-1}} (1-m) < \infty.$$

Also, there are only finitely many times $t \in [0, s]$ with $\sigma_a(Y_t) > 1$, as

$$n(\sigma_a > 1) \leqq \frac{1}{1-e^{-1}} \int (1 - e^{-\sigma_a})\, dn < \infty,$$

so that $S^-(s) < \infty$ a.s., for each $s$.

If $n(U) = 0$ (so that $a$ is a trap), then $m = 1$ so that $S^-(s) = s$ a.s. If $n(U) \neq 0$, then by (i) there is an $\varepsilon > 0$ such that $n(\sigma_a > \varepsilon) > 0$. Thus, there are a.s. infinitely many times $s$ such that $\sigma_a(Y_s) > \varepsilon$, so that in either case, $S^-(s) \to \infty$ as $s \to \infty$, a.s.

If $m>0$, it is clear that $S^-$ is strictly increasing. If $m=0$, then by (vi′)(b) it follows that $n(U)=\infty$, and hence that $\{s; \sigma_a(Y_s)>0\}$ is dense in $[0,\infty)$, a.s.

The last condition is obtained from (ii′), letting $V$ run through a countable base of open neighborhoods of $a$. □

Appealing to the completeness of $\mathscr{F}$, we will without loss of generality assume that $Y \in O$ surely, hence that $X$ is right continuous and by Lemma 1, that $Y_R \in \mathscr{F}_R$ for every $(\mathscr{F}_s)$ stopping time $R$.

If we are given a filtration $(\mathscr{V}_t)_{t\geqq 0}$, a $\sigma$-field $\mathscr{V}_{0-}\subset\mathscr{V}_0$, and a random variable $R$ with values in $[0,\infty]$, recall that $\mathscr{V}_{R-}$ is defined to be the filtration generated by $\mathscr{V}_{0-}$ and by sets of the form $A\cap\{R>s\}$, for $A\in\mathscr{F}_s$ and $s\geqq 0$. In our case, let $\mathscr{F}_{0-}$ be generated by all the $P$-null sets of $\mathscr{F}$. For $R$ a random time, and $r\geqq 0$, put

$$\mathscr{H}^R_{0-}=\mathscr{F}_{R-}$$
$$\mathscr{H}^R_r=\mathscr{F}_{R-}\vee\sigma(Y_R(s);0\leqq s\leqq r).$$

Put

$$\mathscr{G}^0_t=\mathscr{H}^{L_t}_{(t-S^-(L_t))-}$$
$$\mathscr{G}_t=\mathscr{G}^0_{t+}$$

$\mathscr{G}^0_{0-}=\mathscr{G}_{0-}=\mathscr{F}_{0-}$.

**Lemma 4.** (a) *$(\mathscr{G}_t)$ is right continuous, increasing, and each $\mathscr{G}_t$ contains all the $P$-null sets of $\mathscr{F}$. $X_t\in\mathscr{G}_t$ for each $t\geqq 0$.*

(b) *If $T$ is a $(\mathscr{G}_t)$ stopping time then $L_T$ is an $(\mathscr{F}_s)$ stopping time, and*

$$\mathscr{F}_{L_T-}\subset\mathscr{G}_{T-}\subset\mathscr{G}_T\subset\mathscr{F}_{L_T}.$$

(c) *$S^+(s)$ is a $(\mathscr{G}_t)$ stopping time, for $s\geqq 0$. If $T$ is any $(\mathscr{G}_t)$ stopping time such that $S^+(L_T)=T$ then $\mathscr{G}_T=\mathscr{F}_{L_T}$.*

(d) *If $R$ is an $(\mathscr{F}_s)$ stopping time, and $T$ is a $(\mathscr{G}_t)$ stopping time such that $T<S^+(L_T)$ on $\{T<\infty\}$, put*

$$V=\begin{cases}T-S^-(R), & \text{if } L_T=R<\infty\\ \infty, & \text{otherwise.}\end{cases}$$

*Then $V$ is an $(\mathscr{H}^R_{t+})$ stopping time.*

*Proof.* (a) We know that each $L_t$ is an $(\mathscr{F}_s)$ stopping time. Thus, if $t<t'$ then

$$\begin{aligned}\mathscr{G}^0_t\cap\{L_t<L_{t'}\}&\subset\mathscr{H}^{L_t}_\infty\cap\{L_t<L_{t'}\}\\&\subset\mathscr{F}_{L_t}\cap\{L_t<L_{t'}\}\quad(\text{as } Y_{L_t}\in\mathscr{F}_{L_t})\\&\subset\mathscr{F}_{L_{t'}-}\subset\mathscr{G}^0_{t'},\end{aligned}$$

and $\mathscr{G}^0_t\cap\{L_t=L_{t'}\}$ is, by monotone class arguments, and Proposition 18 of [3], generated by

$$\begin{aligned}\mathscr{H}^{L_t}_{0-}\cap\{L_t=L_{t'}\}&\subset\mathscr{H}^{L_{t'}}_{0-}\cap\{L_t=L_{t'}\}\\&\subset\mathscr{G}^0_{t'}\end{aligned}$$

and by

$$\mathscr{H}_r^{L_t}\cap\{t-S^-(L_t)>r\}\cap\{L_t=L_{t'}\}\subset\mathscr{H}_r^{L_{t'}}\cap\{t-S^-(L_{t'})>r\}\cap\{L_t=L_{t'}\}$$
$$\subset\mathscr{H}^{L_{t'}}_{(t-S^-(L_{t'}))-}\subset\mathscr{G}^0_{t'}.$$

Thus, $\mathscr{G}_t^0\subset\mathscr{G}_{t'}^0$.

For $s>t$ we can write $X_t$ as a measurable function of $L_s$, $k_{L_s}(Y)$, and $k_{s-S^-(L_s)}(Y_{L_s})$ (let

$$\hat{Y}_r=\begin{cases}k_{L_s}(Y)(r), & \text{if } r<L_s\\ k_{s-S^-(L_s)}(Y_{L_s}), & \text{if } r=L_s\\ \delta, & \text{if } r>L_s.\end{cases}$$

Then $X_t=x_t(\hat{Y})$.)

As in [3] Proposition 25, the first two are $\mathscr{F}_{L_s-}$ measurable and the latter lies in $\mathscr{H}^{L_s}_{(s-S^-(L_s))-}$.

(b) For $r>t\geqq 0$ we have that

$$\mathscr{F}_{L_t-}\subset\mathscr{G}_t\subset\mathscr{G}_r^0\subset\mathscr{H}_\infty^{L_r}\subset\mathscr{F}_{L_r}.$$

By definition, $L_r\downarrow L_t$ as $t\downarrow r$, hence

$$\mathscr{F}_{L_t}=\bigcap_{r>t}\mathscr{F}_{L_r},$$

so that $\mathscr{G}_t\subset\mathscr{F}_{L_t}$.

If now $T$ is a $(\mathscr{G}_t)$ stopping time, then by the right continuity of $L$,

$$\{L_T<\lambda\}=\bigcup_{q\in\mathbb{Q}}\{T<q\}\cap\{L_q<\lambda\}.$$

But $\{T<q\}\in\mathscr{G}_q\subset\mathscr{F}_{L_q}$, so that since $L_q$ is an $(\mathscr{F}_s)$ stopping time,

$$\{L_T<\lambda\}\in\mathscr{F}_\lambda.$$

Further, if $A\in\mathscr{G}_T$, then

$$A\cap\{T<q\}\in\mathscr{G}_q\subset\mathscr{F}_{L_q}$$

so that

$$A\cap\{T<q\}\cap\{L_q<\lambda\}\in\mathscr{F}_\lambda.$$

Taking the union over $q\in\mathbb{Q}$, we get that $A\cap\{L_T<\lambda\}\in\mathscr{F}_\lambda$, hence that $A\in\mathscr{F}_{L_T}$.

We will prove (c) before showing the remainder of (b); that $\mathscr{F}_{L_T-}\subset\mathscr{G}_{T-}$.

(c) Since $S^-$ and $S^+$ are strictly increasing, and $S^-(L_t)\leqq t\leqq S^+(L_t)$, we see that

$$\{S^+(s)<t\}=\{s<L_t\}\in\mathscr{F}_{L_t-}\subset\mathscr{G}_t,$$

hence $S^+(s)$ is a $(\mathscr{G}_t)$ stopping time.

Similarly, if $S^+(L_T)=T$, then

$$\{L_T<L_t\}=\{T<t\},$$

and hence if $A\in\mathscr{F}_{L_T}$, then by [3] Proposition 16

$$A\cap\{T<t\}=A\cap\{L_T<L_t\}\in\mathscr{F}_{L_t-}\subset\mathscr{G}_t.$$

That is, $A\in\mathscr{G}_T$. Since $\mathscr{F}_{L_T}\supset\mathscr{G}_T$ by the part of (b) already shown, we get that $\mathscr{G}_T=\mathscr{F}_{L_T}$.

To finish the proof of (b), let $A\in\mathscr{F}_s=\mathscr{G}_{S^+(s)}$. Then

$$A\cap\{L_T>s\}=A\cap\{T>S^+(s)\}\in\mathscr{G}_{T-}$$

by [3], Proposition 16 again.

(d)
$$\begin{aligned}\{V<h\}&=\{T<S^-(R)+h,\, L_T\leqq R<\infty\}\backslash\{L_T<R\}\\&=\{T<(S^-(R)+h)\wedge S^+(R)\}\backslash\{L_T<R\},\end{aligned}$$

as $T<S^+(L_T)$ on $\{T<\infty\}$. Because $T$ is a $(\mathscr{G}^0_{t+})$ stopping time,

$$\{L_T<R\}\in\mathscr{F}_{R-}\subset\mathscr{H}^R_h$$

and

$$\{T<(S^-(R)+h)\wedge S^+(R)\}\in\mathscr{G}^0_{[(S^-(R)+h)\wedge S^+(R)]-}.$$

We must therefore show that the latter field lies in $\mathscr{H}^R_h$. It is generated by $\mathscr{G}^0_{0-}=\mathscr{F}_{0-}\subset\mathscr{H}^R_h$, and by

$$\mathscr{G}^0_t\cap\{(S^-(R)+h)\wedge S^+(r)>t\},\qquad\text{for } t\geqq 0.$$

By monotone class arguments, the latter is generated by

$$\mathscr{H}^{L_t}_{0-}\cap\{(S^-(R)+h)\wedge S^+(R)>t\}$$

and by

$$\mathscr{H}^{L_t}_r\cap\{t-S^-(L_t)>r\}\cap\{(S^-(R)+h)\wedge S^+(R)>t\},\qquad\text{for } r\geqq 0.$$

$\sigma_a(Y_R)$ is an $(\mathscr{H}^R_{s+})$ stopping time, since $\sigma_a$ is a $(\mathscr{U}_{s+})$ stopping time, each $\sigma$-field $\mathscr{H}^R_s$ is complete, and $Y_R^{-1}(\mathscr{U}_s)\subset\mathscr{H}^R_s$. $(t-S^-(R))\vee 0$ is also an $(\mathscr{H}^R_{s+})$ stopping time, since $S^-(R)\in\mathscr{F}_{R-}$ as $S^-$ is predictable. Thus

$$\{(S^-(R)+h)\wedge S^+(R)>t\}=\{\sigma_a(Y_R)\wedge h>(t-S^-(R))\vee 0\}\in\mathscr{H}^R_{(\sigma_a(Y_R)\wedge h)-}\subset\mathscr{H}^R_h.$$

Also,

$$\mathscr{F}_{L_t-}\cap\{L_t\leqq R\}\subset\mathscr{F}_{R-}\cap\{L_t\leqq R\}$$

by [3] Proposition 18. Therefore

$$\mathscr{H}^{L_t}_{0-}\cap\{(S^-(R)+h)\wedge S^+(R)>t\}\subset\mathscr{F}_{R-}\cap\{(S^-(R)+h)\wedge S^+(R)>t\}\subset\mathscr{H}^R_h,$$

as $\{(S^-(R)+h)\wedge S^+(R)>t\}\subset\{L_t\leqq R\}$.

