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A Parabolic Matter-Radiation Model of the Universe

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A study of matter-radiation universes under certain supplementary conditions specified in the introduction shows us that the only model of this class compatible with observations is a parabolic universe which at the present time is almost the same as an Einstein-de Sitter model. The numerical values obtained for Hubble's constant, the age of the universe and the matter density at the present time are quite acceptable. We can also obtain some limits for the mass of neutrinos. The advantage of this parabolic model is that it gives the same results as the $t^{2/3}$ model at the present time and what is more could be used in studying problems of the formation of galaxies, after the recombination epoch, where matter and radiation have comparable importance.

1. INTRODUCTION

We are going to study some models of universes which present both matter (density ρ_m, pressure $p_m = 0$) and radiation characterised by a density ρ_r and a pressure $p_r = (c^2/3)\rho_r$ under the form of a cosmological radiation. We suppose that there is no interaction between these two constituents. Therefore, before the recombination time, when matter and radiation interact, it is only a crude approximation. We use a Robertson-Walker metric and a zero cosmological constant. We have therefore the two relationships

$$\begin{cases} \rho_r R^4 = A \\ \rho_m R^3 = B. \end{cases}$$

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A and B are constants and R is the "radius" of the universe. With $\Lambda = 0$, $p = p_r$, $\rho = \rho_r + \rho_m$, the equations of cosmologies could be written

$$\begin{cases} \dfrac{8\pi G}{3}\rho_m\, R^3 = 2kc^2R + 2R\dot{R}^2 + 2R^2\ddot{R} = \dfrac{8\pi G}{3}B \\ \dfrac{8\pi G}{3}\rho_r\; R^4 = -kc^2R^2 - R^2\dot{R}^2 - 2R^3\ddot{R} = \dfrac{8\pi G}{3}A. \end{cases}$$

Restricting ourselves to an expansion, the resolution of these equations gives us

$$t = \frac{1}{c}\int \frac{R\,dR}{\left(-kR^2 + (8\pi G/3c^2)(BR + A)\right)^{1/2}} \tag{1}$$

2. PARABOLIC MODEL: $k = 0$

2.1. Parametric representation

With our notation, we obtain in this case the equation given by Jacobs in 1967 [1]:

$$B^2\sqrt{\frac{8\pi G}{3}}\,t = \frac{2}{3}(BR + A)^{3/2} - 2A(BR + A)^{1/2} + \frac{4}{3}A^{3/2}.$$

The integration constant is determined by the condition $R = 0$ when $t = 0$. We are going to solve this equation. Let $X = (BR + A)^{1/2}$, we have

$$X^3 - 3AX + 2A^{3/2} - B^2\sqrt{6\pi G}\,t = 0.$$

This is a third degree equation whose discriminant is

$$\Delta = 162\pi GB^4 t\left(t - \frac{4}{\sqrt{6\pi G}}\frac{A^{3/2}}{B^2}\right).$$

(i) – Where

$$\Delta < 0 \quad \left(0 < t < t_1 = \frac{4}{\sqrt{6\pi G}}\frac{A^{3/2}}{B^2}\right)$$

there are three real solutions which are as follows (restricting oneself to the case where R and t increase simultaneously and where R is positive):

1st representation:

$$\begin{cases} R = \dfrac{A}{B}\left(4\cos^2\dfrac{\theta}{3} - 1\right) \\ t = K(1 - \cos\theta) \qquad\qquad \theta \in [2\pi, 3\pi] \end{cases}$$

2nd representation:

$$\begin{cases} R = \dfrac{A}{B}\left(4\cos^2\dfrac{\pi+\theta}{3} - 1\right) \\ t = K(1-\cos\theta) \qquad\qquad \theta \in [4\pi, 5\pi] \end{cases}$$

3rd representation:

$$\begin{cases} R = \dfrac{A}{B}\left(4\cos^2\dfrac{\pi-\theta}{3} - 1\right) \\ t = K(1-\cos\theta) \qquad\qquad \theta \in [0, \pi] \end{cases}$$

where

$$K = \frac{2}{\sqrt{6\pi G}}\frac{A^{3/2}}{B^2}.$$

These three solutions are identical and differ only by the choice of the parameter θ.

(ii) – Where $\Delta = 0$, we have a borderline case.

(iii) – Where $\Delta > 0\,(t > t_1)$ there is only one real solution, which is:

4th representation:

$$\begin{cases} R = \dfrac{A}{B}\left(4\,\mathrm{ch}^2\dfrac{\theta}{3} - 1\right) \\ t = K(1+\mathrm{ch}\theta) \qquad\qquad \theta \geq 0. \end{cases}$$

K has the same expression as previously. The study shows that the $\cos\theta$ and the $\mathrm{ch}\theta$ representations are perfectly identical at the point $R = 3A/B$ and $t = t_1$.

