Z. Wahrscheinlichkeitstheorie verw. Gebiete
40, 203–238 (1977)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

Law of the Iterated Logarithm for Sums of Non-Linear Functions of Gaussian Variables that Exhibit a Long Range Dependence*

Murad S. Taqqu
Department of Operations Research, Upson Hall, Cornell University, Ithaca, NY 14853, USA

1. Introduction

In this paper we extend the results of Taqqu (1975) and establish the functional law of the iterated logarithm for interpolated sums of non-Gaussian random variables $G(X_i)$, $i \geqq 1$, that exhibit a long range dependence.

G is such that $G(X_i)$ has mean zero and sufficiently high moments, and $\{X_i, i \geqq 1\}$ is a stationary normalized strongly correlated Gaussian sequence: the correlations $r(k) = EX_1 X_{1+k}$ of $\{X_i, i \geqq 1\}$ decrease to zero like $k^{-D} L(k)$, as $k \to \infty$, where L is a slowly varying function. The exponent $D > 0$ is chosen sufficiently small to ensure that $\operatorname{Var}\left(\sum_{i=1}^{N} G(X_i)\right)$ diverges to infinity, as $N \to \infty$, at a faster rate than N times a slowly varying function. The sequence $\{G(X_i), i \geqq 1\}$ is too strongly dependent for Strassen's functional law of the iterated logarithm to apply. We will show that for a large class of G's, the expansion of the function G in Hermite polynomials leads to the correct law.

We also derive some properties of the self-similar processes associated with the Hermite polynomials; in particular we obtain explicit formulas for the moments of their finite-dimensional distributions. These self-similar processes are related to processes studied by Sinaï (1976) and Dobrushin (1977). They may also be of interest in physics, in the context of the renormalization group approach to critical phenomena. See Ma (1976), Fisher (1974) and Wilson and Kogut (1974) for a description of that approach, and see Jona-Lasinio (1977) for example, for an indication of how limit theorems in probability may be relevant to the renormalization group theory.

The basic definitions and theorems are stated in the following section.

2. Definitions and Statement of the Main Results

Definition 2.1. Let $\frac{1}{2} < H < 1$. A *self-similar process with parameter H* is any real separable process $\{Z(t), t \geqq 0\}$ with stationary increments, satisfying $Z(0) = 0$,

* *AMS 1970 Subject Classification.* Primary: 60F15. Secondary: 33A65, 05C30, 82A25.

and invariant (in law) under the transformations (self-similarity)

$$Z(t) \to c^{-H} Z(c\,t), \qquad c \geqq 0.$$

The sequence $\{W_i = Z(i) - Z(i-1),\ i \geqq 1\}$ is the *self-similar increment process with parameter H*. It is stationary and invariant (in law) under the transformations (block renormalization)

$$W_i \to N^{-H} \sum_{j=iN}^{(i+1)N-1} W_j, \qquad N = 1, 2, \ldots, i \geqq 1.$$

We are interested here in self-similar processes that have finite variances (this restriction excludes the stable processes).

The only Gaussian self-similar process is (up to a scale factor) the *fractional Brownian motion* (FBM) $B_H(t)$. (See Mandelbrot and Van Ness (1968).) $B_H(t)$ is a continuous Gaussian process with stationary increments satisfying $B_H(0) = 0$, $EB_H(t) = 0$ and $EB_H^2(t) = t^{2H}$. Its covariance kernel is

$$\Gamma_H(s, t) = EB_H(s)\, B_H(t) = \tfrac{1}{2}\{s^{2H} + t^{2H} - |s - t|^{2H}\}.$$

Therefore, the only self-similar Gaussian increment process is (up to a scale factor) $\{W_i = B_H(i) - B_H(i-1),\ i \geqq 1\}$. Its correlations are

$$r(k) = EW_i\, W_{i+k} = \tfrac{1}{2}\{|k+1|^{2H} + |k-1|^{2H} - 2|k|^{2H}\},$$

and they satisfy $r(k) \sim H(2H-1)\, k^{2H-2}$ as $k \to \infty$. ($a_k \sim b_k$ means $a_k/b_k \to 1$ as $k \to \infty$.) The long range dependence of $\{W_i\}$ is expressed through this asymptotic power law for $r(k)$, and through the value of the exponent $-1 < 2H - 2 < 0$.

Introduce now the following class $(m)(D, L(\cdot))$ of random sequences.

Definition 2.2. For any positive integer m, $\{X_i, i \geqq 1\} \in (m)(D, L(\cdot))$ if $\{X_i, i \geqq 1\}$ is a mean zero, unit variance, stationary Gaussian sequence satisfying

$$EX_i\, X_{i+k} \sim k^{-D} L(k)$$

as $k \to \infty$, for some given $0 < D < \dfrac{1}{m}$ and some given slowly varying function at infinity $L(s)$, $s \geqq 0$.

Note that $\{X_i, i \geqq 1\}$ is the self-similar Gaussian increment process with parameter $H = 1 - \dfrac{D}{2}$ whenever $0 < D < 1$, $L(s) = H(2H-1)$ and $r(k) = \tfrac{1}{2}\{|k+1|^{2H} + |k-1|^{2H} - 2|k|^{2H}\}$.

The random sequences of interest here are $\{G(X_i), i \geqq 1\}$ where $\{X_i\} \in (m)(D, L(\cdot))$ for some $m \geqq 1$ and where G is a function that satisfies $EG^2(X) < \infty$. (Here and throughout the paper, X denotes an $N(0, 1)$ random variable.) Because $EG^2(X) < \infty$, the function $G(x)$ may be expanded in terms of the Hermite polynomials

$$H_k(x) = (-1)^k e^{\frac{x^2}{2}} \frac{d^k}{dx^k} e^{-\frac{x^2}{2}}, \qquad k = 0, 1, 2, \ldots$$

and the series $\sum_{k=0}^{\infty} \frac{J(k)}{k!} H_k(x)$, with $J(k)=EG(X)H_k(X)$, $k\geqq 0$, converges to $G(x)$ in $L^2\left(\mathbb{R}^1, \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx\right)$. The Hermite polynomials $H_k(x)$ satisfy $EH_k(X)H_q(X)=k!\,\delta_{kq}$. The first few are $H_0(x)=1$, $H_1(x)=x$, $H_2(x)=x^2-1$, $H_3(x)=x^3-3x$. Obviously $EG^2(X)=\sum_{k=0}^{\infty}\frac{J^2(k)}{k!}<\infty$.

The following notion plays a central role throughout the paper.

Definition 2.3. The *Hermite rank* of G is

$$m=\min_{k=0,1,2,\ldots}(J(k)\neq 0).$$

For example, all odd powers have Hermite rank 1. After subtracting their mean $(J(0)\equiv EG(X))$, all even powers have Hermite rank 2. H_m has obviously Hermite rank m.

Introduce now the following classes of functions.

Definition 2.4.

$$\mathscr{G}=\{G\colon EG(X)=0,\ EG^2(X)<\infty\},$$

$$\mathscr{G}_m=\{G\colon G\in\mathscr{G},\ G \text{ has Hermite rank } m\}.$$

Let $\mathscr{G}_\infty=\{G\equiv 0\}$. Note that $\mathscr{G}=\mathscr{G}_\infty\cup\mathscr{G}_1\cup\mathscr{G}_2\cup\cdots$ with $\mathscr{G}_i\cap\mathscr{G}_j=\emptyset$ if $i\neq j$. $\mathscr{G}_0=\emptyset$ because $EG(X)H_0(X)=EG(X)=0$.

Let $m\geqq 1$ and $\{X_i\}\in(m)(D,L(\cdot))$. Then for any $G\in\mathscr{G}_m$

$$\operatorname{Var}\left(\sum_{i=1}^{N}G(X_i)\right)\sim\frac{J^2(m)}{m!}\frac{2}{(1-mD)(2-mD)}N^{-mD+2}L^m(N)$$

as $N\to\infty$. This is a consequence of Lemma 3.1 and Theorem 3.1 of Taqqu (1975) and it provides the normalization factor d_N that will be used in the theorems below. It also indicates that the non-Gaussian sequence $\{G(X_i),\ i\geqq 1\}$ exhibits a long range dependence of a type similar to that of $\{X_i, i\geqq 1\}$. $\left(\text{If } D \text{ were greater than } \frac{1}{m}, \operatorname{Var}\left(\sum_{i=1}^{N}G(X_i)\right) \text{ would be asymptotically proportional to } N.\right)$

Taqqu (1975) studied the weak convergence of

$$\frac{1}{\left(\operatorname{Var}\sum_{i=1}^{N}G(X_i)\right)^{1/2}}\sum_{i=1}^{[Nt]}G(X_i),\qquad 0\leqq t\leqq 1,$$

and showed that to determine the limit, one may replace G by H_m, the m^{th} Hermite polynomial ((weak) reduction theorem). When $m=1$, the limit is $B_H(t)$ with $H=1-\frac{D}{2}$. When $m=2$, it is (up to a scale factor) the non-Gaussian Rosenblatt process $R_D(t)$ defined in Taqqu (1975). These results indicate that the Hermite rank of G is the only attribute of $\{G(X_i), i \geqq 1\}$ that is not lost through block renormalization with $N \to \infty$, and it is that attribute that affects the (weak) limit.

In this paper, we investigate first the strong (i.e. a.s.) behavior of interpolated sums of $G(X_i)$, adequately normalized. The following strong reduction theorem is an important tool.

Theorem 1 (Strong reduction theorem). *Let* $m \geqq 1$, $0 < D < \frac{1}{m}$ *and let* p *be the smallest even integer satisfying* $p > 2\max\left(\frac{1}{D}, \frac{1}{1-mD}\right)$. *Assume* $\{X_i, i \geqq 1\} \in (m)(D, L(\cdot))$ *and let* $d_N^2 \sim N^{-mD+2} L^m(N)$ *as* $N \to \infty$.

Then, for any $G \in \mathscr{G}_m$ *satisfying* $EG^p(X) < \infty$,

$$P\left\{\lim_{N\to\infty} \frac{1}{d_N} \max_{1 \leqq k \leqq N} \left| \sum_{i=1}^{k} \left(G(X_i) - \frac{J(m)}{m!} H_m(X_i)\right)\right| = 0\right\} = 1.$$

A major part of the paper is devoted to the proof of this strong reduction theorem.

Now, for any sequence $\{Y_i, i \geqq 1\}$ of random variables, define the corresponding sequence of *polygonal interpolation functions* $\{Z_N(t), 0 \leqq t \leqq 1, N = 1, 2, \ldots\}$ as

$$Z_N(0) = 0,$$

$$Z_N(t) = \sum_{i=1}^{[Nt]} Y_i + Y_{[Nt]+1}(Nt - [Nt]), \qquad 0 < t \leqq 1.$$

The strong reduction theorem implies that if a functional law of the iterated logarithm holds for the polygonal interpolation functions corresponding to $\{H_m(X_i), i \geqq 1\}$, then a law of the same type holds for polygonal interpolation functions corresponding to $\{G(X_i), i \geqq 1\}$, for any $G \in \mathscr{G}_m$ that satisfies the relatively minor growth restriction

$$\int_{-\infty}^{+\infty} G^p(x)\, e^{-\frac{x^2}{2}}\, dx < \infty.$$

Consider the case $m = 1$. In the appendix, we establish a functional law of the iterated logarithm for polygonal interpolation functions corresponding to the Gaussian sequence $\{X_i, i \geqq 1\}$. Since $H_1(X_i) = X_i$, the strong reduction theorem may be applied to the case $m = 1$, with the following result.

Let $C[0, 1]$ be the space of continuous functions on $[0, 1]$, let

$$d(x, y) = \sup_{0 \leqq t \leqq 1} |x(t) - y(t)|, \qquad x, y \in C[0, 1],$$

and let $d(x, K) = \inf_{y \in K} d(x, y)$ be the distance to $x \in C[0, 1]$ of a subset K of $C[0, 1]$. If $\{x_N, N \geqq 3\}$ is a sequence of elements of $C[0, 1]$, let $C\{x_N\}$ denote the set of their limit points (cluster set).

Theorem 2 ($m = 1$). *Suppose* $0 < D < 1$ *and let* p *be the smallest even integer satisfying* $p > 2\max\left(\frac{1}{D}, \frac{1}{1-D}\right)$. *Assume* $\{X_i\} \in (1)(D, L(\cdot))$ *where* L *is a slowly varying* *function that satisfies*

$$\inf_{[0,s_0]} L(s)>0 \quad and \quad \sup_{[0,s_0]} L(s)<\infty$$

for all $s_0>0$.

Fix $G\in\mathscr{G}_1$ *satisfying* $EG^p(X)<\infty$. *Let* $J(1)=EXG(X)$ *and let* $\{Z_N(t)\}$ *be the sequence of polygonal interpolation functions corresponding to* $\{G(X_i), i\geqq 1\}$.

Then

$$\lim_{N\to\infty} d\left(\frac{Z_N(t)}{\left(\frac{2J^2(1)}{H(2H-1)}N^{2H}L(N)\log\log N\right)^{1/2}}, K_H\right)=0 \text{ a.s.}$$

$$C\left\{\frac{Z_N(t)}{\left(\frac{2J^2(1)}{H(2H-1)}N^{2H}L(N)\log\log N\right)^{1/2}}\right\}=K_H \text{ a.s.}$$

where $H=1-\frac{D}{2}$ *and* K_H *is the unit ball of the reproducing kernel Hilbert space with the reproducing kernel* $\Gamma_H(s,t)$ *of the fractional Brownian motion.*

Refer to the appendix for an interpretation of this result and for a definition of the reproducing kernel Hilbert space.

Functional laws of the iterated logarithm for the cases $m\geqq 2$ have not yet been developed. To establish such laws, it is sufficient to consider polygonal interpolation functions corresponding to $\{H_m(X_i), i\geqq 1\}$ with $\{X_i\}\in(m)(D, L(\cdot))$. However the analysis may be complicated by the fact that the normalization factor may not turn out to be proportional to $(d_N^2 \log\log N)^{1/2}$ with $d_N^2\sim N^{-mD+2}L^m(N)$ as $N\to\infty$.

In the last section of the paper, we develop some properties of the self-similar processes associated with the Hermite polynomials. In particular, we specify their finite-dimensional moments. Assign the Skorokhod topology to $D[0,1]$, the space of function on $[0,1]$ that are right-continuous and have left limits. The main result is

Theorem 3. *Let* $m\geqq 1$, $0<D<\frac{1}{m}$, $p\geqq 2$, $(t_1,\dots,t_p)\in(0,\infty)^p$, $\{X_i\}\in(m)(D,L(\cdot))$ and $d_N^2\sim N^{-mD+2}L^m(N)$ *as* $N\to\infty$.

Then

a) $\displaystyle\lim_{N\to\infty}\frac{1}{d_N^p}\sum_{u_1=1}^{[Nt_1]}\cdots\sum_{u_p=1}^{[Nt_p]} EH_m(X_{u_1})\dots H_m(X_{u_p})$

exists and is equal to

$$\mu_p(t_1,\dots,t_p)=\frac{(m!)^p}{2^{\frac{mp}{2}}\left(\frac{mp}{2}!\right)}\sum\int_0^{t_1}dx_1\dots\int_0^{t_p}dx_p|x_{i_1}-x_{j_1}|^{-D}\dots|x_{i_q}-x_{j_q}|^{-D} \tag{2.1}$$

where $q=\frac{mp}{2}$ *and* $\sum$ *is a sum over all indices* $i_1, j_1, \dots, i_q, j_q$ *such that*

i) $i_1, j_1, \ldots, i_q, j_q \in \{1, 2, \ldots, p\}$,
ii) $i_1 \neq j_1, \ldots, i_q \neq j_q$,
iii) *each number* $1, 2, \ldots, p$ *appears exactly* m *times in* $(i_1, j_1, \ldots, i_q, j_q)$.

b) $$\left(\min_{1\leqq i\leqq p} t_i^2 \max_{1\leqq i\leqq p} t_i^{-mD}\right)^{p/2} EH_m^p(X) \leqq \mu_p \leqq \left(\frac{2}{1-mD}\max_{1\leqq i\leqq p} t_i^{-mD+1}\right)^{p/2} EH_m^p(X).$$

c) $\sum_{p=1}^{\infty} \mu_{2p}^{-1/2p}$ *converges or diverges according to whether* $\sum_{p=1}^{\infty} p^{-m/2}$ *converges or diverges.*

d) *There exists a process* $\bar{Z}_m(t)$, $0\leqq t\leqq 1$, *self-similar with parameter* $H = 1-\frac{mD}{2}$, *a.s. continuous, whose finite-dimensional moments* $E\bar{Z}_m(t_1)\ldots \bar{Z}_m(t_p)$, $p=1, 2, \ldots$, *are* $\mu_1(t_1)=0$ *and* $\mu_p(t_1, \ldots, t_p)$, $p=2, 3, \ldots$.

e) $$\frac{1}{d_N}\sum_{i=1}^{[Nt]} H_m(X_i) \Rightarrow \bar{Z}_m(t)$$

in $D[0,1]$, *as* $N\to\infty$, *when* $m=1$ *or* 2. *The weak convergence holds for a subsequence of* N *when* $m\geqq 3$.