We argue as in part (a) to see that

$$\begin{aligned}&\mathscr{H}^{L_t}_r\cap\{t-S^-(L_t)>r\}\cap\{(S^-(R)+h)\wedge S^+(R)>t\}\cap\{L_t<R\}\\&\quad\subset\mathscr{F}_{R-}\cap\{(S^-(R)+h)\wedge S^+(R)>t\}\subset\mathscr{H}^R_h,\end{aligned}$$

and

$$\begin{aligned}&\mathscr{H}^{L_t}_r\cap\{t-S^-(L_t)>r\}\cap\{(S^-(R)+h)\wedge S^+(R)>t\}\cap\{L_t=R\}\\&\quad\subset\mathscr{H}^R_r\cap\{\sigma_a(Y_R)\wedge h>t-S^-(R)>r\}\\&\quad\subset\mathscr{H}^R_{(\sigma_a(Y_R)\wedge h)-}\subset\mathscr{H}^R_h.\quad\square\end{aligned}$$

**Lemma 5.** *Let $(\Omega_1, \mathscr{F}_t^1)$, $(\Omega_2, \mathscr{F}_t^2)$ be right continuous filtered spaces. Let $Z: \Omega_1 \to \Omega_2$ be such that $Z^{-1} \mathscr{F}_t^2 \supset \mathscr{F}_t^1$ for every t. Then for every $(\mathscr{F}_t^1)$ stopping time $T_1$, there is an $(\mathscr{F}_t^2)$ stopping time $T_2$ such that $T_2 \circ Z = T_1$.*

*Proof.* For $r \in \mathbb{Q}$, let $B_r = \{T_1 < r\}$, and find $A_r \in \mathscr{F}_r^2$ such that $B_r = Z^{-1} A_r$. Set

$$\hat{A}_t = \bigcup_{\substack{r<t \\ r\in\mathbb{Q}}} A_r.$$

Thus, putting $T_2(\omega) = \inf\{t; \omega \in \hat{A}_t\}$, we have that

$$\{T_2 < t\} = \hat{A}_t \in \mathscr{F}_t^2.$$

Also,

$$Z^{-1} \hat{A}_t = \bigcup_{\substack{r<t \\ r\in\mathbb{Q}}} Z^{-1} A_r = \{T_1 < t\}, \qquad \text{for every } t \geqq 0,$$

so that $T_2 \circ Z = T_1$. □

The following lemma would be much simplified if instead of the conditions of Theorem 2, we had assumed that the coordinate process $(W_t)$ was a right process under $(P_0^b)$.

**Lemma 6.** *Let $(P_0^b)$ satisfy* (2.4) *and* (2.5), *and suppose that n is a σ-finite positive measure on $(U, \mathscr{U})$ satisfying* (i), (iv) *and* (v). *Let R be a $(\mathscr{U}_{t+})$ stopping time such that $n(U^a, R=0)=0$. Then the coordinate process $(W_t, \mathscr{U}_t, n, P_0^b)$ is strong Markov at R.*

*Proof.* By replacing $R$ by the $(\mathscr{U}_{t+})$ stopping times

$$R_B = \begin{cases} R & \text{on } B \\ \infty & \text{off } B \end{cases}$$

for $B \in \mathscr{U}_{R+}$, we see that it suffices to show that for $A \in \mathscr{U}$,

$$n(\Theta_R^{-1} A, R < \infty) = \int_{\{R<\infty\}} P_0^{u(R(u))}(A)\, n(du).$$

Let $h > 0$. By (iv),

$$\int_{\{\sigma_a > h\}} f(u, \Theta_h u)\, n(du) = \int_{\{\sigma_a > h\}} \left[\int f(u, v)\, P_0^{u(h)}(dv)\right] n(du), \tag{3.1}$$

for $f$ of the form $1_{B\times C}$ where $B \in \mathscr{U}_h$, $C \in \mathscr{U}$. Thus, this holds for every $f \in \mathscr{U}_h \otimes \mathscr{U}$, $f \geqq 0$.

Put

$$\hat{\mathscr{U}}_t = \mathscr{U}_h \otimes \mathscr{U}_t.$$

Since $n$ is σ-finite, it follows as in Theorem 7.3 of Blumenthal and Getoor [2], that there is a $(\mathscr{U}_{t+}^0)$ stopping time $\tilde{R}$ such that $n(R \neq \tilde{R}) = 0$. Define $Z: U \to U \times U$ by $Z(u) = (u, \Theta_h u)$. Then $Z^{-1}(\hat{\mathscr{U}}_t) \supset \mathscr{U}_{(t+h)+}^0$. (Note that this might fail for the universal completion $\mathscr{U}_{(t+h)+}$.) Thus by Lemma 5, there is a $(\hat{\mathscr{U}}_{t+})$ stopping time $\hat{R}$ with

$$\hat{R}(u, \Theta_h u) = \begin{cases} \tilde{R}(u) - h, & \text{if } \tilde{R}(u) > h \\ \infty & \text{otherwise.} \end{cases}$$

Let

$$f=1_{\{(u,v);\,\hat R(u,v)<\infty,\,\Theta_{\hat R(u,v)}(v)\in A\}}.$$

Since $\hat R$ is a $(\hat{\mathscr{U}}_{t+})$ stopping time, it is immediate that $\hat R(u,\cdot)$ is *a* $(\mathscr{U}_{t+})$ stopping time, for each $u\in U$. Since also $(W_t, \mathscr{U}_t, P_0^c, P_0^b)$ was assumed to be strong Markov for $c\in E$, we can use (3.1) twice to get that

$$\begin{aligned} n(\sigma_a>h, h<R<\infty, \Theta_R^{-1}A) &= \int_{\{\sigma_a>h\}} f(u, \Theta_h u)\, n(du) \\ &= \int_{\{\sigma_a>h\}} \left[\int P_0^{v(\hat R(u,v))}(A)\, 1_{\{\hat R<\infty\}}(u,v)\, P_0^{u(h)}(dv)\right] n(du) \\ &= \int_{\{\sigma_a>h\}} P_0^{\Theta_h u(\hat R(u,\Theta_h u))}(A)\, 1_{\{\hat R<\infty\}}(u, \Theta_h u)\, n(du) \\ &= \int_{\{\sigma_a>h\}} P_0^{u(R(u))}(A)\, 1_{\{h<R<\infty\}}(u)\, n(du). \end{aligned}$$

Letting $h\downarrow 0$, we get that

$$n(0<R<\infty, \Theta_R^{-1}A) = \int_{\{0<R<\infty\}} P_0^{u(R(u))}\, n(du).$$

From (v), we see that

$$\int_{U\setminus U^a} f(u(0))\, g(u)\, n(du) = \int_{U\setminus U^a} f(u(0))\, E_0^{u(0)}(g)\, n(du) \tag{3.2}$$

for nonnegative $f\in\mathscr{E}$, $g\in\mathscr{U}$.

Let the set of branch points be

$$E_{br}=\{b\in E;\, P_0^b(W_0 \neq b)>0\}\in\mathscr{E}.$$

The strong Markov property of $(W_t, \mathscr{U}_t, P_0^c, P_0^b)$ shows that $P_0^c(W_0\in E_{br})=0$ for every $c\in E$. Thus,

$$\begin{aligned} &n(R=0, U\setminus U^a, A) \\ &\quad= \int_{U\setminus U^a} P_0^{u(0)}(R=0, A)\, n(du) \\ &\quad= \int_{U\setminus U^a} \left[\int_{\{R=0\}} P_0^{v(0)}(A)\, P_0^{u(0)}(dv)\right] n(du) \qquad \text{(SMP)} \\ &\quad= \int_{U\setminus U^a} P_0^{u(0)}(A)\, P_0^{u(0)}(R=0)\, n(du) \\ &\quad= \int_{U\setminus U^a} P_0^{u(0)}(A)\, 1_{\{R=0\}}(u)\, n(du) \qquad \text{(by (3.2))}. \end{aligned}$$

Since $n(\{R=0\}\cap U^a)=0$ by hypothesis, we are done. □

The heart of the proof of Theorem 2 lies in the next result, whose proof we defer until later.

**Lemma 7.** *Let $T$ be a $(\mathscr{G}_t)$ stopping time such that $L_T>0$ and $T<S^+(L_T)$ on $\{T<\infty\}$. Let $\mathscr{H}_t$ be the σ-field $\mathscr{F}_{L_{T-}}\otimes\mathscr{U}_t$ on $\Omega\times U$. Then there is an $(\mathscr{H}_{t+})$ stopping time $R$ such that*

$$P\{\omega;\, n\{u;\, R(\omega,u)<\infty\}<\infty\}=1, \tag{3.3}$$

(3.4) $$R(\omega, Y_{L_T}(\omega)) = (T - S^-(L_T))(\omega), \quad \text{if } T(\omega) < \infty,$$

(3.5) $$R(\omega, u) = \infty \quad \text{for every } u \in U, \text{ if } T(\omega) = \infty,$$

(3.6) $$P(Y_{L_T} \in A, T - S^-(L_T) \in B, T < \infty \mid \mathscr{F}_{L_T-})(\omega) = \frac{n\{u; u \in A, R(\omega, u) \in B, R(\omega, u) < \infty\}}{n\{u; R(\omega, u) < \infty\}} \quad \text{for } P\text{-a.e. } \omega,$$

*where* $A \in \mathscr{U}$, $B \in \mathscr{B}$, *and we take the convention that* $\frac{0}{0} = 0 = \frac{\infty}{\infty}$.