To sum up, the group of parametric equations which describe the total evolution of this model is

$$\begin{cases} R = \dfrac{A}{B}\left(4\cos^2\dfrac{\theta}{3} - 1\right) \\ t = K(1-\cos\theta) \qquad\qquad \theta \in [2\pi, 3\pi] \end{cases}$$

where $R \leq 3A/B$ and $t \leq t_1$ and

$$\begin{cases} R = \dfrac{A}{B}\left(4\mathrm{ch}^2\dfrac{\theta}{3} - 1\right) \\ t = K(1+\mathrm{ch}\theta) \qquad\qquad \theta \geq 0. \end{cases}$$

where $R \geq 3A/B$ and $t \geq t_1$. The expression of the K constant is

$$K = \frac{2}{\sqrt{6\pi G}} \frac{A^{3/2}}{B^2}.$$

2.2 Asymptotic behaviour

2.2.1. For $t \rightarrow 0$

By expanding to the first order the trigonometric functions (and choosing the third representation), we have

$$R \sim \frac{2}{\sqrt{3}} \frac{A}{B} \theta \qquad\qquad t \sim \frac{K}{2} \theta^2$$

given that

$$R = \left(\frac{32\pi GA}{3}\right)^{1/4} t^{1/2}.$$

We can see again therefore, with the singularity vicinity, Weinberg's model of pure radiation.

2.2.2. Where $t \rightarrow \infty$

Then with the fourth representation,

$$R \sim \frac{A}{B} \exp(2\theta/3) \qquad\qquad t \sim \frac{A}{B} \exp\theta$$

given that

$$R = (6\pi GB)^{1/3} t^{2/3}$$

and we come across the Einstein-de Sitter model again.

2.3. Bonds with observations

2.3.1. Calculation of matter density at the present time ($\rho_m o$)

The density of actual radiation (where $T_{ro} \simeq 2.7K$) is

$$\rho_r o \simeq 5 \times 10^{-34} \text{g cm}^{-3}.$$

The hinge instant where we can go from a parametric representation for $\cos\theta$ to the one for $\operatorname{ch}\theta$ is

$$t_1 = \frac{4}{\sqrt{6\pi G}} \frac{A^{3/2}}{B^2} = \frac{4}{\sqrt{6\pi G}} \frac{(\rho_{ro} R_o^4)^{3/2}}{(\rho_{mo} R_o^3)^2} = \frac{4}{\sqrt{6\pi G}} \frac{{\rho_{ro}}^{3/2}}{{\rho_{mo}}^2}.$$

Choosing 10^{-29}g cm^{-3} $\geq \rho_{mo} \geq 10^{-31}$g cm^{-3} we have 1.3×10^4 years $\leq t_1 \leq 1.3 \times 10^8$ years. The present time must therefore be studied with the aid of the fourth representation in $\operatorname{ch}\theta$. Now

$$\rho_r = \frac{A}{R^4} \qquad \text{and} \qquad \rho_m = \frac{B}{R^3}.$$

Therefore

$$\frac{\rho_{mo}}{\rho_{ro}} = \frac{B}{A} R_o = 4\mathrm{ch}^2\theta_o/3 - 1$$

and

$$t_0 = K(1 + \mathrm{ch}\theta_o).$$

Table I thus shows that if we want 1.7×10^{10} years $\leq t_o \leq 2.5 \times 10^{10}$ years, plausible values for the age of the universe, we could have

$$\rho_{mo} \sim (1 \text{ to } 3) \times 10^{-30} \mathrm{g\ cm}^{-3}.$$

2.3.2. Study of the critical point

If we choose the extreme values 10^{10} years $\leq t_o \leq 2.7 \times 10^{10}$ years, we find with the ρ_{mo} values of Table I

$$t_1 \sim 10^4 \text{ years to } 10^6 \text{ years}.$$

In order to study the critical point (index c), that is to say the point where matter and radiation densities are equal, it is therefore necessary to use the first representation and we obtain

$$\frac{\rho_{mc}}{\rho_{rc}} = 4\cos^2\theta_c/3 - 1 = 1 \Rightarrow \theta_c = 495°,$$

a unique value included in the interval $[2\pi, 3\pi]$.

With 10^{10} years $\leq t_o \leq 2.7 \times 10^{10}$ years we have, using Table I

$$2 \times 10^4 \text{ years } \leq t_o \leq 10^6 \text{ years}$$

$$8 \times 10^{-21} \mathrm{g\ cm}^{-3} \leq \rho_c \leq 3 \times 10^{-17} \mathrm{g\ cm}^{-3}$$

$$5500K \leq T_c \leq 44,000K.$$

This shows that the recombination of the matter (at a temperature approximating to 3000K) will definitely happen but only in the dominant matter phase.