To normalize the self-similar process $\bar{Z}_m(t)$, divide it by

$$(\mu_2(1,1))^{1/2} = \left(\frac{2(m!)}{(1-mD)(2-mD)}\right)^{1/2}.$$

Note that $0<D<\frac{1}{m} \Rightarrow \frac{1}{2}<H=1-\frac{mD}{2}<1$. Note also that $\mu_p=0$ when mp is odd (no indices satisfy the requirements (i), (ii) and (iii) when mp is odd). The $\mu_p(t_1, \ldots, t_p)$'s are the moments of the Gaussian process $\frac{1}{\sqrt{K}} B_H(t)$ $\left(\text{with } K=\frac{2}{(1-D)(2-D)} \text{ and } H=1-\frac{D}{2}\right)$ when $m=1$, and they are the moments of the non-Gaussian Rosenblatt process when $m=2$ (Taqqu (1975)). When $m\geqq 3$, Carleman's condition $\left(\sum_{p=1}^{\infty} \mu_{2p}^{-\frac{1}{2p}} = \infty\right)$ is not satisfied and the moments μ_p, $p\geqq 1$, may be those of more than one distribution. Some μ_p's are evaluated at the end of Section 6.

In Section 3, we establish conditions for $EG(X_1)\ldots G(X_p)$ to admit an expansion in terms of the correlations of the X_i's, and in Section 4 we develop graph theoretic arguments to obtain adequate bounds on the moments. The results of these sections are summarized in Propositions 3.1 and 4.2 – propositions which may be of interest in other contexts as well.

Theorems 1 and 2 are proved in Section 5 and Theorem 3 is proved in Section 6.[1]

---

[1] *A note on the terminology.* The term *self-similar* was introduced by Kolmogorov in the context of turbulence. In our context, *self-affine* would be more adequate since the time and space scales are not transformed in the same way. The self-similar increment processes are called *automodel* by Sinaï. Some authors use also *stable* or *semi-stable* by analogy with the stable distributions. Kolmogorov (1940) first introduced $B_H(t)$. Mandelbrot and Van Ness (1968) coined the term *fractional Brownian motion* and referred to the increments of $B_H(t)$ as the *discrete fractional Gaussian noise.* Taqqu (1975) introduced the term *Rosenblatt process* because it is Murray Rosenblatt who characterized, in 1961, the corresponding one-dimensional distribution. For other interpretations of self-similarity, see Mandelbrot (1977)

3. Expansion of the Moments $EG_1(X_1) \ldots G_p(X_p)$

We will expand the multivariate normal density function $f(x_1, \ldots, x_p)$ in Hermite polynomials and express the moments $EG_1(X_1) \ldots G_p(X_p)$ in terms of those of the Hermite polynomials, or directly, in terms of the correlations of the X_i's. *Formal* results in this direction can be found in Kibble (1945) or Isserlis (1919) respectively. See also Slepian (1972). Our goal here is to specify conditions for absolute and uniform convergence of the expansion of $f(x_1, \ldots, x_p)$ and to determine the validity of the expansion of $EG_1(x_1) \ldots G_p(x_p)$ in cases where the functions G_j, $j=1, \ldots, p$ are not polynomial.

Definition 3.1. Let $p \geqq 2$ and $0 \leqq \varepsilon \leqq 1$. $(X_1, \ldots, X_p)$ is said to be *ε-standard Gaussian* if it possesses a (possibly singular) p-variate normal distribution and if it satisfies $EX_j=0$, $EX_j^2=1$ and $|EX_i X_j| \leqq \varepsilon$ for all $i,j=1, \ldots, p$ with $i \neq j$. $(X_1, \ldots, X_p)$ is *standard Gaussian* if it is 1-standard Gaussian. Finally, $(X_1, X_2, \ldots)$ is ε-standard Gaussian if $(X_1, \ldots, X_p)$ is ε-standard Gaussian for all $p \geqq 2$. Let $r_{ij}=EX_i X_j$, $i,j=1, \ldots, p$, be the correlations, and as usual, let X denote any $N(0,1)$ random variable.

Definition 3.2. Let $p \geqq 2$, $0 \leqq \varepsilon \leqq 1$ and let $G_1, \ldots, G_p$ be functions satisfying $EG_j^2(X) < \infty$, $j=1, \ldots, p$. Let $J_j(k)=EG_j(X)\, H_k(X)$ $(k \geqq 0; j=1, \ldots, p)$ be the corresponding Hermite coefficients.

Then $(G_1, \ldots, G_p) \in \mathscr{G}_p(\varepsilon)$ if and only if

(a) For any ε-standard Gaussian $(X_1, \ldots, X_p)$,

$$EG_1(X_1) \ldots G_p(X_p) = \sum_{q=0}^{\infty} \sum_{\substack{k_1+\cdots+k_p=2q \\ 0 \leqq k_1, \ldots, k_p \leqq q}} \frac{J_1(k_1)}{k_1!} \cdots \frac{J_p(k_p)}{k_p!} EH_{k_1}(X_1) \ldots H_{k_p}(X_p)$$

(b) $$\sum_{q=0}^{\infty} \sum_{\substack{k_1+\cdots+k_p=2q \\ 0 \leqq k_1, \ldots, k_p \leqq q}} \left| \frac{J_1(k_1)}{k_1!} \cdots \frac{J_p(k_p)}{k_p!} EH_{k_1}(X) \ldots H_{k_p}(X) \right| \varepsilon^q < \infty.$$

Note. $G \in \mathscr{G}_p(\varepsilon)$ means $(G_1, \ldots, G_p) \in \mathscr{G}_p(\varepsilon)$ with $G_1 \equiv G_2 \equiv \cdots \equiv G_p \equiv G$.

When $p=2$ and $G_1 \equiv G_2 \equiv G$, the expansion in (a) reduces to

$$EG(X_1)\, G(X_2) = \sum_{q=0}^{\infty} \left(\frac{J(q)}{q!}\right)^2 EH_q(X_1)\, H_q(X_2),$$

and an application of Schwarz inequality proves that $EG^2(X) < \infty$ is equivalent to $G \in \mathscr{G}_2(1)$. When $p > 2$ however, $EG^p(X) < \infty$ does not guarantee $G \in \mathscr{G}_p(1)$. Consider for example the function $G(x) = \frac{1}{2} \operatorname{sgn}(x)$ ($\operatorname{sgn}(x)$ is the sign function; it takes the value 1 when x is positive and -1 otherwise). $G(X)$ is bounded and all its moments are finite. For that G, $J(k)=0$ when k is even and

$$J(k) = (2\pi)^{-1/2} (-1)^{(k-1)/2} (k-1)!!$$

when k is odd. A direct computation shows that neither (a) nor (b) above are satisfied (take for example $p=4$ and $X_1 = X_2 = X_3 = X_4$). Hence that G does not belong to $\mathscr{G}_p(1)$ for any even $p > 2$.

Given $p \geqq 2$, for what values of ε does $EG_j^2(X) < \infty$, $j=1, \ldots, p$ entail $(G_1, \ldots, G_p) \in \mathscr{G}_p(\varepsilon)$? Proposition 3.1 below provides an answer. We first need a few lemmas.

Lemma 3.1. *Let $p \geqq 2$ and suppose $(X_1, \ldots, X_p)$ standard Gaussian. Then*

$$E|H_{k_1}(X_1) \ldots H_{k_p}(X_p)| \leqq \prod_{j=1}^{p} (p-1)^{k_j/2} \sqrt{k_j!}.$$

Proof. Let $N = -\frac{d^2}{dx^2} + x\frac{d}{dx}$. Then for all integers $k \geqq 0$ and any real $t \geqq 0$, $NH_k = kH_k$ and $e^{-tN}H_k = e^{-tk}H_k$. Let $\|\cdot\|_p$ denote the $L^p(\mathbb{R}^1, \mu)$ norm, with $d\mu(x) = (2\pi)^{-1/2} \exp\left(-\frac{x^2}{2}\right) dx$. On one hand,

$$\|e^{-tN}H_k\|_p = \|e^{-tk}H_k\|_p = e^{-tk}\|H_k\|_p.$$

On the other hand, view e^{-tN} as an operator from $L^2(\mathbb{R}^1, \mu)$ to $L^p(\mathbb{R}^1, \mu)$ and let $\|e^{-tN}\|_{2,p}$ be its norm. Choose t such that $e^{-t} = (p-1)^{-1/2}$. Then $\|e^{-tN}\|_{2,p} \leqq 1$ (Gross (1975), Corollary 4.1). For such a t,

$$\|e^{-tN}H_k\|_p \leqq \|e^{-tN}\|_{2,p}\|H_k\|_2 \leqq \|H_k\|_2 = \sqrt{k!}.$$

Hence,

$$\|H_k\|_p \leqq e^{tk}\sqrt{k!} = (p-1)^{k/2}\sqrt{k!}.$$

Use now the multidimensional Hölder's inequality to obtain

$$E|H_{k_1}(X_1) \ldots H_{k_p}(X_p)| \leqq \prod_{j=1}^{p} (E|H_{k_j}(X)|^p)^{1/p} \leqq \prod_{j=1}^{p} (p-1)^{k_j/2}\sqrt{k_j!}.$$

The bound in Lemma 3.1 is sharp. In fact, when $p=3$ and $k_1 = k_2 = k$ (even), the bound becomes $2^{3k/2}(k!)^{3/2}$, whereas $EH_k^3(X) = (k!)^3 \left(\frac{k}{2}!\right)^{-3} \sim Ck^{-3/4}2^{3k/2}(k!)^{3/2}$ as $k \to \infty$ for some constant C. The bound therefore includes the exponential trend. If, on the other hand, $k_1 = \cdots = k_p = k$ stays fixed, but p tends to infinity through even values of kp, then $EH_k^p(X) \sim EX^{kp} \sim Ck^{-p/4}(p-1)^{kp/2}(k!)^{p/2}$ for some other constant C, and in this case again, the bound includes the exponential trend.

Lemma 3.2. *Let $p \geqq 2$ and $(X_1, \ldots, X_p)$ be standard Gaussian. Then*

$$EH_{k_1}(X_1) \ldots H_{k_p}(X_p) = \begin{cases} \frac{k_1! \ldots k_p!}{2^q(q!)} \sum_1 r_{i_1 j_1} r_{i_2 j_2} \ldots r_{i_q j_q} \\ \quad \text{if } k_1 + \cdots + k_p = 2q \quad \text{and} \quad 0 \leqq k_1, \ldots, k_p \leqq q \\ 0 \quad \text{otherwise} \end{cases}$$

where $\sum_1$ is a sum over all indices $i_1, j_1, i_2, j_2, \ldots, i_q, j_q$ that satisfy

i) $i_1, j_1, i_2, j_2, \ldots, i_q, j_q \in \{1, 2, \ldots, p\}$,
ii) $i_1 \neq j_1, i_2 \neq j_2, \ldots, i_q \neq j_q$,
iii) *there are k_1 indices 1, k_2 indices 2, $\ldots$, k_p indices p.*

Proof. The moment generating function of $(X_1, \ldots, X_p)$ is

$$E\exp\left(\sum_{j=1}^{p} t_j X_j\right) = \exp\left(\frac{1}{2}\sum_{j=1}^{p} t_j^2\right)\exp\left(\frac{1}{2}\sum_{\substack{i=1\\i\neq j}}^{p}\sum_{j=1}^{p} r_{ij} t_i t_j\right)$$

and therefore

$$E\prod_{j=1}^{p}\exp\left(t_j X_j - \frac{1}{2}t_j^2\right) = \exp\left(\frac{1}{2}\sum_{\substack{i=1\\i\neq j}}^{p}\sum_{j=1}^{p} r_{ij} t_i t_j\right).$$

Since $e^{tx - t^2/2}$ is the generating function of the Hermite polynomials $H_q(x)$, $q \geqq 0$, we get

$$E\prod_{j=1}^{p}\sum_{q_j=0}^{\infty}\frac{(t_j)^{q_j}}{q_j!}H_{q_j}(X_j) = \sum_{q=0}^{\infty}\frac{1}{2^q(q!)}\left(\sum_{\substack{i=1\\i\neq j}}^{p}\sum_{j=1}^{p} r_{ij} t_i t_j\right)^q. \tag{1}$$

But

$$\left(\sum_{\substack{i=1\\i\neq j}}^{p}\sum_{j=1}^{p} r_{ij} t_i t_j\right)^q = \sum_{\substack{i_1=1\\i_1\neq j_1}}^{p}\sum_{j_1=1}^{p}\cdots\sum_{\substack{i_q=1\\i_q\neq j_q}}^{p}\sum_{j_q=1}^{p} r_{i_1 j_1}\ldots r_{i_q j_q} t_1^{k_1}\ldots t_p^{k_p}$$

where k_1 is the number of indices of the r's that are equal to $1, \ldots$, and k_p is the number of those indices equal to p. Note that $k_1 + \cdots + k_p = 2q$ and $0 \leqq k_1, \ldots, k_p \leqq q$. In fact,

$$\left(\sum_{\substack{i=1\\i\neq j}}^{p}\sum_{j=1}^{p} r_{ij} t_i t_j\right)^q = \sum_{\substack{k_1+\ldots+k_p=2q\\0\leqq k_1,\ldots,k_p\leqq q}}\left\{\sum\nolimits_1 r_{i_1 j_1}\ldots r_{i_q j_q}\right\} t_1^{k_1}\ldots t_p^{k_p}.$$

Set $|t| = \max(|t_1|, \ldots, |t_p|)$ and use Lemma 3.1, to show that

$$\sum_{q_1=0}^{\infty}\cdots\sum_{q_p=0}^{\infty}\frac{|t_1|^{q_1}}{q_1!}\cdots\frac{|t_p|^{q_p}}{q_p!}E|H_{q_1}(X_1)\ldots H_{q_p}(X_p)| \leqq \left(\sum_{q=0}^{\infty}\frac{|t_p|^{q_p}}{q!}(p-1)^{q/2}\sqrt{q!}\right)^p < \infty.$$

We may therefore take the expectation under the summation signs in (1) and express (1) as

$$\sum_{s=0}^{\infty}\sum_{q_1+\cdots+q_p=s}\left\{E\frac{H_{q_1}(X_1)}{q_1!}\cdots\frac{H_{q_p}(X_p)}{q_p!}\right\}t_1^{q_1}\ldots t_p^{q_p}$$
$$= \sum_{q=0}^{\infty}\sum_{\substack{k_1+\cdots+k_p=2q\\0\leqq k_1,\ldots,k_p\leqq q}}\left\{\frac{1}{2^q q!}\sum\nolimits_1 r_{i_1 j_1}\cdots r_{i_q j_q}\right\}t_1^{k_1}\ldots t_p^{k_p}.$$

An identification of the coefficients of the t's concludes the proof.

We shall now obtain an expansion for the multivariate normal density function. Suppose that $(X_1, \ldots, X_p)$, $p \geqq 2$, is standard Gaussian. Let $x' = (x_1, \ldots, x_p)$ denote a row vector, let I be the $p \times p$ identity matrix, R, the $p \times p$ variance-covariance matrix of $(X_1, \ldots, X_p)$, $|R|$ the determinant of R, and finally let μ_j, $j = 1, \ldots, p$ be the eigenvalues of $R - I$. When $\mu_j > -1$, $j = 1, \ldots, p$ the distribution of $(X_1, \ldots, X_p)$ is non-singular and is characterized by the multivariate density function

$$f(x_1, \ldots, x_p) = (2\pi)^{-p/2}|R|^{-1/2}\exp\{-\tfrac{1}{2}x' R^{-1} x\}.$$

Lemma 3.3. *Suppose* $|\mu_j|<1, j=1, \ldots, p$. *Then*

$$f(x_1, \ldots, x_p) = \sum_{q=0}^{\infty} \sum_{\substack{k_1+\cdots+k_p=2q \\ 0\leq k_1, \ldots, k_p \leq q}} (2\pi)^{-p/2} \left\{\frac{H_{k_1}(x_1)}{k_1!} \cdots \frac{H_{k_p}(x_p)}{k_p!}\right\} \cdot \exp\left(-\frac{1}{2}\sum_{j=1}^{p} x_j^2\right) EH_{k_1}(X_1) \ldots H_{k_p}(X_p).$$

The series converges absolutely and uniformly in $\mathbb{R}^p$.