**Corollary 1.** *Let* $T$ *be a* $(\mathscr{G}_t)$ *stopping time such that* $X_T = a$. *Then* $T = S^+(L_T)$ *a.s.*

*Proof.* By the strong Markov property of $(W_t, \mathscr{U}_t, P_0^c, P_0^b)$ for $c \in E$, and the hypothesis that $P_0^c(\tau_a < \infty, W_t = a$ for every $t \geqq \tau_a) = 1$, we get that $P_0^a(\sigma_a = 0) = 1$, and hence that $P_0^c(W_0 = a, \sigma_a > 0) = 0$ for every $c \in E$. Since also $P(Y_0 \in M) = \int P_0^{Y_0(0)}(M)\, dP$, it follows that $S^+(0) = 0$ a.s. on $\{X_0 = a\}$, hence that $T = S^+(L_T)$ a.s. on $\{L_T = 0\}$. Thus putting

$$B = \{L_T > 0 \text{ and } T < S^+(L_T)\},$$

we may replace $T$ by $T_B$, to get that $L_T > 0$ and $T < S^+(L_T)$ on $\{T < \infty\}$. We need to show that $T = \infty$ a.s.

Apply Lemma 7 to $T$, to get an $(\mathscr{H}_{t+})$ stopping time $R$. For $\omega \in \Omega$, let

$$H(\omega) = \{u; u(0) = a, R(\omega, u) = 0\}.$$

Since $R$ is an $(\mathscr{H}_{t+})$ stopping time, it is immediate that $R(\omega, \cdot)$ is a $(\mathscr{U}_{t+})$ stopping time, hence that $H(\omega) \in \mathscr{U}_{0+}$ for each $\omega \in \Omega$. Since also $H(\omega) \subset U^a$, we have that

$$n \geqq n|_{H(\omega)} \geqq 0,$$

and that $n|_{H(\omega)}$ satisfies (iv). If $n(U) < \infty$, then $n(H(\omega)) \leqq n(U^a) = 0$ by (vi′)(b). If $n(U) = \infty$, then $n(H(\omega)) = 0$ or $\infty$ for each $\omega$, by (vi′)(c) (this is the only place we use this condition!). But $n(H(\omega)) \leqq n\{u; R(\omega, u) < \infty\}$, which is itself finite for $P$-a.e. $\omega$. Thus in either case, $n(H) = 0$ a.s., so

$$P(T < \infty) = E\left[\frac{n(H)}{n(R < \infty)}\right] = 0$$

by (3.6) (recall that $0/0 = 0$). □

Using these three results, we will show

**Corollary 2.** *Let* $T$ *be a* $(\mathscr{G}_t)$ *stopping time, and* $A \in \mathscr{U}$. *Then*

$$P[Y_{L_T}(\cdot + T - S^-(L_T)) \in A, T < \infty] = E[P_0^{X_T}(A), T < \infty].$$

*Proof.* Replacing $T$ by various $(\mathscr{G}_t)$ stopping times $T_B$, it suffices to treat several distinct cases, namely that $X_T = a$ on $\{T < \infty\}$, that $X_T \neq a$ and $L_T > 0$ on $\{T < \infty\}$, and that $X_T \neq a$ and $L_T = 0$ on $\{T < \infty\}$.

In the first case, Corollary 1 shows that the conclusion is trivial.

In the second case, also $T<S^+(L_T)$ on $\{T<\infty\}$, so that we can apply Lemma 7 to obtain $R$ as in that result. It follows that if $f\in\mathscr{F}_{L_T-}\otimes\mathscr{U}$ is bounded, then

(3.7)

$$E[f(\cdot, Y_{L_T}(\cdot)), T<\infty]=\int\left[\frac{1}{n\{u; R(\omega,u)<\infty\}}\int_{\{R(\omega,\cdot)<\infty\}} f(\omega,u)\,n(du)\right]P(d\omega).$$

Take

$$f=1_{\{(\omega,u);\, R(\omega,u)<\infty,\, \Theta_{R(\omega,u)}u\in A\}},$$

to obtain that

$$P(Y_{L_T}(\cdot+T-S^-(L_T))\in A, T<\infty)=E\left[\frac{n(\Theta_R^{-1}(A), R<\infty)}{n(R<\infty)}\right].$$

Since $R$ is an $(\mathscr{H}_{t+})$ stopping time, it is immediate that $R(\omega,\cdot)$ is a $(\mathscr{U}_{t+})$ stopping time, for each $\omega\in\Omega$. Also, since $X_T\neq a$ on $\{T<\infty\}$ we get that $n(U^a, R(\omega,\cdot)=0)=0$ for $P$-a.e. $\omega$. Thus, by Lemma 6,

$$n(\Theta_{R(\omega,\cdot)}^{-1}(A), R(\omega,\cdot)<\infty)=\int_{\{R(\omega,\cdot)<\infty\}} P_0^{u(R(\omega,u))}(A)\,n(du)$$

for $P$-a.e. $\omega$. Therefore by (3.7) again,

$$\begin{aligned}P(Y_{L_T}(\cdot+T-S^-(L_T))\in A, T<\infty)&=E[P_0^{Y_{L_T}(R(\cdot, Y_{L_T}(\cdot)))}(A), T<\infty]\\&=E[P_0^{X_T}(A), T<\infty].\end{aligned}$$

In the third case, that $L_T=0$ and $X_T\neq a$ on $\{T<\infty\}$, then also $T<S^+(L_T)$ on $\{T<\infty\}$. Thus Lemma 4, part (d) applies, to show that $T$ is an $(\mathscr{H}_{t+}^0)$ stopping time. Since $(\mathscr{H}_{t+}^0)$ is a completion of the natural right continuous filtration of the process $X_{\cdot\wedge\sigma_a(X)}$, it is easy to use the hypotheses that $P(Y_0\in A)=\int P_0^{u(0)}(A)\,P(Y_0\in du)$ for $A\in\mathscr{U}$, and that for $c\in E$ the coordinate process $(W_t, \mathscr{U}_t, P_0^c, P_0^b)$ is strong Markov, to conclude that the process

$$(X_{t\wedge\sigma_a(X)}, \mathscr{H}_t^0, P, P_0^b)$$

is strong Markov. This suffices. □

The remainder of Theorem 2 part (a) now follows easily. For $A\in\mathscr{U}$ and $b\in E$ put

$$P^b(A)=\iint 1_A(F(u,v))\,P(Y|_{(0,\infty)}\in dv)\,P_0^b(du).$$

Since $b\mapsto P_0^b(A)$ is $\mathscr{E}$-measurable for $A\in\mathscr{U}$, the same holds for $b\mapsto P^b(A)$. If $T$ is any $(\mathscr{G}_t)$ stopping time, then by the strong Markov property of PPP's (see Itô [9], Theorem 5.1), $\Theta_{L_T}^0(Y)$ is independent of $\mathscr{F}_{L_T}$, with the same law as $Y|_{(0,\infty)}$.

By replacing $T$ with the $(\mathscr{G}_t)$ stopping times

$$T_B=\begin{cases}T & \text{on } B\\ \infty & \text{off } B,\end{cases}$$

where $B \in \mathscr{G}_T$, it follows from Corollary 2 that for $A \in \mathscr{U}$,

$$P(Y_{L_T}(\cdot + T - S^-(L_T)) \in A \mid \mathscr{G}_T) = P_0^{X_T}(A) \quad \text{on } \{T < \infty\}.$$

Since also $\mathscr{G}_T \subset \mathscr{F}_{L_T}$, $Y_{L_T}(\cdot + T - S^-(L_T)) \in \mathscr{F}_{L_T}$, and

$$X_{\cdot + T} = F(Y_{L_T}(\cdot + T - S^-(L_T)), \Theta^0_{L_T} Y),$$

this yields the strong Markov property of $(X_t, \mathscr{G}_t, P, P^b)$ at $T$.

By definition of $(P^b)$, $(X_t, \mathscr{G}_t, P, P^b)$ is a recurrent extension of $(P_0^b)$. The remaining points have already been dealt with, except for showing that $Y$ is the PPP constructed from $X$ as in Itô [9]. Since $P^a(\sigma_a = 0) = 1$, this will follow provided $(L_t)$ satisfies (2.3).

The set of increase of $(L_t)$ is exactly $\{t; X_t = a\}$, since $X_t = a$ only if $t = S^+(s)$ for some $s$. The normalization

$$E[e^{-\sigma_a(X)}] = E\left[\int_0^\infty e^{-t}\, dL_t\right]$$

follows easily from the PPP nature of $Y$, and Theorem 4.5 of Itô [9]. Finally, we can write

$$(X_{\cdot + T}, L_{\cdot + T} - L_T) = (F(Y_{L_T}(\cdot + T - S^-(L_T)), \Theta^0_{L_T} Y), l_\cdot(\Theta^0_{L_T} Y)),$$

so that by Corollary 1, if $T$ is any $(\mathscr{G}_t)$ stopping time with $X_T = a$, then $(X_{\cdot + T}, L_{\cdot + T} - L_T)$ is independent of $\mathscr{G}_T$ with a law not depending on the choice of $T$, as required. Thus, except for the proof of Lemma 7, the proof of part (a) is complete.

*Proof of Lemma* 7. Choose $\delta_k \downarrow 0$ with $n(\sigma_a > \delta_k) > 0$ for each $k$. For $q \in \mathbb{Q}$, $q \geqq 0$, let

$$S_q^k = \inf\{s > q;\ \sigma_a(Y_s) > \delta_k\}.$$

The $S_q^k$ are $(\mathscr{F}_s)$ stopping times. By completeness of $\mathscr{F}$, we may assume without loss of generality that $N(\{\sigma_a > \delta_k\} \times [0, s], Y(\omega)) < \infty$ for every $\omega$, $k$ and $s$, and that $N(\{\sigma_a > \delta_k\} \times [0, \infty), Y(\omega)) = \infty$ for every $\omega$ and $k$. Thus, each $S_q^k$ is surely finite, and (writing $[\![V]\!]$ for the graph $\{(t, \omega); 0 \leqq t < \infty, t = V(\omega)\}$)

$$[\![(L_T)_{\{\sigma_a(Y_{L_T}) > \delta_k\}}]\!] \subset \bigcup_{q \in \mathbb{Q}_+} [\![S_q^k]\!].$$

Also, for any $k, k', q, q'$ we have that

$$S_q^k \leqq S_{q'}^{k'}, \quad \text{if } k \geqq k' \text{ and } q \leqq q', \qquad \{S_q^k = S_{q'}^{k'}\} \in \mathscr{F}_{S_q^k -} \quad \text{if } k \leqq k'.$$

Put

$$R_q^k = (T - S^-(L_T))_{\{L_T = S_q^k\}}$$
$$\mathscr{H}_t^{k,q} = \mathscr{H}_t^{S_q^k}$$
$$\hat{H}_t^{k,q} = \mathscr{F}_{S_q^k -} \otimes \mathscr{U}_t.$$