2.3.3. Expression of Hubble's constant.

At the present time, we find (fourth representation)

$$H = \frac{\dot{R}}{R} = \frac{4}{3}\sqrt{6\pi G}\,\frac{B^2}{A^{3/2}}\,\frac{\mathrm{ch}\theta/3}{(4\,\mathrm{ch}^3\theta/3 - 1)^2}.$$

Table 1

$\theta_0(rd)$	11	11.5	12	12.5	13	13.5	14	14.5	15
$\rho_{mo}(\mathrm{gcm}^{-3})$	7.7×10^{-31}	1.1×10^{-30}	1.5×10^{-30}	2.1×10^{-30}	2.9×10^{-30}	4.1×10^{-30}	5.7×10^{-30}	7.9×10^{-30}	1.1×10^{-29}
t_0 (yrs.)	3.4×10^{10}	2.7×10^{10}	2.3×10^{10}	2.0×10^{10}	1.7×10^{10}	1.4×10^{10}	1.2×10^{10}	1.0×10^{10}	8.5×10^{9}
H_0 ($\mathrm{kms}^{-1}\mathrm{Mpc}^{-1}$)	20	24	28.5	34	39	48	56	65	76

Note to Table I: We can write, using the fourth representation, the present values of the matter density, the age of the universe and Hubble's constant as a function of the parameter θ_o. These values are very close to the values of the Einstein-de Sitter model. We see in particular that if 1.7×10^{10} years $\leq t_o \leq 2.7\times10^{10}$ years, then $\rho_{mo} \sim (1 \text{ to } 3)10^{-30}\mathrm{g\ cm}^{-3}$. Hence the presence of a missing mass is of little importance.

We give the corresponding numerical values of H_o in Table I.
2.3.4. Deceleration parameter

After simplifying, we obtain

$$q = -\frac{\ddot{R}}{RH^2} = \frac{1}{2} + \frac{1}{8\text{ch}^2\theta/3}.$$

Whatever value of t_o is chosen in acceptable limits one thus finds

$$q_o \# 1/2.$$

First remark:
The equations of cosmologies written in the form

$$\begin{cases} \dfrac{8\pi G}{3}\rho_m = 2\dfrac{\dot{R}^2}{R^2} + 2\dfrac{\ddot{R}}{R} = 2H^2(1-q) > 0 \\ \dfrac{8\pi G}{3}\rho_r = -\dfrac{\dot{R}^2}{R^2} - 2\dfrac{\ddot{R}}{R} = H^2(2q-1) > 0 \end{cases}$$

show us that $q \in]1/2, 1[$.
Second remark:
With the previous expressions of H and q, we can obtain the following relationship:

$$H = \left(\frac{\pi G}{6}\right)^{1/2} \frac{B^2}{A^{3/2}} \frac{(2q-1)^{3/2}}{(1-q)^2}. \tag{2}$$

2.3.5. Mattig's relationship

In Robertson Walker models with $k = 0$ [2], we have

$$r = \int_{te}^{to} \frac{cdt}{R(t)}.$$

In our case, with the fourth representation, we obtain

$$r = c\sqrt{\frac{6}{\pi G}} \frac{A^{1/2}}{B} (\text{ch}\,\theta_o/3 - \text{ch}\,\theta_e/3).$$

However, we can write

$$R = \frac{A}{B}(4\text{ch}^2\theta/3 - 1) = 2\frac{A}{B}\frac{1-q}{2q-1}$$

and, with $R_e = R_o/(1+z)$, we arrive at

$$r = c\sqrt{\frac{3}{2\pi G}}\,\frac{A^{1/2}}{B}\,\frac{1}{(2q_o-1)^{1/2}}\left\{1-\left[\frac{(2q_o-1)z+1}{1+z}\right]^{1/2}\right\}.$$

With the expression (2) we have

$$r\,R_o = \frac{c}{H_o(1-q_o)}\left\{1-\left[\frac{1+(2q_o-1)z}{1+z}\right]^{1/2}\right\},$$

which is a generalization of Mattig's relationships [3] for this model of the universe.

Particular case: $q_o = 1/2$

Then

$$r\,R_o = \frac{2c}{H_o}\left[1-\frac{1}{(1+z)^{1/2}}\right],$$

an expression identical to Mattig's relationship in the Einstein-de Sitter model.