Proof. Let $F_q(x_1, \ldots, x_p)$ denote the q-th term of the expansion, and let

$$P_q(t_1, \ldots, t_p) = \int e^{it'x} F_q(x_1, \ldots, x_p)\, d^p x$$

be its p-dimensional Fourier transform. The integration is over $\mathbb{R}^p$, and $t'x = \sum_{j=1}^{p} t_j x_j$. Since the Fourier transform in $\mathbb{R}^1$ of $(2\pi)^{-1/2} H_k(x)\exp(-\frac{1}{2}x^2)$ is $(it)^k \exp(-\frac{1}{2}t^2)$,

$$\begin{aligned} P_q(t_1, \ldots, t_p) &= \sum_{\substack{k_1+\cdots+k_p=2q \\ 0\leq k_1, \ldots, k_p \leq q}} (-1)^q \left(\frac{t_1^{k_1}}{k_1!} \cdots \frac{t_p^{k_p}}{k_p!}\right) \exp\left(-\frac{1}{2}\sum_{j=1}^{p} t_j^2\right) EH_{k_1}(X_1) \ldots H_{k_p}(X_p) \\ &= \sum_{\substack{k_1+\cdots+k_p=2q \\ 0\leq k_1, \ldots, k_p \leq q}} \frac{(-1)^q}{2^q q!} (t_1^{k_1} \ldots t_p^{k_p}) \exp\left(-\frac{1}{2}\sum_{j=1}^{p} t_j^2\right) \sum\nolimits_1 r_{i_1 j_1} \ldots r_{i_q j_q} \end{aligned}$$

where $\sum_1$ is defined as in Lemma 3.2. Hence

$$P_q(t_1, \ldots, t_p) = \frac{(-1)^q}{2^q q!} \left(\sum_{\substack{i=1 \\ i\neq j}}^{p} \sum_{j=1}^{p} r_{ij} t_i t_j\right)^q \exp\left(-\frac{1}{2}\sum_{j=1}^{p} t_j^2\right).$$

Therefore,

$$\begin{aligned} f(x_1, \ldots, x_p) &= (2\pi)^{-p} \int e^{-it'x} \exp(-\tfrac{1}{2} t' R t)\, d^p t \\ &= (2\pi)^{-p} \int e^{-it'x} \sum_{q=0}^{\infty} \frac{(-1)^q}{2^q q!} (t'(R-I)t)^q \exp\left(-\frac{1}{2}\sum_{j=1}^{p} t_j^2\right) d^p t \\ &= (2\pi)^{-p} \int e^{-it'x} \sum_{q=0}^{\infty} P_q(t_1, \ldots, t_p)\, d^p t \\ &= \sum_{q=0}^{\infty} F_q(x_1, \ldots, x_p) \end{aligned}$$

provided term by term integration is legitimate. To see that it is, note that

$$\begin{aligned} \int \sum_{q=0}^{\infty} |P_q(t_1, \ldots, t_p)|\, d^p t &= \sum_{q=0}^{\infty} \frac{1}{2^q q!} \int |t'(R-I)t|^q \exp\left(-\frac{1}{2}\sum_{j=1}^{p} t_j^2\right) d^p t \\ &= \sum_{q=0}^{\infty} \frac{1}{2^q q!} \int \left|\sum_{j=1}^{p} \mu_j s_j^2\right|^q \exp\left(-\frac{1}{2}\sum_{j=1}^{p} s_j^2\right) d^p s \\ &\leq \sum_{q=0}^{\infty} \sum_{\substack{v_1+\cdots+v_p=q \\ v_1, \ldots, v_p \geq 0}} \prod_{j=1}^{p} \frac{|\mu_j|^{v_j}}{2^{v_j} v_j!} \int_{-\infty}^{+\infty} s_j^{2v_j} \exp(\tfrac{1}{2} s_j^2)\, ds_j \end{aligned}$$

$$= \sum_{q=0}^{\infty} \sum_{\substack{v_1+\cdots+v_p=q\\ v_1,\ldots,v_p \geqq 0}} \prod_{j=1}^{p} |\mu_j|^{v_j} \frac{(2v_j)!}{2^{2v_j}(v_j!)^2}$$

$$\leqq \prod_{j=1}^{p} \left\{ \sum_{v=0}^{\infty} |\mu_j|^v \, O(v^{-1/2}) \right\}$$

$$< \infty.$$

Lemma 3.4. *Let* $p \geqq 2$, $0 \leqq \varepsilon < \frac{1}{p-1}$, *and suppose that* $(X_1, \ldots, X_p)$ *is* ε*-standard Gaussian. Then the eigenvalues* $\mu_j, j=1, \ldots, p$ *of* $R-I$ *satisfy* $|\mu_j| < 1$.

Proof. Let $(R-I)^+$ be $(R-I)$ with each element replaced by its absolute value and let A be the $p \times p$ matrix whose off-diagonal elements are equal to ε and whose diagonal elements are equal to 0. Then $(R-I)^+ \leqq A$ in the sense that $A-(R-I)^+$ has non-negative elements. Lemma 2, page 57 of Gantmacher (1959) ensures that all $|\mu_j|$, $j=1, \ldots, p$, are bounded by λ^*, the maximal eigenvalue of A. They are also bounded by 1 since $\lambda^* = (p-1)\varepsilon < 1$.

Proposition 3.1. *Let* $p \geqq 2$, $X \sim N(0,1)$ *and let* $G_1, \ldots, G_p$ *be functions satisfying* $EG_j^2(X) < \infty$, $j=1, \ldots, p$. *Let* $J_j(k) = EG_j(X)\, H_k(X)$, $k \geqq 0$, *be their Hermite coefficients. Finally let* $(X_1, \ldots, X_p)$ *be* ε*-standard Gaussian. Suppose either*

(i) $\varepsilon = 1$ *and* $\sum_{k=0}^{\infty} \frac{J_j(k)}{\sqrt{k!}} (p-1)^{k/2} < \infty, \quad j=1, \ldots, p$

or

(ii) $0 \leqq \varepsilon < \frac{1}{p-1}$.

Then $(G_1, \ldots, G_p) \in \bar{\mathscr{G}}_p(\varepsilon)$.

Proof. We must show that (a) and (b) of Definition 3.2 hold. First suppose (i).

$$\sum_{q=0}^{\infty} \sum_{k_1+\cdots+k_p=2q} \left| \frac{J_1(k_1)}{k_1!} \cdots \frac{J_p(k_p)}{k_p!} \right| E|H_{k_1}(X_1) \ldots H_{k_p}(X_p)|$$

$$\leqq \prod_{j=1}^{p} \left\{ \sum_{k_j=0}^{\infty} \frac{|J_j(k_j)|}{\sqrt{k_j!}} (p-1)^{k_j/2} \right\} < \infty$$

by Lemma 3.1. (b) holds because we may choose $X_1 = \cdots = X_p = X$. The dominated convergence theorem for example, ensures that (a) holds.

Now suppose (ii). Applying Lemmas 3.4 and 3.3, we get

$$EG_1(X_1) \ldots G_p(X_p) = \int_{-\infty}^{+\infty} \ldots \int_{-\infty}^{+\infty} G_1(x_1) \ldots G_p(x_p) f(x_1, \ldots, x_p)\, dx_1 \ldots dx_p$$

$$= \sum_{q=0}^{\infty} \sum_{\substack{k_1+\cdots+k_p=2q\\ 0 \leqq k_1, \ldots, k_p \leqq q}} \left\{ \prod_{j=1}^{p} \frac{1}{k_j!} EG_j(X)\, H_{k_j}(X) \right\} EH_{k_1}(X_1) \ldots H_{k_p}(X_p)$$

$$= \sum_{q=0}^{\infty} \sum_{\substack{k_1+\cdots+k_p=2q\\ 0 \leqq k_1, \ldots, k_p \leqq q}} \left\{ \prod_{j=1}^{p} \frac{1}{k_j!} J_j(k_j) \right\} EH_{k_1}(X_1) \ldots H_{k_p}(X_p)$$

proving (a). The term by term integration was legitimate because by Schwarz inequality,

$$K \equiv \sum_{q=0}^{\infty} \sum_{\substack{k_1+\dots+k_p=2q \\ 0 \leqq k_1,\dots,k_p \leqq q}} \left\{ \prod_{j=1}^{p} \frac{1}{k_j!} E|G_j(X) H_{k_j}(X)| \right\} |EH_{k_1}(X_1) \dots H_{k_p}(X_p)|$$

$$\leqq C \sum_{q=0}^{\infty} \sum_{\substack{k_1+\dots+k_p=2q \\ 0 \leqq k_1,\dots,k_p \leqq q}} (k_1! \dots k_p!)^{-1/2} |EH_{k_1}(X_1) \dots H_{k_p}(X_p)|$$

with $C=(EG_1^2(X) \dots EG_p^2(X))^{1/2}<\infty$. To show that $K<\infty$, we apply Lemmas 3.2 and 3.1. They yield

$$|EH_{k_1}(X_1) \dots H_{k_p}(X_p)| \leqq \varepsilon^{1/2(k_1+\dots+k_p)} EH_{k_1}(X) \dots H_{k_p}(X)$$

$$\leqq \prod_{j=1}^{p} (\varepsilon(p-1))^{k_j/2} \sqrt{k_j!}.$$

Hence, $K \leqq C \left(\sum_{k=0}^{\infty} (\varepsilon(p-1))^{k/2} \right)^p < \infty$. The preceding computation also establishes that (b) holds. This concludes the proof.

The condition $\sum_{k=0}^{\infty} \frac{J(k)}{\sqrt{k!}} (p-1)^{k/2} < \infty$ is automatically satisfied when G is a polynomial. When $p>2$, it is also satisfied for some special functions G, e.g., $G(x)=e^{ax}$ (a real) or $G(x)=\exp\{\frac{1}{2}ax^2\}$ where $0<a<\frac{1}{p}$. In the latter case for example, $J(k)=0$ if k is odd and $J(2k)=(1-a)^{-1/2}\frac{(2k)!}{2^k k!}\left(\frac{a}{1-a}\right)^k$, so that $\frac{J(2k)}{\sqrt{(2k)!}}(p-1)^k = O(k^{-1/4}\alpha^k)$ as $k \to \infty$ for some $\alpha<1$. The condition

$$\sum_{k=0}^{\infty} \frac{J(k)}{\sqrt{k!}} (p-1)^{k/2} < \infty$$

may not be satisfied even when G is bounded, for example when $G(x)=\frac{1}{2}\operatorname{sgn}(x)$. For that example, even (a) and (b) of Definition 3.2 fail, as we have seen earlier. These facts are apparently related to certain properties of the solutions of the heat equation (see Rosenbloom and Widder (1959) for a characterization of these solutions).

Condition (ii) of Proposition 3.1 replaces the restrictions on the functions $G_j, j=1,\dots,p$ stated in condition (i), by restrictions on the size of the correlations. This second approach is more suited to our purposes.

4. Graph-Theoretic Arguments

We shall characterize the expansion of the moments of $(G_1,\dots,G_p) \in \mathscr{G}_p(\varepsilon)$ in terms of multigraphs and use graph-theoretic arguments to obtain adequate bounds on $E\left(\sum_{i=1}^{N} G(X_i)\right)^p$. Since there is no universally accepted graph-theoretic terminology, we shall start with a series of definitions, some of them non-standard. (For a general reference, see Busacker and Saaty (1965) or Harary (1969).)

A *multigraph* $A=(V, E, \Phi)$ consists of a non-empty set of points (vertices) V, a (possibly empty) set of lines (edges) E, and a mapping Φ of E into $V\times V$. The mapping Φ is usually implicit and therefore we merely denote the multigraph by $A=(V, E)$. $p=|V|\geqq 1$ denotes the number of points of A and $q=|E|\geqq 0$ denotes its number of lines. A *labeled* (point-labeled) multigraph has its points distinguished from one another by names. Its lines are not labeled. Hence for us, two labeled multigraphs are identical if for all i and j, the number of lines joining the points labeled i and j is the same in both multigraphs.

For each $p\geqq 1$, let $\mathscr{A}_p$ be the set of all labeled multigraphs with p points, and whose lines do not form loops. (A *loop* is a line that starts and ends at the same point.) However, a multigraph in $\mathscr{A}_p$ may possess *multiple lines*: more than one line may join any two points. $A\in\mathscr{A}_p$ may also be described through its *pair sequence* $\{(i_1, j_1), (i_2, j_2), \dots, (i_q, j_q)\}$, where for each $s=1, \dots, q$, we have $i_s, j_s\in\{1, \dots, p\}$ with $i_s\neq j_s$; a pair (i_s, j_s) symbolizes the existence of a line that joins point i_s to point j_s. Identical pairs indicate the presence of multiple lines. The requirement $i_s\neq j_s$ for all $s=1, \dots, p$ indicates the absence of loops. A line and a point labeled i are *incident* if one of the endpoints of the line is i. Two lines are *adjacent* if they have a point in common. A *path* is a sequence of adjacent lines.

With each multigraph $A\in\mathscr{A}_p$, associate a *multiplicity number* $g(A)$ defined as follows. Number each of the $\binom{p}{2}$ possible pairs of points by $u=1, 2, \dots, \binom{p}{2}$. Let v_u be the number of lines in A joining the pair of points numbered u. If A has q lines, then obviously $v_1+\dots+v_{\binom{p}{2}}=q$. Define then

$$g(A)=\prod_{u=1}^{\binom{p}{2}} \frac{1}{v_u!}.$$

The *degree* of a point is the number of lines incident to that point. Let k_i denote the degree of the point i. Obviously, $k_1+\dots+k_p=2q$.

Let $\mathscr{A}(k_1, \dots, k_p)$ denote the set of all the multigraphs of $\mathscr{A}_p$ whose p points have degrees $k_1, \dots, k_p$ respectively.

Proposition 4.1. *Let* $p\geqq 2$, $0\leqq\varepsilon\leqq 1$, *and suppose* $EG_j^2(X)<\infty$, $j=1, \dots, p$. *Let* $J_j(k)=EG_j(X)H_k(X)$ $(k\geqq 0; j=1, \dots, p)$ *be the Hermite coefficients. Then* $(G_1, \dots, G_p)\in\bar{\mathscr{G}}_p(\varepsilon)$ *if and only if*

(a) *for any ε-standard Gaussian* $(X_1, \dots, X_p)$,

$$EG_1(X_1)\dots G_p(X_p)=\sum_{q=0}^{\infty}\sum_{\substack{k_1+\dots+k_p=2q\\ 0\leqq k_1, \dots, k_p\leqq q}} J_1(k_1)\dots J_p(k_p)\sum_{A\in\mathscr{A}(k_1, \dots, k_p)} g(A)\,R(A)$$

with $R(A)=r_{i_1 j_1} r_{i_2 j_2}\dots r_{i_q j_q}$, *where* $(i_1, j_1), (i_2, i_2), \dots, (i_q, j_q)$ *is the pair sequence of the multigraph* A. ($R(A)=1$ *if* A *has* $q=0$ *lines.*)

(b) $\sum_{q=0}^{\infty} \sum_{\substack{k_1+\cdots+k_p=2q \\ 0 \leqq k_1, \ldots, k_p \leqq q}} |J_1(k_1) \ldots J_p(k_p)| \sum_{A \in \mathscr{A}(k_1, \ldots, k_p)} g(A) \, \varepsilon^q < \infty.$

Proof. Let $p \geqq 2$, $q \geqq 1$, $k_1 + \cdots + k_p = 2q$, $0 \leqq k_1, \ldots, k_p \leqq q$, and let $\sum_1$ be the summation defined as in Lemma 3.2.

Orderings $(i_1, j_1, \ldots, i_q, j_q)$ that differ by a permutation of the indices within a pair (i, j) are covered by the summation $\sum_1$. Their number is 2^q. Orderings that differ by a permutation between different pairs (i, j) are also covered by $\sum_1$. Their number is $\frac{q!}{v_1! \, v_2! \ldots}$ where $v_1, v_2, \ldots$ are the numbers of identical pairs in these orderings. We now interpret $(i_1, j_1), \ldots, (i_q, j_q)$ as the pair sequence of a multigraph A with p points, q lines, and $k_1, \ldots, k_p$, as the respective degrees of the p points of A. Therefore

$$\frac{1}{2^q(q!)} \sum\nolimits_1 r_{i_1 j_1} \ldots r_{i_q j_q} = \sum_{A \in \mathscr{A}(k_1, \ldots, k_p)} g(A) \, R(A).$$

Use Lemma 3.2 to obtain

$$E H_{k_1}(X_1) \ldots H_{k_p}(X_p) = k_1! \ldots k_p! \sum_{A \in \mathscr{A}(k_1, \ldots, k_p)} g(A) \, R(A).$$

Use the definition 3.2 of $\bar{\mathscr{G}}_p(\varepsilon)$ to conclude.

The following corollaries follow from the proof of the proposition.