By Lemma 4(d), $R_q^k$ is an $(\mathscr{H}_{t+}^{k,q})$ stopping time. Use Lemma 5 with

$$\Omega_1 = \Omega, \mathscr{F}_t^1 = \mathscr{H}_{t+}^{k,q}, \qquad \Omega_2 = \Omega \times U, \mathscr{F}_t^2 = \hat{\mathscr{H}}_{t+}^{k,q}, \qquad Z(\omega) = (\omega, Y_{S_q^k}(\omega)),$$

to obtain an $(\hat{\mathscr{H}}_{t+}^{k,q})$ stopping time $\tilde{R}_q^k$ with

$$\tilde{R}_q^k(\omega, Y_{S_q^k}(\omega)) = R_q^k(\omega) \quad \text{for each } \omega\in\Omega.$$

Let

$$\hat{R}_q^k(\omega,u)=\begin{cases}\tilde{R}_{q'}^{k'}(\omega,u), & \text{if } k'\geqq k,\ q'\in\mathbb{Q}_+,\ S_q^k(\omega)=S_{q'}^{k'}(\omega),\\ & \text{and for every } k''\geqq k',\ q''\in\mathbb{Q}_+ \text{ such that } S_{q''}^{k''}(\omega)=S_{q'}^{k'}(\omega),\\ & \text{also } \tilde{R}_{q''}^{k''}(\omega,u)=\tilde{R}_{q'}^{k'}(\omega,u)\\ \infty, & \text{if for every } k'\geqq k,\ q'\in\mathbb{Q} \text{ this fails to hold.}\end{cases}$$

Thus, for every $k, k'\in\mathbb{Z}_+$ and $q, q'\in\mathbb{Q}_+$ we have that

$$\hat{R}_q^k=\hat{R}_{q'}^{k'} \quad \text{on} \quad \{S_q^k=S_{q'}^{k'}\}\times U,$$

and

$$\hat{R}_q^k(\omega, Y_{S_q^k}(\omega))=(T-S^-(L_T))(\omega) \quad \text{for } \omega\in\{L_T=S_q^k\}.$$

We now show that $\hat{R}_q^k$ is an $(\hat{\mathscr{H}}_{t+}^{k,q})$ stopping time. Let $t>0$, and choose an open base $B_1, B_2, \ldots$ of the space $[0,t)$. Then

$$\begin{aligned}\{\hat{R}_q^k<t\}=\bigcup_{\substack{k'\geqq k\\ q'\in\mathbb{Q}_+}}\bigcap_{\substack{k''\geqq k'\\ q''\in\mathbb{Q}_+}}&[(\{S_{q''}^{k''}\neq S_q^k\}\times U)\\ &\cup\bigcap_{i\geqq 1}\bigcup_{j\geqq i}((\{\tilde{R}_{q''}^{k''}\in B_j\}\cap\{S_{q''}^{k''}=S_q^k\}\times U)\\ &\cap(\{\tilde{R}_{q'}^{k'}\in B_j\}\cap\{S_{q'}^{k'}=S_q^k\}\times U))].\end{aligned}$$

Since $\{S_{q'}^{k'}=S_q^k\}\in\mathscr{F}_{S_q^k-}$ whenever $k'\geqq k$, it will thus suffice to show that

$$\hat{\mathscr{H}}_t^{k',q'}\cap(\{S_{q'}^{k'}=S_q^k\}\times U)\subset\hat{\mathscr{H}}_t^{k,q}$$

for every $k'\geqq k$, $q'\in\mathbb{Q}_+$. This holds, since by monotone class arguments,

$$\begin{aligned}(\mathscr{F}_{S_{q'}^{k'}-}\otimes\mathscr{U}_t)\cap(\{S_{q'}^{k'}=S_q^k\}\times U)&=(\mathscr{F}_{S_{q'}^{k'}-}\cap\{S_{q'}^{k'}=S_q^k\})\otimes\mathscr{U}_t\\ &=(\mathscr{F}_{S_q^k-}\cap\{S_{q'}^{k'}=S_q^k\})\otimes\mathscr{U}_t\subset\mathscr{F}_{S_q^k-}\otimes\mathscr{U}_t.\end{aligned}$$

Put

$$R=\bigwedge_{k,q}(\hat{R}_q^k)_{\{L_T=S_q^k\}\times U}.$$

Then $R=\hat{R}_q^k$ on $\{L_T=S_q^k\}\times U$. We will show that $R$ is an $(\mathscr{H}_{t+})$ stopping time. Let $t>0$, $k\in\mathbb{Z}_+$. Then

$$\{R<t\}\cap\{\sigma_a(Y_{L_T})>\delta_k\}\times U=\bigcup_{q\in\mathbb{Q}_+}[\{\hat{R}_q^k<t\}\cap\{S_q^k=L_T\}\times U].$$

As before,

$$\hat{\mathscr{H}}_t^{k,q}\cap\{S_q^k=L_T\}\times U=\mathscr{H}_t\cap\{S_q^k=L_T\}\times U.$$

Since also

$$\{S_q^k=L_T\}=\{q<L_T\leqq S_q^k\}\cap\{\sigma_a(Y_{L_T})>\delta_k\},$$

we get that for each $k$,

$$\{R<t\}\cap\{\sigma_a(Y_{L_T})>\delta_k\}\times U=H_k\cap\{\sigma_a(Y_{L_T})>\delta_k\}\times U,$$

for some set $H_k \in \mathscr{H}_t$. Using that $\sigma_a(Y_{L_T})>0$ on $\{T<\infty\}$, it is easy to see that in fact

$$\{R<t\}=[\bigcap_{j\geqq 1}\bigcup_{k\geqq j} H_k]\setminus\{T=\infty\}\times U\in\mathscr{H}_t,$$

as required.

Properties (3.4) and (3.5) are now immediate. (3.3) will follow from (3.5), (3.6), and the convention that $\infty/\infty=0$. Thus all that remains is to show (3.6).

Let $A\in\mathscr{F}_r$. By the Markov property of $Y$ at $r$, and Theorem 4.4A of Itô [9],

$$\begin{aligned} P(Y_{S_q^k}\in B, S_q^k>r, A) &= P(A, S_q^k>r)\, P(Y_{S^k_{(q-r)}v_0}\in B)\\ &= P(A, S_q^k>r)\,\frac{n(B,\sigma_a>\delta_k)}{n(\sigma_a>\delta_k)}, \end{aligned}$$

for $B\in\mathscr{U}$. That is, for $f=1_{A\times B}$, $A\in\mathscr{F}_{S_q^k-}$, $B\in\mathscr{U}$ we have

$$E[f(\cdot, Y_{S_q^k}(\cdot))]=\frac{1}{n(\sigma_a>\delta_k)}\int(\int_{\{\sigma_a>\delta_k\}} f(\omega,u)\,n(du))\,P(d\omega).$$

Therefore this holds for $f\in\mathscr{F}_{S_q^k-}\otimes\mathscr{U}$, $f\geqq 0$. Take

$$f(\omega,u)=1_C(\omega)\,1_A(u)\,1_B(R_q^k(\omega,u))\,1_{\{R_q^k<\infty\}}(\omega,u),$$

where $C\in\mathscr{F}_{S_q^k-}$, $A\in\mathscr{U}$, $B\in\mathscr{B}$. Then

$$\begin{aligned} &P(C, Y_{L_T}\in A, T-S^-(L_T)\in B, L_T=S_q^k)=E[f(\cdot, Y_{S_q^k}(\cdot))]\\ &=\int_C \frac{n(A, R_q^k(\omega,\cdot)\in B, R_q^k(\omega,\cdot)<\infty, \sigma_a>\delta_k)}{n(\sigma_a>\delta_k)}\,P(d\omega)\\ &=\int_C P(L_T=S_q^k\mid\mathscr{F}_{S_q^k-})(\omega)\,\frac{n(A, R_q^k(\omega,\cdot)\in B, R_q^k(\omega,\cdot)<\infty, \sigma_a>\delta_k)}{n(R_q^k(\omega,\cdot)<\infty, \sigma_a>\delta_k)}\,P(d\omega)\\ &=\int_{C\cap\{L_T=S_q^k\}} \frac{n(A, R_q^k(\omega,\cdot)\in B, R_q^k(\omega,\cdot)<\infty, \sigma_a>\delta_k)}{n(R_q^k(\omega,\cdot)<\infty, \sigma_a>\delta_k)}\,P(d\omega). \end{aligned}$$

Enumerate $\mathbb{Q}_+$ as $q_1, q_2, \ldots$, and let $C\in\mathscr{F}_{L_T-}$. Then there are $C_q^k\in\mathscr{F}_{S_q^k-}$ such that

$$C\cap\{L_T=S_q^k\}=C_q^k\cap\{L_T=S_q^k\}.$$

Thus

$$\begin{aligned} &P(C, Y_{L_T}\in A, T-S^-(L_T)\in B, \sigma_a(Y_{L_T})>\delta_k, T<\infty)\\ &=\sum_j P(C_j^k, Y_{L_T}\in A, T-S^-(L_T)\in B, L_T=S_{q_j}^k, S_{q_i}^k\neq S_{q_j}^k \text{ for } i<j)\\ &=\sum_j \int_{\substack{C_j^k\cap\{S_{q_j}^k=L_T\}\\ \cap\{S_{q_i}^k\neq S_{q_j}^k \text{ for } i<j\}}} \frac{n(A, R_{q_j}^k(\omega,\cdot)\in B, R_{q_j}^k(\omega,\cdot)<\infty, \sigma_a>\delta_k)}{n(R_{q_j}^k(\omega,\cdot)<\infty, \sigma_a>\delta_k)}\,P(d\omega),\\ &\quad \text{as } \{S_{q_j}^k\neq S_{q_i}^k\}\in\mathscr{F}_{S_{q_j}^k-}\\ &=\sum_j \int_{\substack{C\cap\{S_{q_j}^k=L_T\}\\ \cap\{S_{q_j}^k\neq S_{q_i}^k \text{ for } i<j\}}} \frac{n(A, R(\omega,\cdot)\in B, R(\omega,\cdot)<\infty, \sigma_a>\delta_k)}{n(R(\omega,\cdot)<\infty, \sigma_a>\delta_k)}\,P(d\omega)\\ &=\int_{C\cap\{\sigma_a(Y_{L_T})>\delta_k\}} \frac{n(A, R(\omega,\cdot)\in B, R(\omega,\cdot)<\infty, \sigma_a>\delta_k)}{n(R(\omega,\cdot)<\infty, \sigma_a>\delta_k)}\,P(d\omega). \end{aligned}$$

If $A$ is chosen so that $n(A)<\infty$, then the integrand converges boundedly to

$$\frac{n(A, R(\omega,\cdot)\in B, R(\omega,\cdot)<\infty)}{n(R(\omega,\cdot)<\infty)}.$$

Thus

$$P(C, Y_{L_T}\in A, T-S^-(L_T)\in B, T<\infty)=\int_C \frac{n(A, R(\omega,\cdot)\in B, R(\omega,\cdot)<\infty)}{n(R(\omega,\cdot)<\infty)} P(d\omega)$$

whenever $n(A)<\infty$. Since $n$ is $\sigma$-finite, this holds for every $A\in\mathscr{U}$, which yields (3.6). □