2.3.6. Limits for the mass of neutrinos

If it is true that 1.7×10^{10} years $\leq t_o \leq 2.7 \times 10^{10}$ years then it is necessary that (see Table I):

$$10^{-30}\text{g cm}^{-3} \leq \rho_{mo} \leq 3 \times 10^{-30}\text{g cm}^{-3}.$$

But the present baryonic density is

$$\rho_{Bo} \leq 5 \times 10^{-31}\text{g cm}^{-3}.$$

There is a missing mass. Let us suppose that this mass is made up entirely of neutrinos. We have

$$\rho_{mo} = \rho_{Bo} + \rho_{\nu o}$$

where $\rho_{\nu o}$ is the present density of the neutrinos. We have [4]

$$\rho_{\nu o} = 140\Sigma(m_\nu)$$

with $\Sigma(m_\nu) = m_{\nu e} + m_{\nu\mu} + m_{\nu\tau}$.

We suppose that there are three kinds of neutrinos, ν_e, ν_μ and ν_τ. If we choose the upper limit $\rho_{BO} = 5 \times 10^{-31}$g cm^{-3} we get, with the above values of ρ_{mo},

$$5 \times 10^{-31}\text{g cm}^{-3} \leq \rho_{\nu o} \leq 2.5 \times 10^{-30}\text{g cm}^{-3}$$

$$\text{and} \qquad 2\text{e.V.} \leq \Sigma(m_\nu) \leq 10\text{e.V.}$$

If we consider then that $\rho_{BO} = 10^{-31}\text{g cm}^{-3}$ we have

$$9 \times 10^{-31}\text{g cm}^{-3} \leq \rho_{\nu o} \leq 2.9 \times 10^{-30}\text{g cm}^{-3}$$

$$\text{and} \qquad 3.6\text{e.V.} \leq \Sigma(m_\nu) \leq 11.5\text{e.V.}$$

Let us now suppose that these three kinds of neutrinos have the same mass, $\bar{m}_\nu$, included in the interval $0.6\text{e.V.} < \bar{m}_\nu < 4\text{e.V.}$. We can therefore only have neutrinos of small mass.

2.3.7. Conclusion

The model studied here starts off with a $t^{1/2}$ evolution (pure radiation model) in order to finish with a $t^{2/3}$ model (universe of pure matter). The calculations show that the deceleration parameter at the present time is extremely near 1/2; the Mattig relationship is therefore identical to the relationship which applies in the case of the Einstein-de Sitter universe and these two models can't therefore be distinguished from each other by observations based in this relationship. This is understandable because at the present time the pressure term is negligible in relation to the density term ($p_o \sim 10^{-4}\rho_o c^2$).

On the other hand, this model could be useful in studies after the critical point where matter and radiation are almost equally important and particularly in the study of the phenomenon of increase of perturbations of densities and of the problem of the formation of galaxies. What is more, we can set limits for the mass of possible neutrinos.

3. ELLIPTIC MODEL $k = +1$

3.1. Parametric representation

The resolution of the equations of cosmologies gives here

$$ct = -\sqrt{-R^2 + \frac{8\pi G}{3c^2}(BR + A)} - \frac{4\pi G}{3c^2} B \operatorname{Arcsin}\left[\frac{1 - (3c^2/4\pi GB)R}{(1 + (3c^2/2\pi G)(A/B^2))^{1/2}}\right] + ct_e$$

which is exactly the same expression as that obtained by Cohen [5]. When $R = 0$ and $t = 0$, we can say

$$c(t - t_o) = -\sqrt{-R^2 + \frac{8\pi G}{3c^2}(BR + A)} - \frac{4\pi G}{3c^2} B \operatorname{Arccos}\left[\frac{(3c^2/4\pi GB)R - 1}{(1 + (3c^2/2\pi G)(A/B^2))^{1/2}}\right]$$

where

$$t_o = \frac{1}{c^2}\sqrt{\frac{8\pi G}{3}A} + \frac{4\pi G}{3c^3}\,B\,\mathrm{Arccos}\left[-\left(1+\frac{3c^2}{2\pi G}\frac{A}{B^2}\right)^{1/2}\right].$$

Under this form we can find again Friedmann's elliptic model if we take $A = \rho_r R^4 = 0$ (absence of radiation). By comparing with the equations of Friedmann's model we can take as our parametric representation

$$\begin{cases} R = a(1 - \frac{1}{D}\cos\omega) \\ ct = a(\omega - \frac{1}{D}\sin\omega) + \frac{a}{D}\sqrt{1-D^2} - a\,\mathrm{Arccos}D \end{cases}$$

where $\omega \in [\mathrm{Arccos}D, \pi]$, both R and t increase simultaneously, and R is positive. The constant D satisfies the relationship

$$D = \left(1 + \frac{3c^2}{2\pi G}\frac{A}{B^2}\right)^{-1/2} < 1.$$

We have also

$$a = \frac{4\pi G}{3c^2}B$$

which is the scale factor of Friedmann's elliptic and hyperbolic models, and

$$\frac{1-D^2}{D^2} = \frac{3c^2}{2\pi G}\frac{A}{B^2}.$$

3.2. Asymptotic behaviour

With the new variable $\phi = \omega - \mathrm{Arccos}D$, we will have (when $\phi \to 0$)

$$\begin{cases} R \sim \frac{a}{D}\sqrt{1-D^2}\,\phi \\ ct \sim \frac{a}{2D}\sqrt{1-D^2}\,\phi^2 \end{cases}$$

given that

$$R = \left(\frac{32\pi G A}{3}\right)^{1/4} t^{1/2}$$

which is the $t^{1/2}$ Weinberg model expression.