Corollary 4.1. *Let* $A \in \mathscr{A}_{(k_1, \ldots, k_p)}$ *and let* $(i_1, j_1), \ldots, (i_q, j_q)$ *be its pair sequence. Set* $F(A) = f(i_1, j_1; \ldots; i_q, j_q)$. *Then*

$$\frac{1}{2^q(q!)} \sum\nolimits_1 f(i_1, j_1; \ldots; i_q, j_q) = \sum_{A \in \mathscr{A}(k_1, \ldots, k_p)} g(A) \, F(A).$$

Corollary 4.2. $E H_{k_1}(X_1) \ldots H_{k_p}(X_p) = k_1! \ldots k_p! \sum_{A \in \mathscr{A}(k_1, \ldots, k_p)} g(A) \, R(A).$

The following notions play a central role in the sequel. A subgraph of $A = (V, E)$ is a multigraph (V', E') with $V' \subset V$, $E' \subset E$, and whose pair sequence is a subset of that of A. A multigraph is *connected* if every pair of points is joined by a path. A *component* is a maximal connected subgraph. The *rank* of A, $r(A)$, is the number of points of A minus its number of components and the *cycle number* of A is $|E| - r(A)$.

A *forest* $F = (V_F, E_F)$ is a graph with cycle number zero, hence $r(F) = |E_F|$. A connected forest is a *tree*. Hence, if $T = (V_T, E_T)$ is a tree, $r(T) = |E_T| = |V_T| - 1$. $F = (V_F, E_F)$ is a *spanning forest* of $A = (V, E)$ if it is a subgraph of A and if, by adding to E_F a line of E not already contained in E_F, one increases the cycle number of F. Thus $r(A) = r(F) = |E_F|$ because A and F have the same number of points and the same number of components.

The following lemma is a reformulation in terms of multigraphs of a theorem on matroids (Theorem 2b, p. 150) established by Edmonds and Fulkerson (1965).

Lemma 4.1 (Edmonds-Fulkerson). *Let $A=(V,E)\in\mathscr{A}_p$ for some $p\geqq 1$. Then E contains the lines of m line-disjoint forests $F_i=(V,E_i)$, $i=1,\dots,m$, with prescribed sizes $|E_i|\leqq r(A)$, if and only if, for every subgraph $D=(V,E_D)$ with $E_D\subset E$ as line set,*

$$|\bar{E}_D|\geqq\sum_{i=1}^{m}\{|E_i|-\min(|E_i|,r(D))\}$$

($\bar{E}_D$ denotes the complement of E_D in E).

Lemma 4.2. *Let $A=(V,E)\in\mathscr{A}_p$ be a multigraph with $|V|\equiv p\geqq 2$ points such that α points $(0\leqq\alpha\leqq p)$ have a degree at least equal to $m\geqq 1$.*

Then there are m forests $F_i=(V,E_i)$, $i=1,\dots,m$, with $E_i\cap E_j=\emptyset$ $(i,j=1,\dots,m;$ $i\neq j)$, $\bigcup_{i=1}^{m}E_i\subset E$, such that

$$|E_i|=\begin{cases}\left[\dfrac{\alpha}{2}\right] & \text{if } 1\leqq i\leqq\left[\dfrac{m}{2}\right]\\[2ex] \alpha-\left[\dfrac{\alpha}{2}\right] & \text{if } \left[\dfrac{m}{2}\right]+1\leqq i\leqq m.\end{cases}$$

Remark. As a consequence, the multigraph $B=\left(V,\bigcup_{i=1}^{m}E_i\right)$ is a subgraph of A that has p points and

$$\sum_{i=1}^{m}|E_i|=m\alpha-\left[\frac{m\alpha}{2}\right]$$

lines. Formally, $B=\bigcup_{i=1}^{m}F_i$. The structure of the forests F_i is quite simple. When α is even for example, each forest F_i has p points, $\alpha/2$ lines and thus exactly $p-\alpha/2$ components.

Proof of Lemma 4.2. Let $E_D\subset E$ be arbitrary and let $V_D\subset V$ be the set of points that the lines in E_D are incident to. $D=(V_D,E_D)$ is then a subgraph of $A=(V,E)$.

Since $m\geqq 1$, A has at most $p-\alpha+\left[\frac{\alpha}{2}\right]$ components, thus $r(A)\geqq\alpha-\left[\frac{\alpha}{2}\right]$. Therefore $|E_i|\leqq r(A)$ for $i=1,\dots,m$, and Lemma 4.1 applies. It is then sufficient to prove that

$$|\bar{E}_D|\geqq m\alpha-\left[\frac{m\alpha}{2}\right]-\sum_{i=1}^{m}\min(|E_i|,r(D)).$$

The inequality holds trivially if $r(D)\geqq\alpha-\left[\frac{\alpha}{2}\right]$ since $|\bar{E}_D|\geqq 0$. Suppose $r(D)<\alpha-\left[\frac{\alpha}{2}\right]$. It is then sufficient to establish that

$$|\bar{E}_D|\geqq m\alpha-\left[\frac{m\alpha}{2}\right]-mr(D).$$

We shall evaluate separately $r(D)$ and $|\bar{E}_D|$. Since each point of D has a line in E_D incident to it, D can have at most $[\frac{1}{2}|V_D|]$ components. Therefore

$$r(D)=|V_D|-\#\text{ of components of } D\geqq|V_D|-[\tfrac{1}{2}|V_D|]. \tag{1}$$

Note that (1) and the assumption $r(D)<\alpha-\left[\frac{\alpha}{2}\right]$ imply $\alpha-|V_D|>0$.

Now consider $|\bar{E}_D|$. Let $\bar{V}_D$ be the complementary set to V_D in V. $\bar{V}_D$ contains at least $\alpha-|V_D|$ points that have degrees in A greater or equal to m. The sum of the degrees in A of all the points of $\bar{V}_D$ is at least $m(\alpha-|V_D|)$. Only lines in $\bar{E}_D$ are incident to the points of $\bar{V}_D$. But some lines in $\bar{E}_D$ may connect a point in $\bar{V}_D$ to a point in V_D. Therefore

$$\begin{aligned} m(\alpha-|V_D|) &\leqq \sum \text{degrees in } A \text{ of points in } \bar{V}_D \\ &= 2\,(\text{lines with both ends in } \bar{V}_D)+1\,(\text{lines from } V_D \text{ to } \bar{V}_D) \\ &\leqq 2|\bar{E}_D|. \end{aligned} \tag{2}$$

First suppose that $m\alpha$ is even. (1) yields $r(D)\geqq\frac{|V_D|}{2}$ and therefore (2) becomes

$$|\bar{E}_D|\geqq\frac{m\alpha}{2}-mr(D)=m\alpha-\left[\frac{m\alpha}{2}\right]-mr(D).$$

Suppose now that $m\alpha$ is odd, that is both m and α odd. If (2) holds with a strict equality, then $|V_D|$ must be odd. In that case (1) yields $r(D)\geqq\frac{|V_D|}{2}+\frac{1}{2}$ and therefore

$$|\bar{E}_D|=\frac{m\alpha}{2}-\frac{m|V_D|}{2}\geqq\frac{m\alpha}{2}+\frac{1}{2}-mr(D)=m\alpha-\left[\frac{m\alpha}{2}\right]-mr(D).$$

If (2) holds with a strict inequality, then we may use $r(D)\geqq\frac{|V_D|}{2}$ to get

$$|\bar{E}_D|\geqq\frac{m\alpha}{2}-\frac{m|V_D|}{2}+\frac{1}{2}\geqq\frac{m\alpha}{2}+\frac{1}{2}-mr(D)=m\alpha-\left[\frac{m\alpha}{2}\right]-mr(D).$$

This completes the proof.

Definition 4.1. Let $\{r(k), k=0, \pm 1, \pm 2, \ldots\}$ be a sequence of real numbers satisfying $r(k)=r(-k)$. Fix $p\geqq 1$, and let $u_1, u_2, \ldots, u_p$ be positive integers. For each multigraph $A\in\mathscr{A}_p$ that has p points labeled $\{1, \ldots, p\}$, $q\geqq 0$ lines, and the pair sequence $(i_1, j_1), (i_2, j_2), \ldots, (i_q, j_q)$, define

$$R_{(u_1, u_2, \ldots, u_p)}(A)=r(u_{i_1}-u_{j_1})\, r(u_{i_2}-u_{j_2}) \ldots r(u_{i_q}-u_{j_q}).$$

By convention, $R_{(u_1, u_2, \ldots, u_p)}(A)=1$ if $q=0$. Each $r(u)$ will be identified with $EX_i X_{i+u}$ whenever a *stationary* standard Gaussian process $\{X_i, i\geqq 1\}$ is introduced. In that case, $R(A)$ defined in proposition 4.1 is $R_{(u_1, \ldots, u_p)}(A)$ with $(u_1, \ldots, u_p)=(1, \ldots, p)$.

Definition 4.2. Let $\mathscr{T}_p$ be the set of all trees in $\mathscr{A}_p$.

Lemma 4.3. *For all* $m\geqq 1$, $p\geqq 1$, $N\geqq 1$,

$$\sup_{T\in\mathscr{T}_p}\sum_{u_1=1}^{N}\sum_{u_2=1}^{N}\cdots\sum_{u_p=1}^{N}|R^m_{(u_1, u_2, \ldots, u_p)}(T)|\leqq N\left\{\sum_{u=-N}^{N}|r^m(u)|\right\}^{p-1}.$$

Proof. Since $T \in \mathcal{T}_p$ is a tree, it has 1 component, p points (labeled $1, \ldots, p$), $p-1$ lines, a cycle number zero, and hence no loops. To each of the $p-1$ lines we can associate a different point in such a way that the line is incident to the point with which it is associated. The *root* of the tree is the point with which no line is associated and any point may be chosen as the root. We may suppose, without loss of generality, that the root is the point labeled $i=1$ and that the other points are labeled in such a way that

$$\sum_{u_1=1}^{N} \sum_{u_2=1}^{N} \cdots \sum_{u_p=1}^{N} |R^m_{(u_1, u_2, \ldots, u_p)}(T)|$$
$$= \sum_{u_1=1}^{N} \sum_{u_2=1}^{N} |r^m(u_2 - u_{j_2})| \sum_{u_3=1}^{N} |r^m(u_3 - u_{j_3})| \ldots \sum_{u_p=1}^{N} |r^m(u_p - u_{j_p})|$$

where (i, j_i), for $i = 2, \ldots, p$ and $j_i < i$, represents the endpoints of the line associated with the point i (another labeling would lead to a different ordering of the summation signs in the right hand side of the expression). But for any integer $1 \leqq v \leqq N$,

$$\sum_{u=1}^{N} |r^m(u-v)| \leqq \sum_{u=-N}^{N} |r^m(u)|.$$

Hence

$$\sum_{u_1=1}^{N} \cdots \sum_{u_p=1}^{N} |R^m_{(u_1, \ldots, u_p)}(T)| \leqq \sum_{u_1=1}^{N} \prod_{i=2}^{p} \left\{ \sum_{u_i=-N}^{N} r^m(u_i) \right\}$$
$$= N \left\{ \sum_{u=-N}^{N} |r^m(u)| \right\}^{p-1}.$$

Definition 4.3. Let $\mathcal{M}_{p,q}(\alpha, m)$ denote the set of those multigraphs of $\mathcal{A}_p$ that have q lines and for which a number $0 \leqq \alpha \leqq p$ of their p points have a degree at least equal to m.

Lemma 4.4. *Let $m \geqq 1$, $p \geqq 2$, $0 \leqq \alpha \leqq p$ and $q \geqq m\alpha - \left[\frac{m\alpha}{2}\right]$. Then*

$$\sup_{A \in \mathcal{M}_{p,q}(\alpha, m)} \sum_{u_1=1}^{N} \sum_{u_2=1}^{N} \cdots \sum_{u_p=1}^{N} |R_{(u_1, u_2, \ldots, u_p)}(A)|$$
$$\leqq \{\sup_{u \geqq 0} |r(u)|\}^{q - (m\alpha - [\frac{m\alpha}{2}])} \left\{ N^{p - \frac{\alpha}{2}} \left(\sum_{u=-N}^{N} |r^m(u)| \right)^{\frac{\alpha}{2}} \right\} \left\{ \frac{1}{N} \sum_{u=-N}^{N} |r^m(u)| \right\}^{\frac{1}{m}(\frac{m\alpha}{2} - [\frac{m\alpha}{2}])}.$$

Proof. Let $A = (V, E) \in \mathcal{M}_{p,q}(\alpha, m)$. There are m forests $F_i = (V, E_i)$, $i = 1, \ldots, m$ with $E_i \cap E_j = \emptyset$ $(i, j = 1, \ldots, m;\ i \neq j)$, $\bigcup_{i=1}^{m} E_i \subset E$, and $\sum_{i=1}^{m} |E_i| = m\alpha - \left[\frac{m\alpha}{2}\right]$ (Lemma 4.2). Let c_i be the number of components of F_i. $\sum_{i=1}^{m} c_i = mp - m\alpha + \left[\frac{m\alpha}{2}\right]$ because $c_i = p - r(F_i)$ and $\sum_{i=1}^{m} r(F_i) = \sum_{i=1}^{m} |E_i|$.

Let $B = \bigcup_{i=1}^{m} F_i = \left(V, \bigcup_{i=1}^{m} E_i\right)$. Since $\bigcup_{i=1}^{m} E_i \subset E$ and $|E| - \left|\bigcup_{i=1}^{m} E_i\right| = q - \left(m\alpha - \left[\frac{m\alpha}{2}\right]\right)$,

$$|R_{(u_1,\dots,u_p)}(A)| \leqq \{\sup_{u\geqq 0}|r(u)|\}^{q-(m\alpha-[\frac{m\alpha}{2}])} \left| R_{(u_1,\dots,u_p)}\left(\bigcup_{i=1}^m F_i\right)\right|$$

for all $1\leqq u_1,\dots,u_p\leqq N$. Using a generalized Hölder inequality,

$$\sum_{u_1=1}^N\cdots\sum_{u_p=1}^N \left|R_{(u_1,\dots,u_p)}\left(\bigcup_{i=1}^m F_i\right)\right| \leqq \prod_{i=1}^m\left\{\sum_{u_1=1}^N\cdots\sum_{u_p=1}^N |R^m_{(u_1,\dots,u_p)}(F_i)|\right\}^{1/m}.$$

Each of the m forests F_i has p points and c_i components T_{ij}, $j=1,\dots,c_i$. The T_{ij}'s are trees. Let $p_{ij}\geqq 1$ be the number of points of T_{ij}. Obviously $\sum_{j=1}^{c_i} p_{ij}=p$ for all $i=1,\dots,m$. The T_{ij}, $j=1,\dots,c_i$, have disjoint point sets and line sets. Therefore

$$\sum_{u_1=1}^N\cdots\sum_{u_p=1}^N |R^m_{(u_1,\dots,u_p)}(F_i)| = \prod_{j=1}^{c_i}\left\{\sum_{u_1=1}^N\cdots\sum_{u_{p_{ij}}=1}^N |R^m_{(u_1,\dots,u_p)}(T_{ij})|\right\}$$

after an obvious relabeling of the points. Note that each T_{ij} belongs to the set $\mathscr{T}_{p_{ij}}$ defined earlier. Using the previous inequalities in conjunction with Lemma 5.3, we obtain

$$\begin{aligned}
&\sum_{u_1=1}^N\cdots\sum_{u_p=1}^N \left|R_{(u_1,\dots,u_p)}\left(\bigcup_{i=1}^m F_i\right)\right| \\
&\quad\leqq\left\{\prod_{i=1}^m\prod_{j=1}^{c_i}\sup_{T\in\mathscr{T}_{p_{ij}}}\sum_{u_1=1}^N\cdots\sum_{u_{p_{ij}}=1}^N |R^m_{(u_1,\dots,u_p)}(T)|\right\}^{1/m} \\
&\quad\leqq\left\{\prod_{i=1}^m\prod_{j=1}^{c_i}\left(N\left(\sum_{u=-N}^N |r^m(u)|\right)^{p_{ij}-1}\right)\right\}^{1/m} \\
&\quad=\left\{N^{mp-m\alpha+[\frac{m\alpha}{2}]}\left(\sum_{u=-N}^N |r^m(u)|\right)^{m\alpha-[\frac{m\alpha}{2}]}\right\}^{1/m} \\
&\quad=\left\{N^{p-\frac{\alpha}{2}}\left(\sum_{u=-N}^N |r^m(u)|\right)^{\frac{\alpha}{2}}\right\}\left\{\frac{1}{N}\sum_{u=-N}^N |r^m(u)|\right\}^{\frac{1}{m}(\frac{m\alpha}{2}-[\frac{m\alpha}{2}])}
\end{aligned}$$

because $\sum_{j=1}^{c_i} p_{ij}=p$ and $\sum_{i=1}^m c_i=mp-m\alpha+\left[\frac{m\alpha}{2}\right]$. This concludes the proof.

When $p\geqq 2$, let $\sum'^{N}_{u_1,\dots,u_p=1}$ denote the summation over all $(u_1,\dots,u_p)$, $1\leqq u_i\leqq N$, $i=1,\dots,p$, with the restriction that no two u_i's assume identical values.