Part (a) is proven. The proof of part (b) is similar. For $u_0, u_1, u_2, \ldots \in U$, put

$$S_k(u_0, u_1, \ldots)=\sum_{i<k}\sigma_a(u_i), \qquad k\geqq 0,$$

$$F(u_0, u_1, \ldots)(t)=\begin{cases} u_k(t-S_k(u_0, u_1, \ldots)), & \text{if } S_k(u_0, u_1, \ldots)\leqq t<S_{k+1}(u_0, u_1, \ldots)\\ a, & \text{if } t\geqq S_\infty(u_0, u_1, \ldots).\end{cases}$$

Put

$$X_t=F(Y_0, Y_1, \ldots)(t), \qquad S(k)=S_k(Y_0, Y_1, \ldots).$$

For $A\in\mathscr{U}$, put

$$P^b(A)=\begin{cases} \int_U \int_{U\times U\times\ldots} 1_A(F(u, v_1, v_2, \ldots)) \\ \quad \cdot P((Y_1, Y_2, \ldots)\in d(v_1, v_2, \ldots))\, P_0^b(du), & \text{if } b\neq a\\ \int_U \int_{U\times U\times\ldots} 1_A(F(u, v_1, v_2, \ldots)) \\ \quad \cdot P((Y_1, Y_2, \ldots)\in d(v_1, v_2, \ldots))\, n(du), & \text{if } b=a.\end{cases}$$

We let $\mathscr{F}_{-1}$ be the set of $P$-null sets of $\mathscr{F}$. For $R$ a random variable with values in $\{-1, 0, 1, 2, \ldots\}$, $\mathscr{F}_R$ is defined to be the $\sigma$-field generated by sets in $\mathscr{F}_r\cap\{R=r\}$, for $-1\leqq r\leqq\infty$. For $R$ a non negative integer valued random time, we put

$$\mathscr{H}_{0-}^R=\mathscr{F}_{R-1}, \qquad \mathscr{H}_t^R=\mathscr{F}_{R-1}\vee\sigma(Y_R(s); 0\leqq s\leqq t).$$

For $r\geqq 0$, let $L_r=\inf\{k; S(k+1)>r\}$. We put

$$\mathscr{G}_t^0=\mathscr{H}_{(t-S(L_t))-}^{L_t}, \qquad \mathscr{G}_t=\mathscr{G}_{t+}^0, \qquad \mathscr{G}_{0-}=\mathscr{G}_{0-}^0=\mathscr{F}_{-1}.$$

Since $n(U\backslash U^a)=0$, we have that $S(k)\to\infty$, a.s., and that for each $k\geqq 1$, $S(k)<\infty$, $S(k-1)<S(k)$, and $X_{S(k)}=a$, a.s. Delete a null set of $\Omega$ to ensure that these statements hold surely. The discrete-time analogue to Lemma 4 is;

**Lemma 4′.** (a) *$(\mathscr{G}_t)$ is right continuous, increasing, and each $\mathscr{G}_t$ contains all the $P$-null sets of $\mathscr{F}$. $X_t\in\mathscr{G}_t$ for each $t\geqq 0$.*

(b) *If $T$ is a $(\mathscr{G}_t)$ stopping time, then $L_T$ is an $(\mathscr{F}_k)$ stopping time, and $\mathscr{F}_{L_T-1}\subset\mathscr{G}_T\subset\mathscr{F}_{L_T}$. If further $X_T\neq a$ on $\{T<\infty\}$, then $\mathscr{F}_{L_T-1}\subset\mathscr{G}_{T-}$.*

(c) *$S(k)$ is a $(\mathscr{G}_t)$ stopping time, for $k\geqq 0$. If $T$ is a $(\mathscr{G}_t)$ stopping time such that $X_T=a$ on $\{T<\infty\}$, then $\mathscr{F}_{L_T-1}\supset\mathscr{G}_{T-}$.*

(d) *If $R$ is an $(\mathscr{F}_k)$ stopping time, and $T$ is a $(\mathscr{G}_t)$ stopping time, put*

$$V=\begin{cases} T-S(R) & \text{on } \{L_T=R<\infty\} \\ \infty & \text{otherwise.} \end{cases}$$

*Then $V$ is an $(\mathscr{H}^R_{t+})$ stopping time.*

*Proof.* The proofs of part (a), and that for $T$ a $(\mathscr{G}_t)$ stopping time, $L_T$ is an $(\mathscr{F}_k)$ stopping time and $\mathscr{G}_T \subset \mathscr{F}_{L_T}$, are as in Lemma 4. Further,

$$\begin{aligned} \mathscr{F}_{L_T-1} \cap \{T<t\} &= \mathscr{F}_{L_T-1} \cap \{L_T-1 \leqq L_t-1\} \cap \{T<t\} \\ &\subset \mathscr{F}_{L_t-1} \cap \{L_T-1 \leqq L_t-1\} \cap \{T<t\} \\ &= \mathscr{F}_{L_t-1} \cap \{T<t\} \subset \mathscr{G}_t, \end{aligned}$$

so that $\mathscr{F}_{L_T-1} \subset \mathscr{G}_T$.

For $k \geqq 0$ we have that

$$\{S(k) \leqq t\} = \{k \leqq L_t\} \in \mathscr{F}_{L_t-1} \subset \mathscr{G}_t,$$

so that $S(k)$ is a $(\mathscr{G}_t)$ stopping time. Applying the previous computation, we see that

$$\mathscr{F}_k \subset \mathscr{G}_{S(k+1)}.$$

Let $T$ be a $(\mathscr{G}_t)$ stopping time such that $X_T \neq a$ on $\{T<\infty\}$, and let $A \in \mathscr{F}_k$. Then

$$\begin{aligned} A \cap \{L_T-1 \geqq k\} &= A \cap \{L_T \geqq k+1\} \\ &= A \cap \{T > S(k+1)\} \in \mathscr{G}_{S(k+1)} \cap \{T > S(k+1)\} \subset \mathscr{G}_{T-}. \end{aligned}$$

Now let $T$ be a $(\mathscr{G}_t)$ stopping time such that $X_T = a$ on $\{T<\infty\}$. Then $\{T>r\} = \{L_T > L_r\}$. By monotone class arguments, $\mathscr{G}_{T-}$ is generated by

$$\mathscr{G}_{0-} = \mathscr{F}_{-1} \subset \mathscr{F}_{L_T-1},$$

and by

$$\mathscr{G}_r \cap \{T>r\} \subset \mathscr{F}_{L_r} \cap \{L_T-1 \geqq L_r\} \subset \mathscr{F}_{L_T-1},$$

for $r \geqq 0$. Thus part (c) is shown. Part (d) follows as in Lemma 4. □

The discrete time version of Lemma 7 is

**Lemma 7′.** *Let $T$ be a $(\mathscr{G}_t)$ stopping time such that $X_0=a$ on $\{L_T=0,\ T<\infty\}$. Let $\mathscr{H}_t = \mathscr{F}_{L_T-1} \otimes \mathscr{U}_t$. Then there is an $(\mathscr{H}_{t+})$ stopping time $R$ such that*

(3.8) $$R(\omega, Y_{L_T}(\omega)) = (T-S(L_T))(\omega) \quad \textit{if } T(\omega)<\infty.$$

(3.9) $$R(\omega, u) = \infty \quad \textit{for every } u \in U \textit{ if } T(\omega)=\infty.$$

(3.10) $$\begin{aligned} &P(Y_{L_T} \in A,\ T-S(L_T) \in B,\ T<\infty \mid \mathscr{F}_{L_T-1})(\omega) \\ &= \frac{n\{u;\ u \in A,\ R(\omega,u) \in B,\ R(\omega,u)<\infty\}}{n\{u;\ R(\omega,u)<\infty\}} \quad \textit{for } P\textit{-a.e. } \omega, \end{aligned}$$

*where $A \in \mathscr{U}$, $B \in \mathscr{B}$, and we take the convention that $0/0=0$.*

*Proof.* For $k \geqq 0$, let

$$R^k = (T - S(L_T))_{\{L_T = k\}},$$
$$\hat{\mathscr{H}}_t^k = \mathscr{F}_{k-1} \otimes \mathscr{U}_t.$$

By Lemma 4′(d), $R^k$ is an $(\mathscr{H}_{t+}^k)$ stopping time. Use Lemma 5 with

$$\Omega_1 = \Omega, \quad \mathscr{F}_t^1 = \mathscr{H}_{t+}^k, \quad \Omega_2 = \Omega \times U, \quad \mathscr{F}_t^2 = \hat{\mathscr{H}}_{t+}^k \quad \text{and} \quad Z(\omega) = (\omega, Y_k(\omega)),$$

to obtain an $(\hat{\mathscr{H}}_{t+}^k)$ stopping time $\hat{R}^k$ such that for every $\omega \in \Omega$,

$$\hat{R}^k(\omega, Y_k(\omega)) = R^k(\omega).$$

Let

$$R = \bigwedge_k (\hat{R}^k)_{\{L_T = k\} \times U}.$$

Thus for $t \geqq 0$,

$$\{R < t\} = \bigcup_k \{\hat{R}^k < t\} \cap \{L_t = k\} \times U.$$

By monotone class arguments,

$$\hat{\mathscr{H}}_t^k \cap \{L_T = k\} \times U = (\mathscr{F}_{k-1} \cap \{L_T = k\}) \otimes \mathscr{U}_t \subset \mathscr{F}_{L_T - 1} \otimes \mathscr{U}_t = \mathscr{H}_t,$$

so that $R$ is an $(\mathscr{H}_{t+})$ stopping time. Properties (3.8) and (3.9) are immediate. For $k \geqq 1$, $Y_k$ is independent of $\mathscr{F}_{k-1}$ with law $n$, so that

$$E[f(\cdot, Y_k(\cdot))] = \iint f(\omega, u)\, n(du)\, P(d\omega)$$

for $f = 1_{A \times B}$, $A \in \mathscr{F}_{k-1}$, $B \in \mathscr{U}$. Thus this holds for $f \in \mathscr{F}_{k-1} \otimes \mathscr{U}$, $f \geqq 0$. As in the proof of Lemma 7 it follows from this that for $k \geqq 1$, $C \in \mathscr{F}_{k-1}$, $A \in \mathscr{U}$, and $B \in \mathscr{B}$, we have that

$$P(C, Y_{L_T} \in A, T - S(L_T) \in B, L_T = k, T < \infty)$$
$$= \int_{C \cap \{L_T = k\}} \frac{n(A, R^k(\omega, \cdot) \in B, R^k(\omega, \cdot) < \infty)}{n(R^k(\omega, \cdot) < \infty)}\, P(d\omega).$$