3.3. Calculation of the constant A/B^2

In the parabolic case we have shown (2.2.1 above) that the expansions restricted to the first order when one approaches the singularity are

$$\begin{cases} R = \dfrac{2}{\sqrt{3}}\dfrac{A}{B}\theta \\ t = \dfrac{K}{2}\theta^2 \; ; \quad K = \dfrac{2}{\sqrt{6\pi G}}\dfrac{A^{3/2}}{B^2}. \end{cases}$$

They allow Weinberg's model to be rediscovered. In the elliptical case, we have

$$\begin{cases} R = \dfrac{a}{D}\sqrt{1-D^2}\,\phi \\ t = \dfrac{a}{2cD}\sqrt{1-D^2}\,\phi^2 \; ; \quad \dfrac{1-D^2}{D^2} = \dfrac{3c^2}{2\pi G}\dfrac{A}{B^2}. \end{cases}$$

They can equally be said to allow the same model of Weinberg in $t^{1/2}$ to be seen again. The parametric representations in these two cases must therefore be identical, and so we have

$$\frac{2}{\sqrt{3}}\frac{A}{B} = \frac{a}{D}\sqrt{1-D^2} \Rightarrow \frac{A}{B^2} = \frac{2\pi G}{c^2}$$

and

$$\frac{K}{2} = \frac{a}{2cD}\sqrt{1-D^2} \Rightarrow \frac{A}{B^2} = \frac{2\pi G}{c^2}.$$

Thus, the $k = 0$ and $k = +1$ models converge to the same Weinberg's model if we have

$$\frac{A}{B^2} = \frac{2\pi G}{c^2} \simeq 4.7 \times 10^{-28}\text{cm g}^{-1}.$$

But

$$\frac{A}{B^2} = \frac{\rho_{ro}{R_o}^4}{(\rho_{mo}{R_o}^3)^2} = \frac{\rho_{ro}}{(\rho_{mo}R_o)^2}.$$

With $\rho_{ro} \simeq 5 \times 10^{-34}$ g cm^{-3} and $\rho_{mo} \sim 10^{-31}$ g cm^{-3}, this leads us to derive $R_o \sim 10^{28}$cm, which is plausible.

3.4. New study of this model

With the value above

$$\frac{A}{B^2} = \frac{2\pi G}{c^2}$$

we can continue our study of the parametric representation. We obtain

$$D = \left(1 + \frac{3c^2}{2\pi G}\frac{A}{B^2}\right)^{-1/2} = \frac{1}{2}$$

and

$$\begin{cases} R = \dfrac{4\pi G}{3c^2} B(1 - 2\cos\omega) \\ ct = \dfrac{4\pi G}{3c^2} B(\omega - (\pi/3) + \sqrt{3} - 2\sin\omega) \ ; \qquad \omega \in [\pi/3, \pi]. \end{cases}$$

So we obtain subsequently

$$\dot{R} = \frac{dR}{dt} = \frac{dR}{d\omega}\frac{d\omega}{dt} = 2c\frac{\sin\omega}{1 - 2\cos\omega}$$

$$H = \frac{\dot{R}}{R} = \frac{3c^3}{2\pi GB}\frac{\sin\omega}{(1 - 2\cos\omega)^2}$$

$$\ddot{R} = \frac{d\dot{R}}{dt} = \frac{d\dot{R}}{d\omega}\frac{d\omega}{dt} = \frac{3c^4}{2\pi GB}\frac{\cos\omega - 2}{(1 - 2\cos\omega)^3}$$

$$q = -\frac{\ddot{R}}{RH^2} = \frac{2 - \cos\omega}{2\sin^2\omega} \ ; \qquad q \in]1, \infty[.$$

3.5. Conclusion

The following relationships hold true:

$$\rho_r R^4 = A \quad \text{and} \quad \rho_m R^3 = B.$$

Therefore

$$\frac{\rho_m}{\rho_r} = \frac{B}{A} R = \frac{4\pi G}{3c^2}\frac{B^2}{A}(1 - 2\cos\omega)$$

and, with the above value of A/B^2, we obtain for the present time

$$\frac{\rho_{mo}}{\rho_{ro}} = \frac{2}{3}(1 - 2\cos\omega_o) \sim \frac{10^{-31}}{5 \times 10^{-34}} = 200,$$

which is impossible. The greatest possible value of this expression is

$$\frac{\rho_{mo}}{\rho_{ro}} = \frac{2}{3}(1 + 2) = 2$$

which is not consistent with the observations. Thus this elliptic model must be rejected.