Lemma 4.5. *Let $m\geqq 1$, $p\geqq 2$, $0\leqq\varepsilon\leqq 1$, and let $(X_1,\dots,X_N)$, $N\geqq 1$ be stationary ε-standard Gaussian with $r(u)=EX_iX_{i+u}$. Let G_j, $j=1,\dots,p$ be functions satisfying $EG_j^2(X)<\infty$ and $(G_1,\dots,G_p)\in\bar{\mathscr{G}}_p(\varepsilon)$. Assume also that for some $0\leqq\alpha\leqq p$, the functions $G_1,\dots,G_\alpha$ have a Hermite rank at least equal to m. Then there is a finite constant $K=K(p,\varepsilon,G_1,\dots,G_p)$ such that, for all $N\geqq 1$,*

$$\sum_{u_1,\dots,u_p=1}^{N}{}' |EG_1(X_{u_1})\dots G_p(X_{u_p})|$$

$$\leqq K\left\{N^{p-\frac{\alpha}{2}}\left(\sum_{u=1}^{N}|r^m(u)|\right)^{\frac{\alpha}{2}}\right\}\left\{\frac{1}{N}\sum_{u=1}^{N}|r^m(u)|\right\}^{\frac{1}{m}\left\{\frac{m\alpha}{2}-\left[\frac{m\alpha}{2}\right]\right\}}.$$

Proof. Let $J_j(k)=EG_j(X)\,H_k(X)$, $k\geqq 0$, $j=1,\dots,p$. Proposition 4.1 holds since $(G_1,\dots,G_p)\in\bar{\mathscr{G}}_p(\varepsilon)$. Using part (a) of that proposition with the fact that $G_1,\dots,G_\alpha$ have a Hermite rank greater or equal to m, we obtain

$$\sum_{u_1,\dots,u_p=1}^{N}{}' |EG_1(X_{u_1})\dots G_p(X_{u_p})|$$

$$\leqq \sum_{q=m\alpha-\left[\frac{m\alpha}{2}\right]}^{\infty}\ \sum_{\substack{k_1+\dots+k_p=2q\\ m\leqq k_1,\dots,k_\alpha\leqq q\\ 0\leqq k_{\alpha+1},\dots,k_p\leqq q}} |J_1(k_1)\dots J_p(k_p)|\left\{\sum_{A\in\mathscr{A}(k_1,\dots,k_p)} g(A)\right\}$$

$$\cdot\left\{\sup_{A\in\mathscr{M}_{p,q}(\alpha,m)}\ \sum_{u_1,\dots,u_p=1}^{N}{}' |R_{(u_1,\dots,u_p)}(A)|\right\}.$$

$(X_1,\dots,X_N)$ is ε-standard Gaussian and therefore $\sup_{u\geqq 1}|r(u)|\leqq\varepsilon$. $R_{(u_1,\dots,u_p)}(A)$ is a product of terms of the form $r(u_i-u_j)$ with $u_i,u_j\in\{u_1,\dots,u_p\}$. The term $r(0)$ never appears, since under $\sum'$, no two $u_1,\dots,u_p$ are ever equal. For convenience, set temporarily $r(0)=0$. Then

$$\sup_{A\in\mathscr{M}_{p,q}(\alpha,m)}\sum_{u_1,\dots,u_p=1}^{N}{}' |R_{(u_1,\dots,u_p)}(A)|=\sup_{A\in\mathscr{M}_{p,q}(\alpha,m)}\sum_{u_1=1}^{N}\dots\sum_{u_p=1}^{N}|R_{(u_1,\dots,u_p)}(A)|$$

$$\leqq 2^\alpha\,\varepsilon^{q-\left(m\alpha-\left[\frac{m\alpha}{2}\right]\right)}\left\{N^{p-\frac{\alpha}{2}}\left(\sum_{u=1}^{N}|r^m(u)|\right)^{\frac{\alpha}{2}}\right\}\left\{\frac{1}{N}\sum_{u=1}^{N}|r^m(u)|\right\}^{\frac{1}{m}\left(\frac{m\alpha}{2}-\left[\frac{m\alpha}{2}\right]\right)}$$

using Lemma 4.4. Conclude the proof by setting

$$K(p,\varepsilon,G_1,\dots,G_p)$$

$$\equiv 2^\alpha\sum_{q=m\alpha-\left[\frac{m\alpha}{2}\right]}^{\infty}\ \sum_{\substack{k_1+\dots+k_p=2q\\ 0\leqq k_1,\dots,k_p\leqq q}}|J_1(k_1)\dots J_p(k_p)|\sum_{A\in\mathscr{A}(k_1,\dots,k_p)} g(A)\,\varepsilon^{q-\left(m\alpha-\left[\frac{m\alpha}{2}\right]\right)}.$$

$K=K(p,\varepsilon,G_1,\dots,G_p)$ is finite by part (b) of Proposition 4.1.

Proposition 4.2. *Let $m\geqq 1$, $p\geqq 2$, $0\leqq\varepsilon\leqq 1$, and let $\{X_i, i\geqq 1\}$ be stationary ε-Gaussian with $r(u)=EX_iX_{i+u}$. Let G be a function satisfying $E(G(X))^{2(p-[p/2])}<\infty$, $G\in\bar{\mathscr{G}}_p(\varepsilon)$, and whose Hermite rank is at least m.*

Then

i) *there is a finite constant $K_1(p,\varepsilon,G)$ such that, for all $N\geqq 1$,*

$$\left|E\left(\sum_{i=1}^{N}G(X_i)\right)^p\right|\leqq K_1(p,\varepsilon,G)\left\{N\sum_{u=0}^{N}|r^m(u)|\right\}^{\frac{p}{2}}\left\{\frac{1}{N}\sum_{u=0}^{N}|r^m(u)|\right\}^{\frac{1}{m}\left\{\frac{mp}{2}-\left[\frac{mp}{2}\right]\right\}},$$

ii) *if $p>2$, there is a finite constant K_2 such that, for all $N \geqq 1$,*

$$E\left\{\max_{1 \leqq k \leqq N}\left|\sum_{i=1}^{k} G(X_i)\right|\right\}^p \leqq K_2\left\{N \sum_{u=0}^{N}|r^m(u)|\right\}^{p/2}.$$

K_2 is independent of N, but may depend on p, ε, G and on the joint distribution of the X_i's.

Proof. 1. To establish the first part of the proposition, note that

$$\left|E\left(\sum_{i=1}^{N} G(X_i)\right)^p\right| = \left|\sum_{\substack{k_1+\cdots+k_N=p \\ k_1, \ldots, k_N \geqq 0}} \frac{p!}{k_1! \ldots k_N!} E G^{k_1}(X_1) \ldots G^{k_N}(X_N)\right|$$

$$= \left|\sum_{p'=1}^{p} \sum_{1 \leqq u_1<\ldots<u_{p'} \leqq N} \sum_{\substack{k_{u_1}+\cdots+k_{u_{p'}}=p \\ k_{u_1}, \ldots, k_{u_{p'}} \geqq 1}} \frac{p!}{k_{u_1}! \ldots k_{u_{p'}}!} E G^{k_{u_1}}(X_{u_1}) \ldots G^{k_{u_{p'}}}(X_{u_{p'}})\right|$$

$$\leqq C(p) \sum_{p'=1}^{p} \max_{\substack{v_1+\cdots+v_{p'}=p \\ v_1, \ldots, v_{p'} \geqq 1}} \sum_{u_1, \ldots, u_{p'}=1}^{N}{}' \left|E G^{v_1}(X_{u_1}) \ldots G^{v_{p'}}(X_{u_{p'}})\right|$$

where $C(p)$ is a finite constant independent of N and where, under $\sum'$, no two u_i's in $(u_1, \ldots, u_{p'})$ assume identical values.

Suppose $p' \leqq \left[\frac{p}{2}\right]$. Then by Hölder's inequality,

$$\max_{\substack{v_1+\cdots+v_{p'}=p \\ v_1, \ldots, v_{p'} \geqq 1}} \sum_{u_1, \ldots, u_{p'}=1}^{N}{}' \left|E G^{v_1}(X_{u_1}) \ldots G^{v_{p'}}(X_{u_{p'}})\right| \leqq N^{p'} E|G(X)|^p$$

$$\leqq E|G(X)|^p\left\{N \sum_{u=0}^{N}|r^m(u)|\right\}^{\frac{p}{2}}\left\{\frac{1}{N} \sum_{u=0}^{N}|r^m(u)|\right\}^{\frac{1}{m}\left\{\frac{mp}{2}-\left[\frac{mp}{2}\right]\right\}}$$

Fix now $p' \geqq \left[\frac{p}{2}\right]+1$. Then

$$\max_{\substack{v_1+\cdots+v_{p'}=p \\ v_1, \ldots, v_{p'} \geqq 1}} \sum_{u_1, \ldots, u_{p'}=1}^{N}{}' \left|E G^{v_1}(X_{u_1}) \ldots G^{v_{p'}}(X_{u_{p'}})\right|$$

$$= \max_{\alpha_{\min} \leqq \alpha \leqq \alpha_{\max}} \max_{\substack{v_{\alpha+1}+\cdots+v_{p'}=p-\alpha \\ v_{\alpha+1}, \ldots, v_{p'} \geqq 2}} \sum_{u_1, \ldots, u_{p'}=1}^{N}{}' \left|E G(X_{u_1}) \ldots G(X_{u_\alpha}) G^{v_{\alpha+1}}(X_{u_{\alpha+1}}) \ldots G^{v_{p'}}(X_{u_{p'}})\right|$$

where

$$\alpha_{\min}=2p'-p \quad \text{and} \quad \alpha_{\max}= \begin{cases} p'-1 & \text{if } p'<p \\ p' & \text{if } p'=p. \end{cases}$$

Note that $\alpha_{\min} \geqq 1$ and $\alpha_{\max} \leqq p'$.

Choose any values of α and of the exponents $v_{\alpha+1}, \ldots, v_{p'}$ that satisfy the above constraints. For convenience set $G_i(\cdot)=G(\cdot)$, $i=1, \ldots, \alpha$ and $G_i(\cdot)=G^{v_i}(\cdot)$,

$i=\alpha+1, \ldots, p'$. Note that $\max(v_{\alpha+1}, \ldots, v_{\alpha_{p'}}) \leqq p-\left[\frac{p}{2}\right]$ (when $p>2$, this value is reached when $p'=\left[\frac{p}{2}\right]+1$ and $\alpha=\alpha_{\max}\equiv p'-1$, making $v_{p'}=p-\alpha=p-\left[\frac{p}{2}\right]$). The assumption $E(G(X))^{2(p-[p/2])}<\infty$ ensures that $EG_i^2(X)<\infty$ for all $i=1, \ldots, p$. The functions G_i, $1 \leqq i \leqq \alpha$, have a Hermite rank at least equal to $m \geqq 1$. But the functions G_i, $\alpha+1 \leqq i \leqq p'$, may have a Hermite rank as low as 0 since taking powers does not conserve the Hermite rank. Applying Lemma 4.5, we obtain

$$\begin{aligned}
&\sum_{u_1, \ldots, u_{p'}=1}^{N}{}' |EG_1(X_{u_1}) \ldots G_{p'}(X_{u_{p'}})| \\
&\leqq K\left\{N^{p'-\frac{\alpha}{2}}\left(\sum_{u=1}^{N}|r^m(u)|\right)^{\frac{\alpha}{2}}\left\{\frac{1}{N}\sum_{u=1}^{N}|r^m(u)|\right\}^{\frac{1}{m}\left\{\frac{m\alpha}{2}-\left[\frac{m\alpha}{2}\right]\right\}}\right. \\
&\leqq K\left\{N\sum_{u=0}^{N}|r^m(u)|\right\}^{\frac{p}{2}}\left\{\frac{1}{N}\sum_{u=0}^{N}|r^m(u)|\right\}^{\frac{1}{m}\left\{\frac{mp}{2}-\left[\frac{mp}{2}\right]\right\}}
\end{aligned}$$

because $p \geqq \alpha \geqq 2p'-p$. (To check the bound when p is odd, note that $\alpha>2p'-p \Rightarrow p'-\frac{\alpha}{2} \leqq \frac{p}{2}-\frac{1}{2}$, and $\alpha=2p'-p \Rightarrow \frac{m\alpha}{2}-\left[\frac{m\alpha}{2}\right]=\frac{mp}{2}-\left[\frac{mp}{2}\right]$.)

Conclude the proof of the first part of the proposition by setting

$$\begin{aligned}
&K_1(p, \varepsilon, G) \\
&=C(p)\left\{\left[\frac{p}{2}\right] E|G(X)|^p+\sum_{p'=\left[\frac{p}{2}\right]+1}^{p} \max_{\alpha_{\min} \leqq \alpha \leqq \alpha_{\max}} \max_{\substack{v_1+\cdots+v_{p'}=p-\alpha \\ v_1=\cdots=v_\alpha=1 \\ v_{\alpha+1}, \ldots, v_{p'} \geqq 2}} K(p, \varepsilon, G^{v_1}, \ldots, G^{v_{p'}})\right\}.
\end{aligned}$$

2. Let $p>2$, $K'=K_1(p, \varepsilon, G)+(2K_1(p+1, \varepsilon, G))^{\frac{p}{p+1}}$, and let

$$g(N)=(K')^{2/p}\left\{N\sum_{u=-N}^{N}|r^m(u)|\right\}$$

be a function of the positive integers. It is easy to check that g is nondecreasing, satisfies $2g(N) \leqq g(2N)$ and $\frac{g(N)}{g(N+1)} \to 1$ as $N \to \infty$. By the first part of the proposition,

$$E\left|\sum_{i=1}^{N} G(X_i)\right|^p \leqq K_1(p, \varepsilon, G)\left\{N\sum_{u=-N}^{N}|r^m(u)|\right\}^{p/2} \leqq g^{p/2}(N)$$

when p is even, and

$$\begin{aligned}
E\left|\sum_{i=1}^{N} G(X_i)\right|^p &\leqq \left(E\left|\sum_{i=1}^{N} G(X_i)\right|^{p+1}\right)^{\frac{p}{p+1}} \\
&\leqq (2K_1(p+1, \varepsilon, G))^{\frac{p}{p+1}}\left\{N\sum_{u=-N}^{N}|r^m(u)|\right\}^{p/2} \\
&\leqq g^{p/2}(N)
\end{aligned}$$

when p is odd. Moreover, the sequence $\{G(X_i), i \geqq 1\}$ is stationary. Hence

$$E\left|\sum_{i=a+1}^{a+N} G(X_i)\right|^p \leqq g^{p/2}(N)$$

holds for all integers $a \geqq 0$ and $N \geqq 1$. By Serfling's maximal identity (Serfling (1970), Theorem B), there is a constant $K'' < \infty$, independent of N such that

$$E\left\{\max_{1 \leqq k \leqq N}\left|\sum_{i=1}^{k} G(X_i)\right|^p\right\} \leqq K'' g^{p/2}(N)$$

holds for all $N \geqq 1$. Set $K_2 = K' K''$. This concludes the proof.

5. Proof of the Theorems 1 and 2

Proof of Theorem 1. Let $G^*(x) = G(x) - \frac{J(m)}{m!} H_m(x)$ and let m^* be the Hermite rank of G^*. There is nothing to prove if $m^* = \infty$ ($G^* \equiv 0$). Suppose $m^* < \infty$. Let p be the smallest even integer satisfying $p > 2\max\left(\frac{1}{D}, \frac{1}{1-mD}\right)$. Thus $p > 2$. Fix $0 < \varepsilon < \frac{1}{p-1}$. Then by Proposition 3.1, $G^* \in \bar{\mathscr{G}}_p(\varepsilon)$.