We will show this for $k = 0$ as well, as then the argument of Lemma 7 will give (3.10). Let

$$E_{br} = \{b \in E \colon P_0^b(W_0 = b) \neq 1\}.$$

Since $(W_t, \mathscr{U}, P_0^c, P_0^b)$ is strong Markov for each $c \in E$, we get that

$$P_0^c(W_0 \in E_{br}) = 0, \qquad \text{for ever} \quad c \in E.$$

By hypothesis,

$$P(Y_0 \in M) = \int_{U \setminus U^a} P_0^{u(0)}(M)\, P(Y \in du) + n(M)\, P(Y_0 \in U^a),$$

so that as $a \notin E_{br}$ and $n(U \setminus U^a) = 0$, we see that

$$P(Y_0(0) \in E_{br}) = 0.$$

Thus,

$$P(Y_0 \in M \cap U^a) = \int_{U \setminus U^a} P_0^{u(0)}(M \cap U^a)\, P(Y_0 \in du) + n(M)\, P(Y_0 \in U^a)$$
$$= n(M)\, P(Y_0 \in U^a).$$

Since $\mathscr{F}_{-1}$ consists of sets of $P$-measure 0, it follows that for $f \in \mathscr{F}_{-1} \otimes \mathscr{U}, f \geqq 0$,

$$E[f(\cdot, Y_0(\cdot)), Y_0 \in U^a] = E[\int f(\cdot, u)\, n(du)]\, P(Y_0 \in U^a).$$

Taking $f$ as before and using the hypothesis that $X_0 = a$ if $L_T = 0$, we see that for $C \in \mathscr{F}_{-1}$, $A \in \mathscr{U}$, $B \in \mathscr{B}$,

$$\begin{aligned} &P(C, Y_{L_T} \in A, T - S(L_T) \in B, L_T = 0, T < \infty) \\ &\quad = E(C, n(A, R^0 \in B, R^0 < \infty)]\, P(Y_0 \in U^a) \\ &\quad = \frac{E(C, n(A, R^0 \in B, R^0 < \infty)]}{E[n(R^0 < \infty)]}\, P(L_T = 0) \\ &\quad = \int_{C \cap \{L_T = 0\}} \frac{n(A, R^0(\omega, \cdot) \in B, R^0(\omega, \cdot) < \infty)}{n(R^0(\omega, \cdot) < \infty)}\, P(d\omega). \quad \square \end{aligned}$$

Once the following result is established, the remainder of part (b) follows immediately, as before.

**Corollary 2′.** *Let $T$ be a $(\mathscr{G}_t)$ stopping time, and $A \in \mathscr{U}$. Then*

$$\begin{aligned} &P(Y_{L_T}(\cdot + T - S(L_T)) \in A, T < \infty) \\ &\quad = E[P_0^{X_T}(A), X_T \neq a, T < \infty] + n(A)\, P(X_T = a, T < \infty). \end{aligned}$$

*Proof.* As before, it suffices to treat three cases; that on $\{T < \infty\}$, respectively $X_T = a$, $X_T \neq a$ and $L_T > 0$, or $X_T \neq a$ and $L_T = 0$. The second case is handled as before, using Lemmas 7′ and 6. In the third case, we use Lemma 4′(d) to see that $T$ is an $(\mathscr{H}_{t+}^0)$ stopping time. Since $(\mathscr{H}_{t+}^0)$ is a completion of the natural right continuous filtration of the process $X_{\cdot \wedge \tau_a(X)}$, and

$$P(Y_0 \in M) = \int_{U \setminus U^a} P_0^{u(0)}(M)\, P(Y_0 \in du), \qquad \text{for } M \in \mathscr{U}, M \cap U^a = \emptyset$$

it is simple to use the strong Markov property of the processes $(W_t, \mathscr{U}_t, P_0^c, P_0^b)$, for $c \in E$, to conclude that the process

$$(X_{t \wedge \tau_a(X)}, \mathscr{H}_t^0, P, P_0^b)$$

is strong Markov. This suffices.

In the third case, we apply Lemma 7′ to obtain an $R$ satisfying the conclusions of that result. Put $H(\omega) = \{u; R(\omega, u) = 0\}$. Then for $A \in \mathscr{U}$,

$$P(Y_{L_T} \in A, T < \infty) = \int \frac{n(H(\omega) \cap A)}{n(H(\omega))}\, P(d\omega),$$

as $T - S(L_T) = 0$ on $\{T < \infty\}$. Since $R$ is an $(\mathscr{H}_{t+})$ stopping time, also $R(\omega, \cdot)$ is a $(\mathscr{U}_{t+})$ stopping time for each $\omega \in \Omega$, so that $H(\omega) \in \mathscr{U}_{0+}$. Thus $n|_{H(\omega)}$ satisfies (iv), so that by (vi′)(a), it must be a multiple of $n$. Since

$$n(H(\omega))=n|_{H(\omega)}(H(\omega)),$$

we must in fact have $n|_{H(\omega)}=n$ or 0, so that $n(H(\omega))=1$ or 0 for each $\omega$. Thus

$$P(Y_{L_t}\in A,\, T<\infty)=n(A)\,P(T<\infty,\, n(H)=1), \qquad \text{for } A\in\mathscr{U}.$$

Putting $A=U$, we have that $n(H)=1$ a.s. on $\{T<\infty\}$, so that

$$P(Y_{L_T}\in A,\, T<\infty)=n(A)\,P(T<\infty),$$

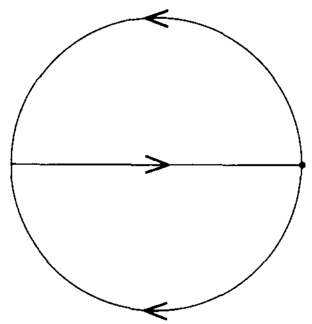
as required. □

## 4. Insufficiency of Conditions (ii) and (vi)

First, we present some examples to show that the conditions (vi) and (ii) do not suffice.

*Example 1.* (vi)(a) is not sufficient:

Let $P_0$ correspond to uniform motion as indicated, with absorption at $a$.




There are two excursion measures $n_1, n_2$ corresponding to strong Markov processes visiting a discretely. $\frac{n_1+n_2}{2}$ satisfies (vi)(a), and gives a Markov process which visits a discretely, but is not strong Markov.

*Example 2.* (vi)(c) is not sufficient:

Consider a Bessel process on $(0,\infty)$ (so that 0 is an entrance, non-exit point), and make the point 1 absorbing. Wrap $(0,1]$ around to make a circle $E$, and let $P_0$ correspond to the resulting process on $E$. Let $a=\{1\}$. $P_0$ corresponds to a continuous process that is absorbed at $a$, but which approaches $a$ only from the counter clockwise direction (say).

We have strong Markov continuous recurrent extensions with a instantaneous, corresponding to making the Bessel process (slowly) reflecting at 1, with various delay coefficients. Let the excursion measure with delay coefficient $m$ be $n_m$, so that

$$m=1-\int(1-e^{-\sigma_a})\,dn_m.$$

There are also continuous 'recurrent extensions' which are not strong Markov, corresponding to stopping the original process at $a$, holding it there an exponential time, and then making it enter $E\setminus\{a\}$ in the counterclockwise direction (this is possible, as 0 is an entrance point for the Bessel process). For $m\in(0,1)$, this gives an 'excursion measure' $n'_m$ such that

$$m=1-\int(1-e^{-\sigma_a})\,dn'_m.$$

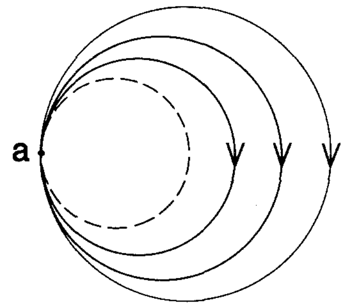
($m$ determines the mean of the holding time at $a$).

Though condition (vi)(c) fails for the $n'_m$, it will hold for any measure $n_m+n'_q$ for which $m+q\geqq 1$ and $m, q\in(0,1)$. These measures are ruled out by (vi')(c).

*Example 3.* (ii) does not suffice:

Let $E$ be the subset of $\mathbb{R}^2$ described in polar coordinates as

$$\bigcup_{R\in(1,2]}\{(r\cos\theta, r\sin\theta);\ r=R\cos\theta\geqq 0\}.$$




Let $a$ be the origin, and let $(P_0^b)$ correspond to uniform clockwise motion around the circles $r=R\cos\theta$, at speed $\pi R(R-1)^{-1}$, with absorption at $a$. For $\mu$ any $\sigma$-finite measure on $E\setminus\{a\}$, $n=P_0^\mu$ satisfies (i), (iv), (v), and (vi')(c). Let $n_0$ be the excursion measure of one dimensional Brownian motion, from 0, and let $M(u)=\max\{|u(t)|;\ 0\leqq t<\infty\}$, $u\in U$. Let $f\colon E\setminus\{a\}\to(0,1]\times(0,\pi)$ be given by

$$f(r\cos\theta, r\sin\theta)=\left(\frac{r}{\cos\theta}-1,\ \frac{\pi}{2}-\theta\right).$$

Then for $\mu(A)=n_0((\sigma_a, M)\in f(A))$, we get properties (ii) and (iii) for free, but (ii') fails, so that the resulting process is not right continuous.

We can replace (ii') by the more appealing condition (ii), provided we assume some regularity of $(P_0^b)$;

**Proposition 2.** (a) *Suppose that* $(P_0^b)$ *satisfies* (2.4) *and* (2.5), *and that* $n$ *satisfies* (i), (iii), (iv) *and* (v). *Suppose also that*

(4.1) *for every open neighbourhood* $V$ *of* $a$, *there is an open neighborhood* $V'$ *of* $a$, $V'\subset V$, *such that*

$$\sup_{b\in V'}\frac{P_0^b((W_t)\text{ leaves }V)}{E_0^b(1-e^{-\sigma_a})}<\infty$$

$\left(\text{\textit{with the convention that} } \frac{0}{0}=0\right)$. *Then* (ii) *holds if and only if* (ii') *does.*

(b) *Conversely, suppose that for every initial measure* $\mu$ *with* $\mu\{a\}=0$ *and* $E_0^\mu[1-\exp(-\sigma_a)]<1$, *the measure* $P_0^\mu$ *is the excursion measure of a right continuous process. Then* (4.1) *holds.*

*Proof.* (a) Let $V$, $V'$ be as above, and assume that (ii) holds. Then

$$n((W_t)\text{ leaves }V)\leqq n(W_0\notin V')+n(W_0\notin V', (W_t)\text{ leaves }V).$$