4. HYPERBOLIC MODEL: $k = -1$

4.1. General information

Cohen [5] discovered a logarithmic solution of eq. (1) where $k = -1$. Here we have written it in another form as ($k = -1$)

$$X = R^2 + \frac{8\pi G}{3c^2} BR + \frac{8\pi G}{3c^2} A.$$

The discriminant of this second degree trinomial is

$$\Delta = \left(\frac{8\pi G}{3c^2} B\right)^2 \left(1 - \frac{3c^2}{2\pi G} \frac{A}{B^2}\right).$$

We are going to move on to the integration of the relationship (1) by distinguishing between the three cases $\Delta = 0$, $\Delta < 0$ and $\Delta > 0$.

4.2. When $\Delta = 0$; $(A/B^2) = (2\pi G/3c^2)$.

We thus obtain

$$ct = R - \frac{4\pi GB}{3c^2} \ln \left(1 + \frac{3c^2}{4\pi GB} R\right),$$

choosing $R = 0$ when $t = 0$, which determines the constant of integration.

4.2.1. Asymptotic behaviour

The expansion of the logarithm term when $R \to 0$ gives

$$R = \left(\frac{8\pi GB}{3c^2}\right)^{1/2} t^{1/2}.$$

If we want to find Weinberg's model again

$$R = \left(\frac{32\pi GA}{3}\right)^{1/4} t^{1/2},$$

the following must be true:

$$\frac{A}{B^2} = \frac{2\pi G}{3c^2},$$

which is compatible with our initial hypothesis. This model is therefore consistent with the $t^{1/2}$ model near the singularity.

For $t \to \infty$, we have the analogue of Milne's universe: $R = ct$.

4.2.2. Study of this model

Let $Y = 1 + (3c^2/4\pi GB)R \geq 1$; when $R \to 0$, $Y \to 1$. We have

$$ct = \frac{4\pi G}{3c^2} B(Y - \ln Y - 1),$$

and therefore

$$\dot{R} = \frac{dR}{dt} = \frac{dR}{dY}\frac{dY}{dt} = c\frac{Y}{Y-1} > 0$$

and

$$H = \frac{\dot{R}}{R} = \frac{3c^3}{4\pi GB}\frac{Y}{(Y-1)^2}.$$

In the same way

$$\ddot{R} = \frac{d\dot{R}}{dt} = \frac{d\dot{R}}{dY}\frac{dY}{dt} = -\frac{3c^4}{4\pi GB}\frac{Y}{(Y-1)^3} < 0$$

then

$$q = -\frac{\ddot{R}}{RH^2} = \frac{1}{Y}\ ; \quad q \in]1, 0[.$$

4.2.3. Comparison with the observations

We have

$$\frac{A}{B^2} = \frac{\rho_{ro}}{\rho_{mo}{}^2 R_o{}^2} = \frac{2\pi G}{3c^2} \simeq 1.6 \times 10^{-28} \mathrm{cm\ g}^{-1}$$

with $\rho_{ro} \simeq 5 \times 10^{-34}$g cm^{-3} and $\rho_{mo} \simeq 10^{-31}$g cm^{-3}, and we obtain $R_o \sim 10^{28}$ cm, which is suitable. As one must admit that $H_o < 100$ km s^{-1} Mpc^{-1}, the calculation shows us that we should get $\rho_{mo} < 2\times 10^{-31}$g cm^{-3} which is incompatible with observational data and the universally admitted existence of dark matter (if we impose $t_o \geq 1.7 \times 10^{10}$ years, we should have $\rho_{mo} \leq 10^{-31}$g cm^{-3}). The case $\Delta = 0$ must therefore be rejected.

4.3. When $\Delta < 0$; $(A/B^2) > (2\pi G/3c^2)$.

4.3.1. General considerations

When we integrate (1) we get

$$c(t - t_o) = \sqrt{R^2 + \frac{8\pi G}{3c^2}(BR + A)} - \frac{4\pi G}{3c^2} B \operatorname{Argsh}\left[\frac{1 + (3c^2/4\pi GB)R}{((3c^2/2\pi G)(A/B^2) - 1)^{1/2}}\right]$$

and

$$t_o = -\frac{1}{c}\sqrt{\frac{8\pi G}{3c^2}A + \frac{4\pi G}{3c^3}B\text{Argsh}\left(\frac{3c^2}{2\pi G}\frac{A}{B^2} - 1\right)^{1/2}}.$$