1) We first show that the conclusion of the theorem holds when $\{X_i\} \in (m)(D, L(\cdot))$ is such that $\sup_{u \geqq 1} |r(u)| < \varepsilon$. Choose an arbitrary $a > 0$. By the Borel-Cantelli lemma, it is sufficient to prove

$$\sum_{N=1}^{\infty} P\left\{\frac{1}{d_N} \max_{1 \leqq s \leqq N}\left|\sum_{i=1}^{s} G^*(X_i)\right| > a\right\} < \infty.$$

By Čebyšev's inequality,

$$\begin{aligned} P\left\{\frac{1}{d_N} \max_{1 \leqq s \leqq N}\left|\sum_{i=1}^{s} G^*(X_i)\right| > a\right\} &\leqq a^{-p} d_N^{-p} E\left(\max_{1 \leqq s \leqq N}\left|\sum_{i=1}^{s} G^*(X_i)\right|\right)^p \\ &\leqq K_2 a^{-p}\left\{d_N^{-2} N \sum_{u=0}^{N} |r^{m^*}(u)|\right\}^{p/2} \end{aligned}$$

using Proposition 4.2. K_2 a constant independent of N. We now use the fact that $r(k) \sim k^{-D} L(k)$ as $k \to \infty$. Suppose first $0 < D < \frac{1}{m^*}$. Then as $N \to \infty$

$$\begin{aligned} d_N^{-2} N \sum_{u=0}^{N} |r^{m^*}(u)| &\sim d_N^{-2} (1 - m^* D)^{-1} N^{-m^* D + 2} L^{m^*}(N) \\ &\sim (1 - m^* D)^{-1} N^{-(m^* - m) D} L^{m^* - m}(N) \\ &= O(N^{-D} L^{m^* - m}(N)) \end{aligned}$$

since $m^* > m$. When $D \geqq \frac{1}{m^*}$, $L_1(N) = \sum_{u=0}^{N} |r^{m^*}(u)|$ is slowly varying as $N \to \infty$, and

$$d_N^{-2} N \sum_{u=0}^{N} |r^{m^*}(u)| \sim N^{-(1-mD)} L_1(N) L^{-m}(N).$$

Since $0<D<\frac{1}{m}$, both D and $1-mD$ are positive. The choice of p ensures that both $Dp/2$ and $(1-mD)p/2$ are strictly greater than 1, so that

$$\sum_{N=1}^{\infty}\left\{d_N^{-2} N \sum_{u=0}^{N}|r^{m*}(u)|\right\}^{p/2}<\infty.$$

Therefore, the conclusion of the theorem holds whenever $\{X_i\}\in(m)(D, L(\cdot))$ is such that $\sup_{u\geq 1}|r(u)|<\varepsilon$.

2) Now merely suppose $\{X_i\}\in(m)(D, L(\cdot))$, as in the statement of the theorem. Since $r(u)=EX_iX_{i+u}$ tends to 0 as $u\to\infty$, there exists an integer $n=n(\varepsilon)>1$ such that $|r(u)|<\varepsilon$ for all $u\geq n$.

Let $1\leq s\leq N$ be arbitrary and define j^* by $j^*=s-\left[\frac{s}{n}\right]n$ when $\frac{s}{n}$ is not an integer and by $j^*=n$ when $\frac{s}{n}$ is an integer. Then

$$\begin{aligned}\left|\sum_{i=1}^{s} G^*(X_i)\right| &\leq \sum_{j=1}^{j^*}\left|\sum_{k=0}^{\left[\frac{s}{n}\right]} G^*(X_{j+kn})\right| + \sum_{j=j^*+1}^{n}\left|\sum_{k=0}^{\left[\frac{s}{n}\right]-1} G^*(X_{j+kn})\right| \\ &\leq \sum_{j=1}^{n} \max_{1\leq s'\leq\left[\frac{N}{n}\right]}\left|\sum_{k=0}^{s'} G^*(X_{j+kn})\right|.\end{aligned}$$

We may replace the left hand side by its maximum over $1\leq s\leq N$. Increasing also the right hand side, we get

$$\max_{1\leq s\leq N}\left|\sum_{i=1}^{s} G^*(X_i)\right|\leq \sum_{j=1}^{n}\max_{1\leq s'\leq N}\left|\sum_{k=1}^{s'} G^*(X_{j+(k-1)n})\right|.$$

For each $j=1, \ldots, n$, the sequence $\{X_{j+(k-1)n}, k\geq 1\}$ belongs to $(m)(D, L_2(\cdot))$ with $L_2(u)=n^{-D}L(un)\sim n^{-D}L(u)$ as $u\to\infty$. All its correlations (excluding variances) are bounded by ε in absolute value. Therefore

$$\begin{aligned}\lim_{N\to\infty}\frac{1}{d_N}\max_{1\leq s\leq N}\left|\sum_{i=1}^{s} G^*(X_i)\right| &\leq \sum_{j=1}^{n}\lim_{N\to\infty}\frac{1}{d_N}\max_{1\leq s\leq N}\left|\sum_{k=1}^{s} G^*(X_{j+(k-1)n})\right| \\ &=0 \quad \text{a.s.}\end{aligned}$$

using the conclusion of the first part of the proof.

Proof of Theorem 2. Let $\{Z_N^*(t), 0\leq t\leq 1, N=1, 2, \ldots\}$ be the sequence of polygonal interpolation functions corresponding to $\left\{G(X_i)-\frac{J(m)}{m!}H_m(X_i), i\geq 1\right\}$. For each $N\geq 1$, $Z_N^*(t)$ is polygonal and therefore

$$\sup_{0\leq t\leq 1}|Z_N^*(t)|=\max_{1\leq k\leq N}\left|\sum_{i=1}^{k}\left(G(X_i)-\frac{J(m)}{m!}H_m(X_i)\right)\right|.$$

The conclusion is now a direct consequence of Theorem 1 and of Corollary A2 (part I) of the appendix.

6. Proof of Theorem 3

A multigraph $A\in\mathscr{A}_p$ is *m-regular* if all the points of A have degree m.

Given $m\geqq 0$, $p\geqq 1$, let $\mathscr{A}_p(m)$ denote the set of all multigraphs of $\mathscr{A}_p$ that are m-regular. $\mathscr{A}_p(m)$ is empty when mp is odd. When mp is even, any $A\in\mathscr{A}_p(m)$ has $q=\frac{mp}{2}$ lines.

Now, for given $A\in\mathscr{A}_p$, $(t_1,\ldots,t_p)\in[0,\infty)^p$ and $D>0$, define

$$S_D(A;t_1,\ldots,t_p)=\int_0^{t_1}dx_1\ldots\int_0^{t_p}dx_p|x_{i_1}-x_{j_1}|^{-D}\ldots|x_{i_q}-x_{j_q}|^{-D}$$

where $(i_1,j_1),\ldots,(i_q,j_q)$ is the pair sequence of A.

Lemma 6.1. *Let* $m\geqq 1$, $p\geqq 2$, mp *even,* $(t_1,\ldots,t_p)\in(0,\infty)^p$, $0<D<\frac{1}{m}$ *and let* $L\colon[0,\infty)\to(0,\infty)$ *be a slowly varying function at infinity. Suppose that* $\{r(k), k=0,\pm 1,\ldots\}$ *satisfies* $|r(k)|\leqq 1$ *and* $r(k)=r(-k)$ *for all values of* k, *and satisfies* $r(k)\sim k^{-D}L(k)$ *as* $k\to\infty$. *Let* $d_N^2\sim N^{-mD+2}L^m(N)$ *as* $N\to\infty$.

Then for all $A\in\mathscr{A}_p(m)$,

$$\lim_{N\to\infty}\frac{1}{d_N^p}\sum_{u_1=1}^{[Nt_1]}\cdots\sum_{u_p=1}^{[Nt_p]}R_{(u_1,\ldots,u_p)}(A)=S_D(A;t_1,\ldots,t_p),$$

and

$$(\min_{1\leqq i\leqq p}t_i^2\max_{1\leqq i\leqq p}t_i^{-mD})^{p/2}\leqq S_D(A,t_1,\ldots,t_p)\leqq\left(\frac{2}{1-mD}\max_{1\leqq i\leqq p}t_i^{-mD+1}\right)^{p/2}.$$

Proof. Let $x=(x_1,\ldots,x_p)\in\mathbb{R}^p$. For each integer $1\leqq s\leqq p$, and real numbers a and b, $a\leqq b$, define the indicator function

$$I_x^{(s)}(a,b)=\begin{cases}1 & \text{if } a<x_s<b \text{ and } -\infty<x_i<\infty,\ i\neq s,\ i=1,\ldots,p\\ 0 & \text{otherwise.}\end{cases}$$

Let $A\in\mathscr{A}_p(m)$, $q=\frac{mp}{2}$, $t=\max(t_1,\ldots,t_p)$ and let λ be the Lebesgue measure on $\mathbb{R}^p$ restricted to the hypercube $C=[0,t+1]^p$. λ is thus a finite measure. For all $N\geqq 1$ and $x\in[0,t+1]^p$ define

$$J_{x;N}(A)=\left\{\prod_{s=1}^{p}I_x^{(s)}\left(\frac{1}{N},\frac{[Nt_s]}{N}+\frac{1}{N}\right)\right\}\left\{\frac{1}{d_N^p}\prod_{s=1}^{q}r([Nx_{i_s}]-[Nx_{j_s}])\right\}$$

and

$$J_x(A)=\left\{\prod_{s=1}^{p}I_x^{(s)}(0,t_s)\right\}\left\{\prod_{s=1}^{q}|x_{i_s}-x_{j_s}|^{-D}\right\},$$

where $(i_1,j_1),\ldots,(i_q,j_q)$ is the pair sequence of the multigraph A. Then

$$\frac{1}{d_N^p}\sum_{u_1=1}^{[Nt_1]}\cdots\sum_{u_p=1}^{[Nt_p]}R_{(u_1,\ldots,u_p)}(A)=\int_C J_{x;N}(A)\,d\lambda(x)$$

and

$$S_D(A; t_1, \ldots, t_p) = \int_C J_x(A)\, d\lambda(x).$$

Now, $\lim_{N\to\infty} J_{x;N}(A) = J_x(A)$ for almost every x in C. Also, for any $\delta > 0$,

$$\begin{aligned}\sup_{N\geqq 1} \int_0 d\lambda(x)\, J_{x;N}^{1+\delta}(A) &\leqq \sup_{N\geqq 1} \frac{1}{d_N^p} \sum_{u_1=1}^{[Nt_1]} \cdots \sum_{u_p=1}^{[Nt_p]} |R_{(u_1, \ldots, u_p)}(A)| \\ &\leqq \sup_{N\geqq 1} \left\{ N d_N^{-2} \sum_{u=-[Nt]}^{[Nt]} |r^m(u)| \right\}^{p/2} \\ &< \infty\end{aligned}$$

by Lemma 4.4. $J_{N,x}$ is therefore uniformly integrable in C. Hence

$$\lim_{N\to\infty} \int J_{x;N}(A)\, d\lambda(x) = \int J_x(A)\, d\lambda(x)$$

which establishes the first part of the lemma. To establish the second part, note that

$$\begin{aligned}(t_1 \ldots t_p) t^{-Dq} \leqq S_D(A; t_1, \ldots, t_p) \leqq S_D(A; t, \ldots, t) &\leqq \lim_{N\to\infty} \left\{ N d_N^{-2} \sum_{u=-[Nt]}^{[Nt]} |r^m(u)| \right\}^{p/2} \\ &= \left(\frac{2t^{-mD+1}}{1-mD} \right)^{p/2}.\end{aligned}$$

Proof of Theorem 3. a) If mp is odd, $EH_m(X_{u_1}) \ldots H_m(X_{u_p}) = 0$ by Lemma 3.2 and, in expression (2.1), no indices satisfy the requirements i), ii) and iii). Hence $\mu_p(t_1, \ldots, t_p) = 0$ when mp is odd. Now suppose mp even. Using Corollary 4.2, get

$$\begin{aligned}&\frac{1}{d_N^p} \sum_{u_1=1}^{[Nt_1]} \cdots \sum_{u_p=1}^{[Nt_p]} EH_m(X_{u_1}) \ldots H_m(X_{u_p}) \\ &\quad = (m!)^p \sum_{A\in\mathscr{A}_p(m)} g(A) \frac{1}{d_N^p} \sum_{u_1=1}^{[Nt_1]} \cdots \sum_{u_p=1}^{[Nt_p]} R_{(u_1, \ldots, u_p)}(A).\end{aligned}$$

As $N \to \infty$, this converges to

$$\begin{aligned}&(m!)^p \sum_{A\in\mathscr{A}_p(m)} g(A)\, S_D(A; t_1, \ldots, t_p) \\ &\quad = \frac{(m!)^p}{2^{\frac{mp}{2}} \left(\frac{mp}{2}!\right)} \sum \int_0^{t_1} dx_1 \ldots \int_0^{t_p} dx_p |x_{i_1} - x_{j_1}|^{-D} \ldots |x_{i_q} - x_{j_q}|^{-D} = \mu_p\end{aligned}$$

using Lemma 6.1 and Corollary 4.1.

b) To get the bounds on μ_p, use the bounds on $S_D(A; t_1, \ldots, t_p)$ that are given in Lemma 6.1 and use the relation $(m!)^p \sum_{A\in\mathscr{A}_p(m)} g(A) = EH_m^p(X)$ which follows from Corollary 4.2.

c) $EH_m^p(X) = 0$ if mp is odd. Suppose mp even and $m \geqq 2$, and write $H_m(X) = \sum_{s=0}^m b_s X^s$. Note that $b_m = 1$ and $b_{m-1} = 0$. Expand $E\left(\sum_{s=0}^m b_s X^s\right)^p$, isolate the term EX^{mp} and bound the other expectations by $EX^{(m-2)p}$, to obtain $EH_m^p(X) = EX^{mp} + B(p)$ where $|B(p)| \leqq \left(\sum_{s=0}^{m} |b_s|\right)^p EX^{(m-2)p}$. An application of Stirling's formula shows that $|B(p)| = o(EX^{mp})$ as $p \to \infty$. Hence, for $m \geqq 1$, mp even, and $p \to \infty$,

$$EH_m^p(X) \sim EX^{mp} = \frac{(mp)!}{2^{\frac{mp}{2}} \left(\frac{mp}{2}\right)!} \sim \{C(m)\, 2^{-\frac{1}{2p}} p^{-\frac{m}{2}}\}^{-p}$$

where $C(m)$ is a constant depending on m. Thus

$$\sum_{p=1}^{\infty} \mu_{2p}^{-\frac{1}{2p}} \underset{\cap}{\cup} \sum_{p=1}^{\infty} \{EH_m^{2p}(X)\}^{-\frac{1}{2p}} \underset{\cap}{\cup} \sum_{p=1}^{\infty} p^{-\frac{m}{2}}.$$

d) and e) The μ_p's are moments because they are limits of moments. The sequence $Z_{N,m}(t) = \frac{1}{d_N} \sum_{i=1}^{[Nt]} H_m(X_i)$, $N \geqq 1$, is tight in $D[0,1]$ (Taqqu (1975), Lemma 2.1[2]), thus relatively compact. There exists therefore a subsequence $Z_{N',m}(t)$ that converges weakly, to $\bar{Z}_m(t)$ say, as $N' \to \infty$. The finiteness of the μ_p's (part (b)) ensures uniform integrability of any linear combination of $Z_{N,m}(t_1), \ldots, Z_{N,m}(t_p)$, $p \geqq 1$, and therefore the moments of the finite-dimensional distributions of $Z_{N',m}(t)$ converge to those of $\bar{Z}_m(t)$ as $N' \to \infty$. Use part (a) to conclude that the μ_p's are the moments of the finite-dimensional distributions of $\bar{Z}_m(t)$. That $\bar{Z}_m(t)$ is self-similar with parameter $H = 1 - \frac{mD}{2}$ and a.s. continuous follows from Theorem 2.1 of Taqqu (1975). When $m=1$ or 2, Carleman's condition is satisfied (part (c)) and the moments μ_p determine the finite-dimensional distributions uniquely. In these cases, $Z_{N,m}(t) \Rightarrow \bar{Z}_m(t)$ as $N \to \infty$.

Corollary 6.1. *Expression* (2.1) *of Theorem* 3 *is equivalent to*

$$\mu_p(t_1, \ldots, t_p) = (m!)^p \sum_{A \in \mathscr{A}_p(m)} g(A)\, S_D(A; t_1, \ldots, t_p). \tag{6.1}$$

Evaluation of some μ_p*'s.*

When $m=1$ and p is even, expression (2.1) yields

$$\mu_p(1, \ldots, 1) = \frac{p!}{2^{\frac{p}{2}} \left(\frac{p}{2}!\right)} \left(\int_0^1 \int_0^1 |x_1 - x_2|^{-D}\, dx_1\, dx_2\right)^{p/2} = \frac{p!}{2^{\frac{p}{2}} \left(\frac{p}{2}!\right)} \left(\frac{2}{(1-D)(2-D)}\right)^{p/2}$$

and one recognizes the even moments of a normal distribution.

When $m=2$ and $p \geqq 2$,

$$\mu_p(1, \ldots, 1) = 2^p p! \sum_{\substack{v_2, \ldots, v_p \geqq 0 \\ 2v_2 + \cdots + pv_p = p}} \prod_{j=2}^{p} \frac{1}{2^{v_j}(v_j!)} \left(\frac{I(j)}{j}\right)^{v_j}$$

[2] *Errata.* In Taqqu (1975), Theorem 2.1, replace the denominator in (i) by its square root; also each a, in the last four lines of the proof of Lemma 2.1 (with the exception of the a in $J_N(a, t_2, t, t_1)$) should read a', where $\frac{1}{2H} < a' < a$

where

$$I(j)=\int_0^1 dx_1 \dots \int_0^1 dx_j |x_1-x_2|^{-D}|x_2-x_3|^{-D} \dots |x_{j-1}-x_j|^{-D}|x_j-x_1|^{-D}.$$

The easiest way to derive this expression is to use the fact that the cumulants are $\kappa_1=0$, $\kappa_k=2^{k-1}(k-1)!\, I(k)$, $k\geqq 2$ when $m=2$ (see Taqqu (1975)).