The first term is finite by (ii), and the second term is

$$\begin{aligned}
& n(W_0\in V'\setminus\{a\}, (W_t)\text{ leaves }V)\\
&\quad +\lim_{\delta\downarrow 0} n(W_0=a,\ W_t\in V'\text{ for }t\in[0,\delta],\ \sigma_a>\delta,\ (W_t)\text{ leaves }V)\\
&= \int_{\{W_0\in V'\setminus\{a\}\}} P_0^{u(0)}((W_t)\text{ leaves }V)\,n(du)\\
&\quad +\lim_{\delta\downarrow 0}\int_{\{W_0=a,\ W_t\in V'\text{ for }t\in[0,\delta],\ \sigma_a>\delta\}} P_0^{u(\delta)}((W_t)\text{ leaves }V)\,n(du)\\
&\leqq \sup_{\substack{b\in V'\\ b\neq a}}\frac{P_0^b((W_t)\text{ leaves }V)}{E_0^b(1-e^{-\sigma_a})}\Big[\int_{\{W_0\in V'\setminus\{a\}\}} E_0^{u(0)}(1-e^{-\sigma_a})\,n(du)\\
&\quad +\liminf_{\delta\downarrow 0}\int_{\{W_0=a,\ W_t\in V'\text{ for }t\in[0,\delta],\ \sigma_a>\delta\}} E_0^{u(\delta)}(1-e^{-\sigma_a})\,n(du)\Big]\\
&= \sup_{\substack{b\in V'\\ b\neq a}}\frac{P_0^b((W_t)\text{ leaves }V)}{E_0^b(1-e^{-\sigma_a})}\Big[\int_{\{W_0\in V'\setminus\{a\}\}} (1-e^{-\sigma_a})\,dn\\
&\quad +\liminf_{\delta\downarrow 0}\int_{\{W_0=a,\ W_t\in V'\text{ for }t\in[0,\delta],\ \sigma_a>\delta\}} (1-e^{-\sigma_a-\delta})\,dn\Big]\\
&\leqq \sup_{\substack{b\in V'\\ b\neq a}}\frac{P_0^b((W_t)\text{ leaves }V)}{E_0^b(1-e^{-\sigma_a})}\int(1-e^{-\sigma_a})\,dn<\infty.
\end{aligned}$$

(b) Assume (4.1) fails for some open neighborhood $V$ of $a$. Then there are $b_k\in E\setminus\{a\}$, $b_k\to a$ such that

$$a_k=\frac{P_0^{b_k}((W_t)\text{ leaves }V)}{E_0^{b_k}(1-e^{-\sigma_a})}\to\infty.$$

By passing to a subsequence, we may assume that $a_k\geqq k$ for each $k$. Let

$$\lambda_k=(k^2\,E_0^{b_k}(1-e^{-\sigma_a}))^{-1},$$

$$\mu=\sum_{k=2}^{\infty}\lambda_k\,\varepsilon_{b_k}$$

(where $\varepsilon_b$ is the point mass concentrated at $b$). Then

$$E_0^\mu(1-e^{-\sigma_a})=\sum_{k=2}^{\infty}k^{-2}<1,$$

while

$$\begin{aligned}
P_0^\mu((W_t)\text{ leaves }V) &= \sum_{k=2}^{\infty}\lambda_k\,a_k\,E_0^{b_k}(1-e^{-\sigma_a})\\
&\geqq \sum_{k=2}^{\infty}k^{-1}=\infty. \qquad\square
\end{aligned}$$

The corresponding condition on $(P_0^b)$ under which (vi′) may be replaced by (vi) is that the class of positive measures $n$ satisfying (i), (iii), (iv) and $n(U\setminus U^a) = 0$ consists of either multiples of a single probability measure, or consists completely of infinite measures. It is perhaps worth mentioning that though this condition fails for Example 3, (4.1) is not in general a consequence of this condition. As example we can replace the space $E$ of Example 3 by

$$E' = E \cap \{(x, y);\, y \leqq \sqrt{2x}\}.$$

In this case there are no measures satisfying (i) and (iv), and concentrated on $U^a$, since no path $r = R\cos\theta$, $R \in (1, 2]$ lies entirely within $E'$, whereas by Proposition 2(b), there do exist $n$ satisfying (i), (ii), (iii), (iv), (v), (vi′) but not (ii′).

## 5. Proof of Proposition 1

Condition (ii′) is clearly necessary. Assume that $a$ is instantaneous ($n(U) = \infty$), but (vi′)(c) fails. We will find a set $H \in \mathscr{U}_{0+}$ such that $0 < n(H) < \infty$, and $H \subset U^a \cap \{\sigma_a > 0\}$. Let $H^0 \in \mathscr{U}_{0+}^0$ satisfy $H^0 \subset H$ and $n(H \setminus H^0) = 0$. By completeness of each $\mathscr{F}_t$, we obtain easily that for $h > 0$, the $(U, \mathscr{U}_h^0)$-valued process $(X_{t+}.)$ is progressively measurable, for the filtration $(\mathscr{F}_{t+h})_{t \geqq 0}$. Thus,

$$T = \inf\{t > 0;\, X_{t+}. \in H^0\}$$

is an $(\mathscr{F}_{t+h+})_{t \geqq 0}$ stopping time, for every $h > 0$, so that it is in fact an $(\mathscr{F}_{t+})$ stopping time. $T$ is finite almost surely since $n(H^0) > 0$, and $X_{T+}. \in H^0$ a.s., since $n(H^0) < \infty$. This contradicts the strong Markov property of $(X_t, \mathscr{F}_t, P, P^b)$, as $a$ is instantaneous, yet $X_T = a$ and $\sigma_a(Y_{L_T}) > 0$.

Similarly, in the case that $X$ visits $a$ discretely, but (vi′)(a) fails, we will obtain a contradiction to the strong Markov property of $X$ by finding a set $H \in \mathscr{U}_{0+}$ such that $H \subset U^a \cap \{\sigma_a > 0\}$ and $0 < n(H) < 1$.

In both cases, the argument would be simplified if we had assumed that $X$ was a right process, and we had the Ray-Knight compactification at our disposal.

In the first case, let $D$ be the set of dyadic rational numbers. For $\varepsilon, v > 0$ let

$$B_{\varepsilon,v} = \{b \in E;\, P_0^b(\sigma_a \geqq v) \geqq \varepsilon\} \in \mathscr{E}$$

$$H_{\varepsilon,v} = \bigcup_{\eta > 0} \{W_0 = a,\, W_t \in B_{\varepsilon,v} \text{ for } t \in D \cap (0, \eta),\, \sigma_a > 0\}.$$

We will show that for some $\varepsilon, v > 0$ we have

$$0 < n(H_{\varepsilon,v}) < \infty,$$

so that $H_{\varepsilon,v}$ satisfies the above conditions.

For $\varepsilon$ and $v$ fixed, let

$$\tau(u) = \inf\{t > 0;\, t \in D \text{ and } u(t) \notin B_{\varepsilon,v}\}.$$

Then $\tau$ is a $(\mathscr{U}_{t+})$ stopping time. For $\rho \in D \cap (0, \infty)$, we have $\{\tau > \rho\} \in \mathscr{U}_{\rho+}$, so that by Lemma 6,

$$\begin{aligned} n(\sigma_a \geqq v) &\geqq n(\tau > \rho, \sigma_a \geqq \rho + v) \\ &= \int_{\{\tau > \rho\}} P_0^{u(\rho)}(\sigma_a \geqq v)\, n(du) \\ &\geqq \varepsilon n(\tau > \rho). \end{aligned}$$

Letting $\rho \downarrow 0$, $\rho \in D$ we obtain that $n(\sigma_a \geqq v) \geqq \varepsilon n(H_{\varepsilon, v})$. Thus $n(H_{\varepsilon, v}) < \infty$ for every $\varepsilon, v > 0$.

Conversely, since (vi′)(c) fails, there is some measure $n'$ satisfying (iv) and (v) such that $n \geqq n' \geqq 0$, and $0 < n'(U^a) < \infty$. Fix $\varepsilon, v > 0$ such that $n(H_{\varepsilon, v}) = 0$. Then also $n'(H_{\varepsilon, v}) = 0$. Thus, for $\eta > 0$,

$$n'(W_0 = a, W_t \in B_{\varepsilon, v} \text{ for every } t \in (0, \eta) \text{ such that } t = j2^{-k} \text{ for some } j) \downarrow 0 \text{ as } k \to \infty.$$

That is,

$$\sum_{j=1}^{[2^k \eta - 1]} n'(W_0 = a, W_{j2^{-k}} \notin B_{\varepsilon, v}, W_{i2^{-k}} \in B_{\varepsilon, v} \text{ for } 1 \leqq i < j) \uparrow n'(U^a) \qquad \text{as } k \to \infty.$$

Thus, for $\eta \in (0, v)$

$$\begin{aligned} & n'(W_0 = a, \sigma_a \geqq 2v) \\ &= \lim_{k \to \infty} \sum_{j=1}^{[2^k \eta - 1]} n'(W_0 = a, \sigma_a \geqq 2v, W_{j2^{-k}} \notin B_{\varepsilon, v}, W_{i2^{-k}} \in B_{\varepsilon, v} \text{ for } 1 \leqq i < j) \\ &= \lim_{k \to \infty} \sum_{j=1}^{[2^k \eta - 1]} \int_{\substack{\{W_0 = a, \sigma_a \geqq j2^{-k}, W_{j2^{-k}} \notin B_{\varepsilon, v}, \\ W_{i2^{-k}} \in B_{\varepsilon, v} \text{ for } 1 \leqq i < j\}}} P_0^{u(j2^{-k})}(\sigma_a \geqq 2v - j2^{-k})\, n'(du) \\ &\leqq \varepsilon n'(U^a). \end{aligned}$$

Since $n'(U^a) \in (0, \infty)$, this cannot happen for every $\varepsilon, v > 0$, so that indeed $n(H_{\varepsilon, v}) > 0$ for some $\varepsilon, v$.

In the second case, suppose that $n$ is a probability measure concentrated on $U^a$, yet (vi′)(a) fails. Then we may find $\gamma_1, \gamma_2 \in (0, 1)$ and probability measures $n_1$ and $n_2$, each concentrated on $U^a$ and satisfying (i), (iv) and (v), such that $n_1 \neq n_2$ and $n = \gamma_1 n_1 + \gamma_2 n_2$. Since $n_1 \neq n_2$, we obtain from Lemma 6 that there is an open neighbourhood $V$ of $a$, a set $A \in \mathscr{E}$ with $A \subset E \setminus V$; and numbers $\lambda_1, \lambda_2$ such that

$$n_1(W_\tau \in A) > \lambda_1 > \lambda_2 > n_2(W_\tau \in A),$$

where $\tau(u) = \inf\{t > 0;\ u(t) \notin V\}$. Let $V'$ be an open neighborhood of $a$, with $\bar{V}' \subset V$. Let

$$B = \{b \in V';\ P_0^b(W_\tau \in A) \geqq \lambda_1\}.$$

Let $D$ be the dyadic rationals, and put

$$H = \bigcup_{\eta > 0} \{W_0 = a, \sigma_a > 0, W_t \in B \text{ for every } t \in D \cap (0, \eta)\}.$$

Then as before we obtain that

$$\lambda_2 > n_2(W_\tau \in A) \geqq \lambda_1 n_2(H),$$

so that $n_2(H) < \lambda_2/\lambda_1 < 1$, and hence

$$n(H) = \gamma_1 n_1(H) + \gamma_2 n_2(H) < \gamma_1 + \gamma_2 = 1.$$

Conversely, if $n(H)=0$ then also $n_1(H)=0$, so that as before,

$$\lambda_1 < n_1(W_\tau \in A) \leqq \lambda_1 n_1(U^a) = \lambda_1,$$

which is impossible. □

## 6. Variations on Lemma 7

We turn to the variations of Lemma 7 alluded to in Sect. 2. Let $E$ be a topological space and $\mathscr{E}$ its Borel field. Recall that a function $K(x, dy)$ is called a *kernel*, if for each $x \in E$, $K(x, \cdot)$ is a probability measure on $(E, \mathscr{E})$, and if $K(\cdot, A) \in \mathscr{E}$ whenever $A \in \mathscr{E}$.