4.3.2. Parametric representation

This time

$$\begin{cases} R = a\left(\dfrac{1}{D}\text{Sh}\,\omega - 1\right) \\ ct = a\left(\dfrac{1}{D}\text{Ch}\,\omega - \omega\right) + a\,\text{Argsh}\,D - \dfrac{a}{D}\sqrt{D^2+1} \end{cases}$$

with $\omega \in [\text{Argsh}\,D, \infty[$ so that R and t increase simultaneously. The constant D is written thus:

$$D = \left(\frac{3c^2}{2\pi G}\frac{A}{B^2} - 1\right)^{-1/2}$$

and also

$$a = \frac{4\pi G}{3c^2}B$$

$$\frac{1+D^2}{D^2} = \frac{3c^2}{2\pi G}\frac{A}{B^2}.$$

4.3.3. Asymptotic behaviour

4.3.3.1. When $t \to 0$

We change the variable $\phi = \omega - \text{Argsh}\,D$; when $\phi \to 0$, then

$$\begin{cases} R \sim \dfrac{a}{D}\sqrt{D^2+1}\,\phi \\ ct \sim \dfrac{a}{2D}\sqrt{D^2+1}\,\phi^2 \end{cases}$$

so

$$R = \left(\frac{32\pi GA}{3}\right)^{1/4} t^{1/2},$$

which is the expression for Weinberg's model.

4.3.3.2. For $t \to \infty$

Then

$$\begin{cases} R \sim \dfrac{a}{2D}\,e^{\omega} \\ t \sim \dfrac{a}{2cD}\,e^{\omega} \end{cases}$$

so $R = ct$, which is identical to the expression valid for a Milne universe.

4.3.4. Calculation of the constant A/B^2

We proceed in the same way as for the elliptic model (3.3 above). For the limited developments near the singularity, the constants of the parametric representations must be equal when $k = 0$ and $k = -1$. This time

$$\frac{A}{B^2} = \frac{2\pi G}{c^2}.$$

So in order to make sure that the parabolic and hyperbolic models (where $\Delta < 0$) converge as they approach the same Weinbrg model, it is necessary that

$$\frac{A}{B^2} = \frac{2\pi G}{c^2}$$

which agrees with our initial hypothesis.

4.3.5. New study of this model

With the above value of A/B^2, we get

$$D = \left(\frac{3c^2}{2\pi G}\frac{A}{B^2} - 1\right)^{-1/2} = \frac{1}{\sqrt{2}}.$$

Then

$$\begin{cases} R = \dfrac{4\pi G}{3c^2} B(\sqrt{2}\mathrm{Sh}\omega - 1) \\ ct = \dfrac{4\pi G}{3c^2} B[(\sqrt{2}\mathrm{Ch}\omega - \omega + \mathrm{Argsh}(1/\sqrt{2}) - \sqrt{3}] \end{cases}$$

with $\omega \in \mathrm{Argsh}[(1/\sqrt{2}), \infty[$. Subsequently we have

$$\dot{R} = \frac{dR}{dt} = \frac{dR}{d\omega}\frac{d\omega}{dt} = c\sqrt{2}\frac{\mathrm{Ch}\,\omega}{\sqrt{2}\mathrm{Sh}\,\omega - 1}$$

$$H = \frac{\dot{R}}{R} = \frac{3c^3\sqrt{2}}{4\pi GB}\frac{\mathrm{Ch}\,\omega}{(\sqrt{2}\mathrm{Sh}\,\omega - 1)^2}$$

$$\ddot{R} = \frac{d\dot{R}}{dt} = \frac{d\dot{R}}{d\omega}\frac{d\omega}{dt} = \frac{3\sqrt{2}c^4}{4\pi GB}\frac{\mathrm{Sh}\,\omega + \sqrt{2}}{(\sqrt{2}\mathrm{Sh}\,\omega - 1)^3}$$

$$q = -\frac{\ddot{R}}{RH^2} = \frac{1}{\sqrt{2}}\frac{\mathrm{Sh}\,\omega + \sqrt{2}}{\mathrm{Ch}^2\omega}; \qquad q \in]1, 0[.$$

4.3.6. Comparison with the observations

We have

$$\frac{\rho_m}{\rho_r} = \frac{B}{A}R = \frac{4\pi G}{3c^2}\frac{B^2}{A}(\sqrt{2}\mathrm{Sh}\,\omega - 1).$$

So when

$$\frac{A}{B^2} = \frac{\rho_{ro}}{{\rho_{mo}}^2 {R_o}^2} = \frac{2\pi G}{c^2} = 4.7 \times 10^{-28} \text{cm g}^{-1}$$

we obtain $R_o \sim 10^{28}$cm, if $\rho_{mo} \sim 10^{-31}$g cm^{-3}. As before, the calculation demonstrates that to have $H_o < 100\,\text{km s}^{-1}\text{Mpc}^{-1}$, it is necessary that $\rho_{mo} < 10^{-31}$g cm^{-3}, which contradicts the observations (if we impose $t_o \geq 1.7 \times 10^{10}$ years we should have $\rho_{mo} \leq 6 \times 10^{-32}$g cm^{-3}). The case

$$\Delta < 0 \qquad \left(\frac{A}{B^2} > \frac{2\pi G}{3c^2}\right)$$

must therefore be rejected as well.