When $p=2$ and $m\geqq 1$, expression (2.1) yields

$$\mu_2(t_1,t_2)=\frac{(m!)^2}{2^m m!}\left\{2^m \int_0^{t_1}\int_0^{t_2} |x_1-x_2|^{-mD}\, dx_1\, dx_2\right\}$$
$$=\frac{m!}{(1-mD)(2-mD)}\{t_1^{2-mD}+t_2^{2-mD}-|t_1-t_2|^{2-mD}\}.$$

Often, it is easier to use expression (6.1). When $p=3$ and m is even, $\mathscr{A}_3(m)$ contains only one multigraph, and hence (6.1) yields

$$\mu_3(t_1,t_2,t_3)=\frac{(m!)^3}{\left(\frac{m}{2}!\right)^3}\int_0^{t_1}\int_0^{t_2}\int_0^{t_3} |x_1-x_2|^{-\frac{mD}{2}}|x_2-x_3|^{-\frac{mD}{2}}|x_3-x_1|^{-\frac{mD}{2}} dx_1\, dx_2\, dx_3.$$

Suppose now $p=4$ and $m\geqq 1$. Any multigraph of $\mathscr{A}_4(m)$ has $\binom{p}{2}=6$ pairs of points. Let $v_1, v_2, \dots, v_6$ be the respective number of pair duplications. $v_1+v_2+\dots+v_6$ must be equal to the total number of lines, that is to $\frac{mp}{2}=2m$. We may define the v's such that $v_1=v_4$, $v_2=v_5$ and $v_3=v_6$ since each point must have degree m. Using (6.1), we get

$$\mu_4(t_1,t_2,t_3,t_4)=\sum_{\substack{v_1,v_2,v_3\geqq 0\\ v_1+v_2+v_3=m}} \frac{(m!)^4}{(v_1!\,v_2!\,v_3!)^2}\int_0^{t_1} dx_1 \dots \int_0^{t_4} dx_4 |(x_1-x_2)(x_3-x_4)|^{-v_1 D}$$
$$\cdot |(x_2-x_4)(x_3-x_1)|^{-v_2 D}|(x_4-x_1)(x_2-x_3)|^{-v_3 D}.$$

Appendix:

In this appendix, we extend results of Oodaira (1973a), (1973b), and establish functional laws of the iterated logarithm for certain Gaussian processes and for sums of non-Gaussian moving averages of independent and identically distributed random variables, whose covariance kernel, adequately normalized, converges to the covariance kernel of fractional Brownian motion. In particular, we establish the result referred to in the proof of Theorem 2 (Section 5).

Let $\Gamma(s,t)$, $s, t\in[0,1]$ be a continuous covariance kernel. The functions $\Gamma(s,\cdot)$, $s\in[0,1]$ belong then to $C[0,1]$. The reproducing kernel Hilbert space (RKHS)

$\mathscr{H}(\Gamma)$, with Γ as reproducing kernel, is the completion of the vector space spanned by the functions $\Gamma(s,\cdot)$, $s\in[0,1]$, and endowed with the scalar product

$$\langle \sum_i c_i \Gamma(s_i,\cdot), \sum_j c'_j \Gamma(s'_j,\cdot)\rangle = \sum_i \sum_j c_i c'_j \Gamma(s_i, s'_j)$$

(see Neveu (1965), page 84). Let

$$K=\{h\in\mathscr{H}(\Gamma): \langle h,h\rangle^{1/2}\leqq 1\}$$

be the unit ball of $\mathscr{H}(\Gamma)$. One may identify $\mathscr{H}(\Gamma)$, and hence K, with subsets of $C[0,1]$. K is then compact in the $C[0,1]$ sup-norm topology (see for example, Oodaira (1972), Lemma 3, or Kuelbs (1976), Lemma 2.1 (iv)).

The functional laws of the iterated logarithm established below state that certain functions z_n, $n\geqq 3$ of $C[0,1]$ are contained in an ε-neighborhood of K when n is large, and that the functions that are limit points of the sequence $\{z_n\}$ fill up the set K. This is formally expressed as $\lim_{n\to\infty} d(z_n, K)=0$ and $C\{z_n\}=K$, where $d(\cdot,\cdot)$ denotes the sup-norm distance in $C[0,1]$, and where $C\{z_n\}$ represents the cluster set of the sequence z_n, $n\geqq 3$.

Theorem A1. *Let $\Gamma(s,t)$, $0\leqq s,t\leqq 1$ be a strictly positive definite covariance kernel with $\Gamma(t,t)$ strictly increasing to $\Gamma(1,1)$. Let K be the unit ball of the RKHS $\mathscr{H}(\Gamma)$, and finally, let $0<H<1$. Suppose that $\{Z(t), t\geqq 0\}$ is Gaussian, has continuous covariance kernel, satisfies $Z(0)=0$ and also*

(C-1) $$\lim_{r\to\infty} \sup_{0\leqq s,t\leqq 1} \left| \frac{EZ(rs)Z(rt)}{r^{2H}L(r)} - \Gamma(s,t)\right| = 0.$$

(C-2) *There is a non-negative, strictly increasing and continuous function ϕ on R^+ satisfying $\int_1^\infty \phi(e^{-u^2})\,du<\infty$, such that*

$$E(Z(rs)-Z(rt))^2\leqq \phi^2(|s-t|)\, r^{2H}L(r),$$

for all $0\leqq s,t\leqq 1$ and $r\geqq 0$.

(C-3) $$\lim_{\substack{n\to\infty\\ \frac{m}{n}\to\infty}} E\frac{Z(ms)}{m^H L^{1/2}(m)}\,\frac{Z(nt)}{n^H L^{1/2}(n)}=0$$

for all $0\leqq s,t\leqq 1$.

Then the sequence of functions $\{Z(nt), 0\leqq t\leqq 1, n\geqq 1\}$, belongs almost surely to $C[0,1]$ and

$$\lim_{n\to\infty} d\left(\frac{Z(nt)}{(2n^{2H}L(n)\log\log n)^{1/2}}, K\right)=0 \quad \text{a.s.} \tag{1}$$

$$C\left\{\frac{Z(nt)}{(2n^{2H}L(n)\log\log n)^{1/2}}\right\}=K \quad \text{a.s.} \tag{2}$$

Proof. (C-2) and Fernique (1964) ensure that the sequence Z_n belongs to $C[0,1]$ with probability 1. Introduce the lacunary sequence $n_u = [c^u]$, $u = 1, 2, \ldots$, defined for arbitrary $c > 1$.

$$\lim_{\substack{u\to\infty \\ v-u\to\infty}} EZ_{n_u}(s)\, Z_{n_v}(t) = \lim_{\substack{n_u\to\infty \\ \frac{n_v}{n_u}\to\infty}} EZ_{n_u}(s)\, Z_{n_v}(t) = 0$$

follows from (C-3). $\{Z_{n_u}(\cdot), u \geqq 1\}$ satisfies the conditions of Corollary 4.1 of Carmona and Kôno (1976) and therefore

$$\lim_{u\to\infty} d\left(\frac{Z_{n_u}}{(2\log\log n_u)^{1/2}}, K\right) = 0 \quad \text{a.s.} \tag{1'}$$

$$C\left\{\frac{Z_{n_u}}{(2\log\log n_u)^{1/2}}\right\} = K \quad \text{a.s.} \tag{2'}$$

since $\log\log n_u \sim \log u$ as $u \to \infty$. However the conditions of the corollary of Theorem 1 of Oodaira (1973 a) are also satisfied. Hence for any $\varepsilon > 0$, there is a $c = c(\varepsilon) > 1$ sufficiently close to 1 such that

$$P\Bigg\{\omega\colon \exists\, u(\varepsilon,\omega) \text{ such that for all } u > u(\varepsilon,\omega),$$
$$\sup_{n_u \leqq n \leqq n_{u+1}} \left\| \frac{Z_n(\cdot,\omega)}{(2\log\log n)^{1/2}} - \frac{Z_{n_u}(\cdot,\omega)}{(2\log\log n_u)^{1/2}} \right\|_{C[0,1]} < \varepsilon \Bigg\} = 1.$$

(1) follows from (1′) since ε is arbitrary and K is closed in $C[0,1]$. (2) follows from (2′) and (1) because

$$K = C\left\{\frac{Z_{n_u}}{(2\log\log n_u)^{1/2}}\right\} \subset C\left\{\frac{Z_n}{(2\log\log n)^{1/2}}\right\} \subset K \quad \text{a.s.}$$

This concludes the proof.

Of particular interest is the covariance kernel Γ_H of the fractional Brownian motion $B_H(t)$, $0 < H < 1$. $B_H(t)$ was defined for $\frac{1}{2} < H < 1$ at the beginning of Section 2, but that definition extends to values of H satisfying $0 < H < 1$ (see Mandelbrot and Van Ness (1968)). For example, $B_{1/2}(t)$ is Brownian motion. For $0 < H < 1$,

$$\Gamma_H(s,t) = EB_H(s)\, B_H(t) = \tfrac{1}{2}\{s^{2H} + t^{2H} - |s-t|^{2H}\}.$$

Let $\mathscr{H}(\Gamma_H)$ be the RKHS with Γ_H as reproducing kernel, and let K_H denote the unit ball of $\mathscr{H}(\Gamma_H)$. The following result is a consequence of Theorem A 1.

Corollary A1. *Let $0 < H < 1$. Then the process $Z(t) = B_H(t)$ satisfies the conclusion of Theorem A 1 with $K = K_H$.*

The presence of regularly varying functions in the correlation kernel introduces technical difficulties.

Lemma A1 (de Haan (1970), page 52). *If $U\colon \mathbb{R}^+ \to \mathbb{R}^+$ is ρ-varying at infinity $(-\infty < \rho < \infty)$, then the relation*

$$\lim_{x\to\infty}\frac{U(xt)}{U(x)}=t^{\rho}$$

holds uniformly on intervals of the form (t_0, t_1) *with* $0<t_0<t_1<\infty$. *If* $\rho<0$, *the restriction* $t_1<\infty$ *can be dropped. If* $\rho>0$ *and* U *is bounded on bounded intervals we may take* $t_0=0$.

These properties of U are often used in the sequel but they are not strong enough for our purposes.

Definition. Let $-\infty<\rho<\infty$. $V:\mathbb{R}^1\to\mathbb{R}^1$ is *smoothly ρ-varying* if it is ρ-varying at infinity, and if, for arbitrary $0<t_0, t_1<1$, there are constants $x_0>0$ and $K=K(x_0, t_0, t_1)>0$ such that

$$\left|\frac{V(x)-V\left(x\left(1-\frac{t}{x}\right)\right)}{V(x)}\right|\leqq\frac{t}{x}K$$

for all $x\geqq x_0$ and $t_0\leqq t\leqq t_1 x$.

Example. $U(x)=x^{\rho}(\log(x+3)+x^{\gamma}\sin x)$ is ρ-varying at infinity for given $\gamma\leqq 0$. It satisfies $U(x)\sim x^{\rho}\log(x+3)$ when $x\to\infty$. When $0<t_0\leqq t\leqq t_1 x<x$,

$$x\frac{U(x)-U(x-t)}{U(x)}\sim t\,0(1)+\frac{x^{\gamma+1}}{\log(x+3)}2\sin\frac{t}{2}\cos\frac{2x-t}{2}$$

as $x\to\infty$. Hence $U(x)$ is smoothly ρ-varying when $\gamma\leqq -1$ but is not smoothly ρ-varying when $\gamma>-1$. Note however that for $\gamma\leqq-\frac{\rho}{2}$, $U(x)$ may be expressed as $U(x)=V(x)+O(V^{1/2}(x))$ for $x\to\infty$, where $V(x)=x^{\rho}\log(x+3)$ is smoothly ρ-varying.

Lemma A 2. *Let* $v(x)$ *be a* $(\rho-1)$*-varying function at infinity with* $\rho>0$, *which is bounded on bounded intervals. Then* $V(x)=\sum_{k=1}^{[x]}v(k)$ *is smoothly ρ-varying and satisfies* $V(x)\sim\frac{1}{\rho}xv(x)$ as $x\to\infty$.

Proof. $V(x)$ is ρ-varying at infinity since $V(x)\sim\frac{1}{\rho}xv(x)$ as $x\to\infty$. Consider now $|V(x)-V(x-t)|$ and suppose without loss of generality that $x>1$ and $[x-t]<[x]$. Then

$$\begin{aligned}|V(x)-V(x-t)|&\leqq\sum_{k=[x-t]+1}^{[x]}v(k)\\&\leqq([x]-[x-t])\max_{[x-t]+1\leqq k\leqq[x]}v(k)\\&\leqq(t+1)\max_{1-\frac{t}{x}\leqq s\leqq 1+\frac{1}{x}}v([sx])\\&\leqq(t+1)\max_{1-t_1\leqq s\leqq 2}v([sx])\\&=(t+1)\,v([s_x x])\end{aligned}$$

for some $1-t_1 \leqq s_x \leqq 2$ and $0<t_1<1$. But s_x is bounded away from zero and infinity. Applying Lemma A1, we get $v([s_x x]) \sim v(s_x x) \sim s_x^{\rho-1} v(x) = 0(1) v(x)$ as $x \to \infty$. Hence for $0<t_0 \leqq t \leqq t_1 x$ and $x \to \infty$,

$$x \left| \frac{V(x)-V(x-t)}{V(x)} \right| = (t+1)\, O(1) \frac{x v(x)}{V(x)} = (t+1)\, O(1) = \left(t+\frac{t}{t_0}\right) O(1) = t\, O(1).$$

Convention. The slowly varying function $L(x)$ appearing in the sequel satisfies

$$\inf_{[0,x_0]} L(x) > 0 \text{ and } \sup_{[0,x_0]} L(x) < \infty$$

for all $x_0 > 0$.

Theorem A2. *Let $\{X_i, i \geqq 1\}$ be a sequence of stationary real Gaussian random variables with mean zero. Let $0<H<1$, and assume that as $r \to \infty$,*

$$E\left(\sum_{i=1}^{[r]} X_i\right)^2 = V(r) + O(r^H L^{1/2}(r)),$$

with $V(r) \sim r^{2H} L(r)$ and $V(r)$ smoothly varying. Define $Z(0)=0$ and

$$Z(t) = \sum_{i=1}^{[t]} X_i + X_{[t]+1}(t-[t]), \quad t \geqq 0.$$

Then the process $Z(t)$ satisfies the conclusion of Theorem A1 with $K=K_H$.

Proof. It is sufficient to prove that conditions (C-1), (C-2) and (C-3) of Theorem A1 are satisfied. Let $U(r)=r^{2H}L(r)$. Calculations show that

$$E(Z(s)-Z(t))^2 = V(|s-t|) + O(U^{1/2}|s-t|) \quad \text{as } |s-t| \to \infty, \tag{1}$$

$$E(Z(s)-Z(t))^2 \sim U(|s-t|) \quad \text{as } |s-t| \to \infty, \tag{2}$$

$$E(Z(s)-Z(t))^2 \leqq C\, U(|s-t|) \tag{3}$$

for some constant $C>0$ and all $s, t \geqq 0$.

1) To establish (C-1), it is enough to prove that

$$\lim_{r\to\infty} \sup_{0 \leqq s,t \leqq 1} \left| \frac{E(Z(rs)-Z(rt))^2}{U(r)} - |s-t|^{2H} \right| = 0.$$

Let $0 \leqq s, t \leqq 1$ and suppose without loss of generality that $s \neq t$. As $r \to \infty$, $\frac{U(r|s-t|)}{U(r)}$ tends to $|s-t|^{2H}$ uniformly for all $0 \leqq s, t \leqq 1$, since $H>0$ and L is bounded on bounded intervals (Lemma A1). Hence for arbitrary $\varepsilon>0$, there is an $R(\varepsilon)$ such that $\left|\frac{U(r|s-t|)}{U(r)} - |s-t|^{2H}\right| < \varepsilon$, for all $r > R(\varepsilon)$ and $0 \leqq s, t \leqq 1$. Now, decompose the square $S=\{0 \leqq s, t \leqq 1\}$ into two disjoint subsets $S_1(r) = \left\{|s-t| > \frac{T(\varepsilon)}{r}\right\}$ and

$S_2(r)=\left\{|s-t|\leqq\frac{T(\varepsilon)}{r}\right\}$ where $T(\varepsilon)$ is such that, for any $|rs-rt|>T(\varepsilon)$,

$$\left|\frac{E(Z(rs)-Z(rt))^2}{U(r|s-t|)}-1\right|<\varepsilon.$$

(2) ensures that such a $T(\varepsilon)$ exists. The triangle inequality yields

$$\begin{aligned}&\left|\frac{E(Z(rs)-Z(rt))^2}{U(r)}-|s-t|^{2H}\right|\\&\quad\leqq\left|\frac{E(Z(rs)-Z(rt))^2}{U(r|s-t|)}-1\right|\left|\frac{U(r|s-t|)}{U(r)}\right|+\left|\frac{U(r|s-t|)}{U(r)}-|s-t|^{2H}\right|\\&\quad\leqq\varepsilon(1+\varepsilon)+\varepsilon,\end{aligned}$$

for all $r\geqq R(\varepsilon)$ and $(s,t)\in S_1(r)$. When $(s,t)\in S_2(r)$, (3) yields

$$\begin{aligned}\left|\frac{E(Z(rs)-Z(rt))^2}{U(r)}-|s-t|^{2H}\right|&\leqq\frac{C\,U(r|s-t|)}{U(r)}+|s-t|^{2H}\\&\leqq C\frac{(T(\varepsilon))^{2H}L(r|s-t|)}{r^{2H}L(r)}+\left(\frac{T(\varepsilon)}{r}\right)^{2H}\\&\leqq C\frac{(T(\varepsilon))^{2H}}{r^{2H}L(r)}\sup_{0\leqq u\leqq T(\varepsilon)}L(\varepsilon)+\left(\frac{T(\varepsilon)}{r}\right)^{2H},\end{aligned}$$

which tends to zero as r tends to infinity since L is bounded on bounded intervals.