**Proposition 3.** *Let $(\Omega, \mathscr{F}, P)$ be a probability space. Let $(Y_t)$ be a càdlàg process with values in $E$ and suppose it is adapted to a filtration $(\mathscr{F}_t)$ of $(\Omega, \mathscr{F})$. Let $K(x, dy)$ be a kernel on $E$. Let $Q$ be a countable ordered set, and for each $q \in Q$, let $S_q$ be an $(\mathscr{F}_t)$ stopping time. Suppose that the following conditions hold:*

*If $q \leqq q'$ then $S_q \leqq S_{q'}$ and $\{S_q \leqq S_{q'}\} \in \mathscr{F}_{S_{q-}}$.*

$$P(Y_{S_q} \in A \mid \mathscr{F}_{S_{q-}}) = K(Y_{S_{q-}}, A) \text{ a.s.,} \quad \text{for } q \in Q,\ A \in \mathscr{E}.$$

*Let $T$ be an $(\mathscr{F}_t)$ stopping time such that*

$$[\![T]\!] \subset \bigcup_{q \in Q} [\![S_q]\!], \quad \{T = S_q\} \in \mathscr{F}_{S_{q-}} \vee \sigma(Y_{S_q}), \quad \text{for } q \in Q.$$

*Then there is an $\mathscr{F}_{T-}$ measurable random subset $H$ of $E$ (that is, $\{(\omega, x); x \in H(\omega)\} \in \mathscr{F}_{T-} \otimes \mathscr{E}$), such that*

$$P(Y_T \in A \mid \mathscr{F}_{T-}) = \frac{K(Y_{T-}, H \cap A)}{K(Y_{T-}, H)} \quad \text{a.s.}$$

*for any $A \in \mathscr{E}$.*

*Note.* If $(Y_t)$ is a standard process, $(\mathscr{F}_t)$ is its natural (right continuous) filtration, $Q \subset [0, \infty)$, and $S_q = q + S_0 \circ \theta_q$, where $S_0$ is a terminal time, then the results of Weil [25], and Walsh and Weil [23] show that such a kernel $K$ exists, and may be expressed in terms of a Lévy system for $(Y_t)$.

*Proof.* Since

$$\{T = S_q\} \in \mathscr{F}_{S_{q-}} \vee \sigma(Y_{S_q}),$$

there is a function $\tilde{f}_q: \Omega \times E \to \{0, 1\}$, measurable with respect to $\mathscr{F}_{S_{q-}} \otimes \mathscr{E}$, with

$$\{T = S_q\} = \{\omega; \tilde{f}_q(\omega, Y_{S_q}(\omega)) = 1\}.$$

Put

$$f_q = \bigvee_r [\tilde{f}_r \wedge 1_{\{S_q = S_r\} \times E}].$$

Then $f_q \in \mathscr{F}_{S_q-} \otimes \mathscr{E}$, since

$$\begin{aligned}\{\tilde{f}_r \wedge 1_{\{S_q=S_r\}\times E} = 1\} &= \{\tilde{f}_r = 1\} \cap \{S_q = S_r\} \times E \in (\mathscr{F}_{S_r-} \otimes \mathscr{E}) \cap (\{S_q = S_r\} \times E)\\ &= (\mathscr{F}_{S_r-} \cap \{S_q = S_r\}) \otimes \mathscr{E}\\ &= (\mathscr{F}_{S_q-} \cap \{S_q = S_r\}) \otimes \mathscr{E} \subset \mathscr{F}_{S_q-} \otimes \mathscr{E},\end{aligned}$$

as $\{S_q = S_r\} \in \mathscr{F}_{S_q-}$. Also, $f_q = f_r$ on $\{S_q = S_r\} \times E$, and

$$\{T = S_q\} = \{\omega; f_q(\omega, Y_{S_q}(\omega)) = 1\}.$$

Put

$$f = \bigvee_q [f_q \wedge 1_{\{S_q = T\} \times E}].$$

Because $f_q = f_r$ on $\{S_q = S_r\} \times E$, it follows that $f = f_q$ on $\{S_q = T\} \times E$. We will show that $f \in \mathscr{F}_{T-} \otimes \mathscr{E}$. This follows as above, once we show that $\{S_q = T\} \in \mathscr{F}_{T-}$. To show this, let $r < q$. Since

$$\{S_r = S_q = T\} \in \mathscr{F}_{S_r-},$$

in fact

$$\{S_r = S_q = T\} \in \mathscr{F}_{S_r-} \cap \{S_r \leqq T\} \subset \mathscr{F}_{T-} \cap \{S_r \leqq T\}.$$

Thus there is a set $\Lambda_r$ in $\mathscr{F}_{T-}$ such that

$$\{S_r = S_q = T\} = \Lambda_r \cap \{S_r \leqq T\}.$$

Set

$$\Lambda = \bigcap_{r<q} [\{S_r < T\} \cup (\{S_r \leqq T\} \cap \Lambda_r)].$$

Since $\Lambda$ and $\{S_q \geqq T\}$ both lie in $\mathscr{F}_{T-}$, we need only show that $\Lambda = \{S_q \leqq T\}$. We obtain the inclusion "$\supset$" since if $r < q$ and $S_q(\omega) \leqq T(\omega)$, but $S_r(\omega) \geqq T(\omega)$, we must have that $S_r(\omega) = S_q(\omega) = T(\omega)$, and hence $\omega \in \Lambda_r$. Conversely, if $S_q(\omega) > T(\omega)$, then there is an $r < q$ with $S_r(\omega) = T(\omega)$. We cannot have $\omega \in \Lambda_r$, hence $\omega \notin \Lambda$ as required.

Let

$$H_q(\omega) = \{y; f_q(\omega, y) = 1\},$$
$$H(\omega) = \{y; f(\omega, y) = 1\}.$$

We have assumed that

$$E[g(\cdot, Y_{S_q}(\cdot))] = \int [\int g(\omega, y) K(Y_{S_q-}(\omega), dy)] P(d\omega)$$

for $g$ of the form $1_{A \times B}$, $A \in \mathscr{F}_{S_q-}$, $B \in \mathscr{E}$. This therefore extends to general $g \in \mathscr{F}_{S_q-} \otimes \mathscr{E}$, $g \geqq 0$. Take

$$g = 1_{A \times B} \cdot f_q,$$

where $A \in \mathscr{F}_{S_q-}$, $B \in \mathscr{E}$. Then

$$\begin{aligned} P(A, Y_T \in B, S_q = T) &= E(g(\cdot, Y_{S_q}(\cdot))] \\ &= \int_A \Big[\int_B f_q(\omega, y)\, K(Y_{S_q-}(\omega), dy)\Big]\, P(d\omega) \\ &= \int_A P(S_q = T \mid \mathscr{F}_{S_q-})(\omega)\, \frac{K(Y_{S_q-}(\omega), B \cap H_q(\omega))}{K(Y_{S_q-}(\omega), H_q(\omega))}\, P(d\omega) \\ &= \int_{A \cap \{S_q = T\}} \frac{K(Y_{S_q-}, B \cap H_q)}{K(Y_{S_q-}, H_q)}\, dP. \end{aligned}$$

Enumerate $Q$ as $q_1, q_2, \ldots$, and let $A \in \mathscr{F}_{T-}$. Then there are $A_q \in \mathscr{F}_{S_q-}$ such that

$$A \cap \{T = S_q\} = A_q \cap \{T = S_q\}.$$

Thus

$$\begin{aligned} P(A, Y_T \in B) &= \sum_j P(A_{q_j}, Y_T \in B, S_{q_j} = T, S_{q_i} \neq S_{q_j} \text{ for } i < j) \\ &= \sum_j \int_{A_{q_j} \cap \{S_{q_j} = T,\, S_{q_i} \neq S_{q_j} \text{ for } i<j\}} K(Y_{S_{q_j}-}, B \cap H_{q_j}) / K(Y_{S_{q_j}-}, H_{q_j})\, dP \\ &= \sum_j \int_{A \cap \{S_{q_j} = T,\, S_{q_i} \neq S_{q_j};\ \text{for } i<j\}} K(Y_{T-}, B \cap H) / K(Y_{T-}, H)\, dP \\ &= E\left[A, \frac{K(Y_{T-}, H \cap B)}{K(Y_{T-}, H)}\right]. \quad \square \end{aligned}$$

A similar result holds in discrete time. For clarity, the proof will be given with a discrete state space, but the general version can easily be obtained by modifying the preceding proof.

**Proposition 4.** *Let $E$ be a countable set, and $(Y_n)_{n \geqq 1}$ an $E$-valued Markov chain with transition probabilities $P(x; dy)$. Let $T$ be a stopping time with respect to the natural filtration of $(Y^n)$. Then there is an $\mathscr{F}_{T-1}$ measurable random subset $H$ of $E$ such that*

$$P(Y_T \in B \mid \mathscr{F}_{T-1}) = \frac{P(Y_{T-1}; B \cap H)}{P(Y_{T-1}; H)}, \qquad \text{for } B \subset E.$$

*Proof.* Atoms of $\mathscr{F}_{T-1}$ are of the form

$$A = \{T = n, Y_1 = y_1, \ldots, Y_{n-1} = y_{n-1}\}.$$

Since $T$ is a stopping time with respect to the natural filtration, it follows that for $y \in E$,

$$\begin{aligned} &\{T = n, Y_1 = y_1 \ldots Y_{n-1} = y_{n-1}\} \cap \{Y_n = y\} \\ &\qquad = \emptyset \quad \text{or} \quad \{Y_1 = y_1, \ldots, Y_{n-1} = y_{n-1}, Y_n = y\}. \end{aligned}$$

Put

$$H(\omega) = \{y; A \cap \{Y_T = y\} \neq \emptyset],$$

where $A$ is the atom of $\mathscr{F}_{T-1}$ containing $\omega$. Thus, for every atom $A$ of $\mathscr{F}_{T-1}$,

$$P(A, Y_T = y) = P(Y_1 = y_1 \ldots Y_{n-1} = y_{n-1})\, P(Y_{T-1}; H \cap \{y\}), \quad \text{on} \quad A$$

so that

$$\frac{P(A, Y_T = y)}{P(A)} = \frac{P(Y_{T-1}; H \cap \{y\})}{P(Y_{T-1}; H)}, \quad \text{on} \quad A. \quad \square$$

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