4.4. When $\Delta > 0$; $(A/B^2) < (2\pi G/3c^2)$

4.4.1. General information

By integrating eq. (1), we get

$$c(t - t_o) = \sqrt{R^2 + \frac{8\pi G}{3c^2}(BR + A)} - \frac{4\pi G}{3c^2} B \text{Argch}\left[\frac{1 + (3c^2/4\pi GB)R}{(1 - (3c^2/2\pi G)(A/B^2))^{1/2}}\right]$$

with

$$t_o = -\frac{1}{c}\sqrt{\frac{8\pi G}{3c^2} A} + \frac{4\pi G}{3c^3} B \text{Argch}\left(1 - \frac{3c^2}{2\pi G}\frac{A}{B^2}\right)^{-1/2}.$$

which with $A = \rho_r R^4 = 0$ amounts to Friedmann's hyperbolic model.

4.3.2.Parametric representation

By comparison with Friedmann's hyperbolic model, we choose

$$\begin{cases} R = a\left(\dfrac{1}{D}\text{Ch}\omega - 1\right) \\ ct = a\left(\dfrac{1}{D}\text{Sh}\omega - \omega\right) - \dfrac{a}{D}\sqrt{D^2 - 1} + a\,\text{Argch}\,D \end{cases}$$

with $\omega \in [\text{Argch}\,D, \infty[$ so that $R \geq 0$ and R and t cross simultaneously. The constant D must satisfy the relationship:

$$D = \left(1 - \frac{3c^2}{2\pi G}\frac{A}{B^2}\right)^{-1/2}.$$

We can just as well have

$$a = \frac{4\pi G}{3c^2} B$$
$$\frac{D^2 - 1}{D^2} = \frac{3c^2}{2\pi G} \frac{A}{B^2}.$$

4.3.3. Asymptotic behaviour

We choose $\phi = \omega - \operatorname{Argch} D$ and, when $\phi \to 0$

$$\begin{cases} R \sim \dfrac{a}{D} \sqrt{D^2 - 1}\, \phi \\ ct \sim \dfrac{a}{2D} \sqrt{D^2 - 1}\, \phi^2 \end{cases}$$

so

$$R = \left(\frac{32\pi G A}{3} \right)^{1/4} t^{1/2}.$$

We arrive at Weinberg's model again. For the same reasons as in 4.3.3.2. above, when $t \to \infty$, we have the analogue of Milne's universe.

4.4.4. Calculation of the constant A/B^2

As in 4.3.4. above, we must have $A/B^2 = 2\pi G/c^2$. But here, by hypothesis

$$\frac{A}{B^2} < \frac{2\pi G}{3c^2} \qquad (\Delta > 0).$$

There is therefore a contradiction and the case $\Delta > 0$ is to be rejected.

4.4.5. Conclusion

In this study of the hyperbolic model, we have shown that the cases $\Delta = 0$ and $\Delta < 0$ are to be rejected, because they are incompatible with the observational data. The case $\Delta > 0$ isn't acceptable because we find a value of the constant A/B^2 which is in contradiction with the initial hypothesis. We can therefore conclude that the hyperbolic model has to be eliminated.

5. OVERALL CONCLUSION

In this article we have studied homogeneous and isotropic universes which satisfy the Robertson-Walker metric with the supplementary condition of a zero cosmological constant. These are models where matter with zero pressure and radiation are present but don't interact with each other. We have shown that the elliptic and hyperbolic models aren't acceptable. We can deduce therefore that, with our hypothesis, the universe is parabolic and infinite. The study of this parabolic model has shown that at the present time we are very close to an Einstein-de Sitter model and at this time can't distinguish between the two. Several years ago, Dabrowski and Stelmach [6] studied open models similar to ours but with a cosmological constant (the closed case was studied by Coquereaux and Grossmann in Ref. 7). For the particular case studied here ($\Lambda = 0$), we can confirm their conclusion that the radiation pressure introduces no essential distrinction with the Friedmann-Lemaître models (where $p_r = 0$). These differences have a solely quantitative character. So we obtain here, with $\Lambda = 0$, open infinite models without inflection point, as in Friedmann's universe. But the quantitative differences seem essential if we try to establish the curvature of the universe, which is indeterminate in the Friedmann models. It is fixed here because the parabolic case is the best choice, among these models, allowed by the observational data.

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