2) To prove (C-2), choose $0<\delta<2H$, and set $W(r)=r^{\delta}L(r)$. By (3),

$$E(Z(rs)-Z(rt))^2\leqq C\frac{W(r|s-t|)}{W(r)}|s-t|^{2H-\delta}r^{2H}L(r)$$

for all $0\leqq s,t\leqq 1$ and $r\geqq 0$. Since $\lim_{r\to\infty}\frac{W(ru)}{W(r)}=u^{\delta}$ uniformly in $0\leqq u\leqq 1$ (Lemma A1), there exists for arbitrary $\varepsilon>0$, an $R(\varepsilon)$ such that $\frac{W(r|s-t|)}{W(r)}\leqq|s-t|^{\delta}+\varepsilon\leqq 1+\varepsilon$, for all $r>R(\varepsilon)$ and $0\leqq s,t\leqq 1$. On the other hand, for all $0<r\leqq R(\varepsilon)$ and $0\leqq s,t\leqq 1$

$$\frac{W(r|s-t|)}{W(r)}=|s-t|^{\delta}\frac{L(r|s-t|)}{L(r)}\leqq C_1(\varepsilon),$$

where

$$C_1(\varepsilon)=\Big(\sup_{0\leqq u\leqq R(\varepsilon)}L(u)\Big)\Big(\inf_{0\leqq u\leqq R(\varepsilon)}L(u)\Big)^{-1}<\infty.$$

Hence (C-2) holds with $\phi^2(s)=\{C\max(1+\varepsilon,C_1(\varepsilon)\}\,s^{2H-\delta}$.

3) To prove (C-3), we set $m=na$ and show that for all $0\leqq s,t\leqq 1$,

$$\lim_{\substack{n\to\infty\\a\to\infty}}\frac{EZ(nas)Z(nt)}{U^{1/2}(na)\,U^{1/2}(n)}=0.$$

That relation holds trivially when either s or t is equal to zero since $Z(0)=0$. Suppose now $s>0$ and $t>0$ and without loss of generality, suppose also $a>a_0\equiv 2\frac{t}{s}$. But

$$2\,EZ(nas)\,Z(nt)=EZ^2(nt)+\{EZ^2(nas)-E[Z(nas)-Z(nt)]^2\},$$

and, because of (1),

$$\frac{2\,EZ(nas)\,Z(nt)}{U^{1/2}(na)\,U^{1/2}(n)}=F_1(n,a)+F_2(n,a)+F_3(n,a)\,F_4(n,a),$$

where

$$F_1(n,a)=\frac{EZ^2(nt)}{U^{1/2}(na)\,U^{1/2}(n)},$$

$$F_2(n,a)=\frac{O(V^{1/2}(nas))+O(V^{1/2}(nas-nt))}{U^{1/2}(na)\,U^{1/2}(n)},$$

$$F_3(n,a)=\frac{(as)^{-1}\,V(nas)}{U^{1/2}(na)\,U^{1/2}(n)},$$

$$F_4(n,a)=as\frac{V(nas)-V(nas-nt)}{V(nas)}.$$

$\lim_{\substack{n\to\infty\\a\to\infty}} F_1(n,a)=0$ holds because

$$F_1(n,a)\sim\frac{U(nt)}{U^{1/2}(na)\,U^{1/2}(n)}=\frac{U^{-1/2}(na)}{U^{-1/2}(n)}\frac{U(nt)}{U(n)}\sim a^{-H}\,t^{2H}$$

as $n\to\infty$, uniformly in $a\in[a_0,\infty)$ (Lemma A1). Consider now $\lim_{\substack{n\to\infty\\a\to\infty}} F_2(n,a)$ and recall that $V(r)\sim U(r)$ as $r\to\infty$. For sufficiently large n and for all $a>a_0$, there are positive constants C_1 and C_2 such that

$$F_2(n,a)\leqq\frac{U^{1/2}(nas)}{U^{1/2}(na)}\left\{\frac{C_1}{U^{1/2}(n)}+\frac{U^{1/2}\left(nas\left(1-\frac{t}{as}\right)\right)}{U^{1/2}(nas)}\,\frac{C_2}{U^{1/2}(n)}\right\}.$$

But $\lim_{n\to\infty} U^{-1/2}(n)=0$, $\lim_{n\to\infty}\frac{U^{1/2}(nas)}{U^{1/2}(na)}=s^H$ uniformly in $a\in[a_0,\infty)$, and, as $n\to\infty$, $\frac{U\left(nas\left(1-\frac{t}{as}\right)\right)^{1/2}}{U(nas)}$ tends to $\left(1-\frac{t}{as}\right)^H$ uniformly in $a\in[a_0,\infty)$ since $\frac{1}{2}\leqq 1-\frac{t}{as}\leqq 1$ (Lemma A1). Therefore, $\lim_{\substack{n\to\infty\\a\to\infty}} F_2(n,a)=0$. Moreover,

$$\lim_{\substack{n\to\infty\\a\to\infty}} F_3(n,a)=\lim_{\substack{n\to\infty\\a\to\infty}}\frac{U^{1/2}(nas)}{U^{1/2}(na)}\lim_{\substack{n\to\infty\\a\to\infty}}\frac{(as)^{-1}U^{1/2}(nas)}{U^{1/2}(n)}$$
$$=s^H\lim_{\substack{n\to\infty\\a\to\infty}}\frac{(nas)^{H-1}L^{1/2}(nas)}{n^{H-1}L^{1/2}(n)}$$
$$=0,$$

using Lemma A1 and $H-1<0$.

We now exhibit a uniform upper bound for $|F_4(n,a)|$. Since $V(r)$ is smoothly $2H$-varying there are constants $r_0>2$ and $K=K(r_0)>0$ such that

$$r\left|\frac{V(r)-V(r-\tau)}{V(r)}\right|\leqq \tau K_1$$

holds for all $r\geqq r_0$ and $1\leqq\tau\leqq\frac{r}{2}$. But for all $a\geqq a_0$ and $n\geqq\frac{r_0}{a_0 s}$, we have $nas\geqq r_0$ and $1\leqq nt\leqq\frac{1}{2}nas$, and therefore

$$|F_4(n,a)|=\frac{1}{n}nas\left|\frac{V(nas)-V(nas-nt)}{V(nas)}\right|\leqq\frac{ntK_1}{n}=tK_1.$$

This proves (C-3) and concludes the proof of the theorem.

Corollary A 2. *Let $\{X_i,\ i\geqq 1\}$ be a sequence of real stationary Gaussian random variables with mean 0 and covariances $r(k)=EX_iX_{i+k}$, $k\geqq 0$. Suppose either*

$$\text{(I)}\quad\begin{cases}1/2<H<1\\ r(k)\sim k^{2H-2}L(k)\quad\text{as } k\to\infty\end{cases}$$

or

$$\text{(II)}\quad\begin{cases}0<H<1/2\\ r(k)\sim -k^{2H-2}L(k)\quad\text{as } k\to\infty\\ r(0)+2\sum_{k=1}^{\infty}r(k)=0.\end{cases}$$

Define $Z(t)=\sum_{i=1}^{[t]}X_i+X_{[t]+1}(t-[t])$, $t>0$, with $Z(0)=0$.

Then the process $Z(t)$ satisfies the conclusion of Theorem A1 with $K=K_H$ and with $L(t)$ replaced by $\frac{L(t)}{H|2H-1|}$.

Proof. Under either assumption (I) or (II),

$$EZ^2(n)=\sum_{i=1}^{n}\sum_{j=1}^{n}r(i-j)=r(0)+\sum_{m=2}^{n}u(m)$$

where

$$u(m)=r(0)+2\sum_{k=1}^{m-1}r(k)\sim\frac{2}{|2H-1|}m^{2H-1}L(m)$$

as $m\to\infty$. For example, under assumption (II), $u(m)=-2\sum_{k=m}^{\infty} r(k)\sim\frac{2m^{2H-1}L(m)}{|2H-1|}$ as $m\to\infty$ (de Haan (1970), page 15). By Lemma A2, as $r\to\infty$. $EZ^2([r])$ is smoothly $2H$-varying and satisfies $EZ^2([r])\sim\frac{2r^{2H}L(r)}{2H|H-1|}$. An application of Theorem A2 completes the proof.

We now obtain a law of the iterated logarithm for non-Gaussian moving averages $\{X_j, j\geqq 1\}$ by using a result of Oodaira (1973b).

Corollary A3. *Let $\{\xi_k, -\infty<k<+\infty\}$ be a sequence of independent and identically distributed random variables with mean 0 and variance 1. Let $X_j=\sum_{k=-\infty}^{+\infty} c_{k-j}\xi_k$, $j=1,2,\dots$ with $\sum_{k=-\infty}^{\infty} c_k^2<\infty$. Suppose in addition*

(a) *$E|\xi_k|^{2i}<\infty$, for some $i\geqq 2$,*

(b) *$\sum_{m=1}^{k} EX_1X_{1+m}\sim\sigma^2k^{2H-1}$ as $k\to\infty$, with $\sigma^2>0$ and $\frac{1}{i}<H<1$,*

(c) *$\sum_{k=0}^{n}|c_k|=O(n^\beta)$, $\sum_{k=-n}^{0}|c_k|=O(n^\beta)$, with $\beta<H-\frac{1}{4}$,*

(d) *$\sum_{k=\delta+1}^{\infty}\left(\sum_{j=1}^{n}c_{k-j}\right)^2=O(n^{\lambda(\delta)})$, $\sum_{k=-\infty}^{-n^\delta-1}\left(\sum_{j=1}^{n}c_{k-j}\right)^2=O(n^{\lambda(\delta)})$,*

with $\lambda(\delta)<H-\frac{2}{i}$ for some $\delta<\frac{H}{\beta+1/4}$.

Define $Z(t)=\sum_{j=1}^{[t]}X_j+X_{[t]+1}(t-[t])$, $t>0$, with $Z(0)=0$.

Then the process $Z(t)$ satisfies the conclusion of Theorem A1 with $K=K_H$ and $L(t)=\frac{\sigma^2}{H}$.

Proof. Let $\{B(t), t\geqq 0\}$ and $\{B(t), t\leqq 0\}$ be two independent Brownian motion processes satisfying $B(0)=0$. Define

$$X_j^*=\sum_{k=-\infty}^{+\infty}c_{k-j}(B(k)-B(k-1)),\qquad j=1,2,\dots,$$

and let $Z^*(t)=\sum_{j=1}^{[t]}X_j^*+X_{[t]+1}^*(t-[t])$, $t>0$ with $Z^*(0)=0$. Note that $E(Z^*(n))^2=EZ^2(n)$ for all $n\geqq 1$. By Lemma A2, as $r\to\infty$, $E(Z^*([r]))^2$ is smoothly $2H$-varying and satisfies $E(Z^*([r]))^2\sim\frac{\sigma^2}{H}r^{2H}$. By Theorem A2, the conclusion of this corollary holds for Z^*, and therefore it holds for Z since, under the conditions of the corollary, Skorokhod representation yields

$$\lim_{n\to\infty}\sup_{0\leqq t\leqq 1}\left|\frac{Z(nt)}{(n^{2H}\log\log n)^{1/2}}-\frac{Z^*(nt)}{(n^{2H}\log\log n)^{1/2}}\right|=0$$

(Oodaira (1973b)).

Example. $X_j=\sum_{k=1}^{\infty}k^{H-3/2}\xi_{j-k}$, $\frac{1}{2}<H<1$, with $E\xi_k^{2i}<\infty$ for some $i>\frac{1}{1-H}$.

References

1. Busacker, R.G., Saaty, T.L.: Finite Graphs and Networks: An Introduction with Applications. New York: McGraw-Hill 1966
2. Carmona, R., Kôno, N.: Convergence en loi et lois du logarithme itéré pour les vecteurs gaussiens. To appear in Z. Wahrscheinlichkeitstheorie verw. Geb. (1977)
3. De Haan, L.: On regular variation and its application to the weak convergence of sample extremes. Math. Centre Tracts **32**, Math. Centre, Amsterdam (1970)
4. Dobrushin, R.L.: Gaussian and their subordinated automodel random generalized fields. To appear in Ann. Probability (1977)
5. Edmonds, J., Fulkerson, D.R.: Transversals and matroid partition. J. Res. National Bureau of Standards **69B** (Math. and Math. Phys.), 147-153 (1965)
6. Fernique, X.: Continuité des processus gaussians. C.R. Acad. Sc. Paris **258**, 6058-6059 (1964)
7. Fisher, M.E.: The renormalization group in the theory of critical behavior. Rev. Modern Phys. **46**, 597-616 (1974)
8. Gantmacher, F.R.: The theory of matrices, Vol. 2. New York: Chelsea 1959
9. Gross, L.: Logarithmic Sobolev inequalities. Amer. J. Math. **97**, 1061-1083 (1975)
10. Harary, F.: Graph theory. Reading: Addison-Wesley 1969
11. Isserlis, L.: On a formula for the product-moment coefficient of any order of a normal frequency distribution in any number of variables. Biometrika **12**, 134-139 (1919)
12. Jona-Lasinio, G.: Probabilistic approach to critical phenomena. Preprint n. 32. Instituto di Fisica "G. Marconi." Università di Roma (1977)
13. Kibble, W.F.: An extension of a theorem of Mehler's on Hermite polynomials. Proc. Cambridge Philos. Soc. **41**, 12-15 (1945)
14. Kolmogorov, A.N.: Wienersche Spiralen und einige andere interessante Kurven in Hilbertschen Raum. C.R. (Doklady) Acad. Sci. URSS (N.S.), **26**, 115-118 (1940)
15. Kuelbs, J.: A strong convergence theorem for Banach space valued random variables. Ann. Prob. **4**, 744-771 (1976)
16. Ma, S.: Modern Theory of Critical Phenomena. (Frontiers in Physics). Reading: Benjamin 1976
17. Mandelbrot, B.: Fractals: Form, Chance and Dimension. San Francisco: W.H. Freeman 1977
18. Mandelbrot, B., Van Ness, J.W.: Fractional Brownian motions, fractional noises and applications. SIAM Rev. **10**, 422-437 (1968)
19. Neveu, J.: Mathematical foundations of the calculus of probability. San Francisco: Holden-Day 1965
20. Oodaira, H.: On Strassen's version of the law of iterated logarithm for Gaussian processes. Z. Wahrscheinlichkeitstheorie verw. Geb. **21**, 289-299 (1972)
21. Oodaira, H.: The law of the iterated logarithm for Gaussian processes. Ann. Prob. **1**, 954-967 (1973a)
22. Oodaira, H.: The log log law for certain dependent random sequences. Proc. Second Japan-USSR Symp. Prob. Theory. Lecture notes in Math. **330**, 355-369. Berlin: Springer Verlag (1973b)
23. Rosenbloom, P.C., Widder, D.V.: Expansions in terms of heat polynomials and associated functions. Trans. Amer. Math. Soc. **92**, 220-266 (1959)
24. Serfling, R.J.: Moment inequalities for the maximum cumulative sum. Ann. Math. Stat. **41**, 1227-1234 (1970)
25. Sinaï, Ya.G.: Automodel probability distributions (In Russian). Teor. Veroyatnost. i Ee Primenen. **21**, 63-80 (1976)
26. Slepian, D.: On the symmetrized Kronecker power of a matrix and extensions of Mehler's formula for Hermite polynomials. SIAM J. Math. Anal. **3**, 606-616 (1972)
27. Taqqu, M.S.: Weak convergence to fractional Brownian motion and to the Rosenblatt process. Z. Wahrscheinlichkeitstheorie verw. Geb. **31**, 287-302 (1975)
28. Wilson, K., Kogut, J.: The renormalization group and the ε expansion. Physics reports **12C**, 75-200 (1974)

Received December 10, 1976