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On Prabhu's Factorization of Lévy Generators

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I. Introduction

Let $X(t)$ be a Lévy process, i.e. a Hunt process with stationary, independent increments. Let A denote the infinitesimal generator of $X(t)$, with domain taken as C^∞, the class of infinitely differentiable functions $u(x)$ such that u and all its derivatives are bounded. Prabhu showed [2] that for each $\theta>0$, $\theta I-A$ can be factored in terms of the generators of a subordinator and the negative of a subordinator, with the same domain,

$$\theta I-A=(\theta_1 I-A^{\theta+})(\theta_2 I-A^{\theta-}), \tag{1}$$

where $\theta=\theta_1\theta_2$. The factorization is unique. To obtain (1) Prabhu factored the resolvent of A and inverted the resulting equation. The operators $A^{\theta+}, A^{\theta-}$ he identified in terms of processes associated with the maximum and minimum processes of $X(t)$.

In this note a direct statement about the structure of $X(t)$ is obtained from Eq. (1). For background, terminology, and examples we refer the reader to Fristedt's recent monograph [1] and to Prabhu's paper [2].

II. Notation and Constructions

If $X(t)$ is a Lévy process, its infinitesimal generator A takes the form

$$A\,u(x)=\int [u(x+y)-u(x)-\chi(y)\,u'(x)]\,\nu(dy)+\gamma\,u'(x)+\Gamma u''(x),$$

where

$$\chi(y)=\begin{cases} -1, & y<-1\\ y, & |y|\leqq 1\\ 1, & y>1.\end{cases}$$

The measure ν, called the Lévy measure, together with the constants γ and Γ determine $X(t)$ up to equivalence. At 0, ν is defined to be 0.

Let $S^\theta(t)$ denote the Lévy process with Lévy measure $e^{-\theta t}t^{-1}\,dt$, $t>0$, and no drift, i.e. $\gamma=\int\chi(y)\,\nu(dy)$, for each $\theta>0$. This family of processes has the useful property that the transition measure $F(\tau,dt)$ at $\tau=1$ is $\theta e^{-\theta t}\,dt$. If the semigroup of $X(t)$ is denoted by T_t, the resolvent at θ is $R_\theta=\int_0^\infty e^{-\theta t}\,T_t\,dt$, and θR_θ is the semigroup at $t=1$ of the process $X(S^\theta(t))$. Prabhu used this property of the process $S^\theta(t)$ in his proof of (1).

The process $X(S^\theta(t))$ has generator A^θ of the form

$$A^\theta u(x)=\int_{-\infty}^{\infty}[u(x+y)-u(x)]\,v_\theta(dy), \tag{2}$$

where

$$v_\theta(dy)=\int_0^\infty e^{-\theta t}\,t^{-1}F(t,dy)\,dt, \qquad y\neq 0,$$

and $F(t, dy)$ is the transition measure of $X(t)$. From (2) we see that

$$X(S^\theta(t))=X(S^\theta(t))^+ + X(S^\theta(t))^- \tag{3}$$

where the processes on the right are independent with generators given by (2) with the integral restricted to $(0, \infty)$ and $(-\infty, 0)$ respectively. We note that for each of these processes, $\Gamma=0$ and γ is $\int \chi(y)\,v_\theta(dy)$ with integral over the appropriate interval.

Two-dimensional Lévy processes $Z^+(\tau)$ and $Z^-(\tau)$ were constructed from the maximum and minimum processes of $X(t)$ by Fristedt in [1]. Let

$$M^+(t)=\sup_{0\leqq s\leqq t} X(s), \qquad M^-(t)=\inf_{0\leqq s\leqq t} X(s).$$

Let $L(t)$ be the local time at zero of $M^+(t)-X(t)$, and $t^+(\tau)=\inf\{s\colon L(s)\geqq\tau\}$. Then $Z^+(\tau)=(t^+(\tau), M^+(t^+(\tau))$ is a 2-dimensional Lévy process, and so is $Z^-(\tau)=(t^-(\tau), -M^-(t^-(\tau)))$, constructed similarly.

III. Relation of a Lévy Process to Its Maximum and Minimum Processes

Theorem. *Let $X(t)$ be a Lévy process and $\theta>0$. There exist $\theta_1, \theta_2>0$ such that $\theta=\theta_1\theta_2$ and*

$$X(S^\theta(t))=Y^{\theta+}(S^{\theta_1}(t))+Y^{\theta-}(S^{\theta_2}(t)), \tag{4}$$

where

(i) *the terms on the right in (2) are independent processes and each $S^{\theta_i}(t)$ is independent of each $Y^{\theta\cdot}$,*

(ii) *$Y^{\theta\cdot}(t)$ is a Lévy process with Lévy measure $\int_0^\infty e^{-\theta t}\,v.(dt, dx)$, and $v.(dt, dx)$ is the Lévy measure of the process $Z^\cdot$ defined above; $\cdot$ is $+$ or $-$.*

The processes $Y^{\theta+}(S^{\theta_1}(t))$ and $Y^{\theta-}(S^{\theta_2}(t))$ are equivalent to $X(S^\theta(t))^+$ and $X(S^\theta(t))^-$, respectively.

Proof. The Eq. (1) is equivalent to the resolvent equation

$$R_\theta=U_{\theta_1}^+\,U_{\theta_2}^-$$

where $U_{\theta_1}^+, U_{\theta_2}^-$ are the resolvents at θ_1, θ_2 of the processes with generators $A^{\theta+}, A^{\theta-}$. These processes are identified [2] as $Y^{\theta+}$, $Y^{\theta-}$ as given in (ii). Since $\theta=\theta_1\theta_2$,

$$\theta R_\theta=\theta_1\,U_{\theta_1}^+\,\theta_2\,U_{\theta_2}^+. \tag{5}$$

As noted in Section II, θR_θ is the Laplace transform of the semigroup at $t=1$ associated with the process $X(S^\theta(t))$, where $S^\theta(t)$ is defined above. Similarly,

$\theta_1 U_{\theta_1}^+$ and $\theta_2 U_{\theta_2}^+$ are the semigroups at $t=1$ associated with $Y^{\theta+}(S^{\theta_1}(t))$ and $Y^{\theta-}(S^{\theta_2}(t))$. A Lévy process is determined by its semigroup at one positive value of t. Using the semigroup property, commutativity, and continuity, we see that the semigroup of $X(S^{\theta}(t))$ is the product of those of $Y^{\theta+}(S^{\theta_1}(t))$ and $Y^{\theta-}(S^{\theta_2}(t))$. Relation (4) follows.

From the known properties of subordinated processes (see [1]) it is easily seen that the constant γ for each process on the right in (4), as was noted above for each process on the right in (3), is $\int \chi(y)\, \nu(dy)$ where $\nu(dy)$ is the Lévy measure of the process in question. In each case $\Gamma=0$. The two independent processes on the right in (4) have Lévy measures with support in $(0,\infty)$, $(-\infty,0)$ respectively, and these measures are the restrictions to $(0,\infty)$, $(-\infty,0)$ of ν_θ. The processes $X(S^{\theta}(t))^+$ and $X(S^{\theta}(t))^-$, have, respectively, the same identifying ν, γ, Γ as $Y^{\theta+}(S^{\theta_1}(t))$ and $Y^{\theta-}(S^{\theta_2}(t))$, and so are equivalent processes.

References

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