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Hitting of Submanifolds by Diffusions

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Abstract. It is proved that a $(d-2)$ or lower dimensional C^2-submanifold is a polar set for a nondegenerate d-dimensional diffusion process. A similar result is established also for diffusions in a closed half space with reflecting boundary conditions.

1. Introduction

Let $W(t)$ be a d-dimensional Brownian motion, where $d \geqq 2$. Note that its two dimensional projection $\pi W(t)$ is a two dimensional Brownian motion. It is known that a two (or higher) dimensional Brownian motion does not hit any specified point (see p. 634 of Doob (1984)). As $\pi W(t)$ does not hit any specified point in $\mathbb{R}^2$, it follows that $W(t)$ does not hit any $(d-2)$-dimensional subspace of $\mathbb{R}^d$ specified in advance.

This suggests the following problem: Can a non degenerate d-dimensional diffusion process $X(t)$ hit a $(d-2)$ or lower dimensional C^2-submanifold of $\mathbb{R}^d$ specified in advance? This problem is considered in the present article. As may be expected in view of the preceding paragraph, the answer is in the negative. Indeed, it is proved in Sect. 2 below that any $(d-2)$ or lower dimensional C^2-submanifold of $\mathbb{R}^d$ is a *polar set* for a non degenerate d-dimensional diffusion process $X(t)$. This is done in two stages. First it is shown that a $(d-2)$ or lower dimensional subspace is a polar set for $X(t)$; and it is equivalent to showing that the origin is a polar set for $\pi X(t)$ where π is the appropriate projection (see Theorems 2.3 and 2.4 below). The point here is that $\pi X(t)$ is not a Markov process and hence the machinery of probabilistic potential theory is not applicable. We get around this difficulty by estimating the probability of the process $\pi X(t)$ hitting the outer shell of the annulus $\{c<|y|<n\}$ before hitting the inner shell (such a technique has been used in studying recurrence of diffusions); and then shrink the inner shell to the origin. Next, let M be a k-dimensional C^2-submanifold where $k \leqq d-2$. We can write $M = \cup M_i$ where each M_i is diffeomorphic to the k-dimensional unit ball. To prove that M is a polar set it is enough to prove that each M_i is a polar set. For this the diffeomorphism between

M_i and the k-dimensional unit ball is extended to the whole of $\mathbb{R}^d$; we look at the transformed diffusion under this diffeomorphism and use the first part (Lemma 2.6 and Theorem 2.7).

In the case of a $(d-1)$-dimensional submanifold, the situation is quite different. For, a nondegenerate diffusion escapes out of a bounded set in finite time; consequently it hits the $(d-1)$-dimensional submanifold ∂D with probability one, if it starts from inside D, where D is any sphere in $\mathbb{R}^d$. Even if it starts from outside D, in view of the support theorem (p. 168 of Stroock and Varadhan (1979)), it hits ∂D with positive probability. Also, for recurrent diffusions the probability of hitting ∂D is unity, whatever be the starting point (see Bhattacharyya (1978)).

In Sect. 3 we extend the results of Sect. 2 to diffusion processes in the closed half space $\bar{G}=\{x\in\mathbb{R}^d: x_1\geqq 0\}$, where $d\geqq 2$, with reflecting boundary conditions. Similar techniques as in Sect. 2 are employed; they work mainly because the extra term involved in estimating Prob. ($\pi X(t)$ hits $\partial B(0:n)$ before $\partial B(0:c)$) vanishes under normal reflection at the boundary; and the case of oblique reflection at the boundary is reduced to the case of normal reflection using an earlier result of the author (see inequality (3.3) and the proof of Theorem 3.3).

It would be interesting to have similar results for diffusions with boundary conditions in more general domains. However, for an arbitrary domain, it is not clear how to obtain estimates (like (3.3)) which are well behaved with respect to even normal reflection. Of course, for any domain which is diffeomorphic to the half space our results clearly hold.

A remark is perhaps in order. Polar sets are the negligible sets of potential theory. For state spaces which have nice geometric structures, it is natural to seek sufficient conditions in terms of the geometry of the space for a set to be polar. However, such a problem does not seem to have been studied even for the Brownian motion in $\mathbb{R}^d$, where $d\geqq 3$ (let alone diffusions or diffusions with boundary conditions). The only result known in this direction seems to be that any $(d-2)$ or lower dimensional affine subspace is a polar set for the d-dimensional Brownian motion (see Doob (1984) or Bliedtner and Hansen (1986)). In view of the methods developed in the present article, the difficulty in extending such a result to submanifolds is due to the following: Under a diffeomorphism the Brownian motion is transformed into a diffusion process (with non constant coefficients) whose projections are not Markov processes. Thus, even for 3 or higher dimensional Brownian motion our result does not seem to be previously known.

2. Diffusions in $\mathbb{R}^d$

Let $d\geqq 2$. We have the diffusion coefficients a, b defined on $\mathbb{R}^d$ satisfying the following conditions.

Conditions. (I). (*I1*) For each $x\in\mathbb{R}^d$, $a(x)=((a_{ij}(x)))_{1\leqq i,j\leqq d}$ is a $d\times d$ real symmetric positive deinite matrix; $a(\cdot)$ is continuous; there exist constants λ_0 and λ_1 such that $0<\lambda_0\leqq\lambda_1<\infty$ and any eigenvalue of $a(x)\in[\lambda_0, \lambda_1]$ for all $x\in\mathbb{R}^d$;

(I2) $b(\cdot)=(b_1(\cdot), \ldots, b_d(\cdot))$ is a bounded and continuous $\mathbb{R}^d$ valued function on $\mathbb{R}^d$.

Define the elliptic operator L by

$$Lf(x)=\frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2 f(x)}{\partial x_i\,\partial x_j}+\sum_{i=1}^{d} b_i(x)\frac{\partial f(x)}{\partial x_i}. \tag{2.1}$$

Let $\Omega=C([0,\infty):\mathbb{R}^d)$ be endowed with the topology of uniform convergence on compacta. Let $X(t)$ denote the t-th coordinate map on Ω; let $\mathscr{B}_t=\sigma\{X(s): 0\leqq s\leqq t\}$, $t\geqq 0$, be the natural filtration in Ω.

Let $\{P_x: x\in\mathbb{R}^d\}$ be the diffusion corresponding to L. That is, for each $x\in\mathbb{R}^d$, P_x is the unique probability measure on Ω such that

(i) $P_x(\{X(0)=x\})=1$

(ii) for any $f\in C_b^2(\mathbb{R}^d)$

$$f(X(t))-\int_0^t Lf(X(s))\,ds \tag{2.2}$$

is a P_x-martingale with respect to $\mathscr{B}_t$.

It is a basic theorem of Stroock and Varadhan (1979) that such a diffusion exists under conditions *I*. Moreover the process $X(t)$ is strong Markov and strong Feller under $\{P_x\}$.

We shall denote by $B_k(x:r)$ the open ball in $\mathbb{R}^k$ with centre at x and radius r.

Lemma 2.1. *Let conditions I hold. Let $k<d$; let π: $\mathbb{R}^d\to\mathbb{R}^{d-k}$ be the canonical projection onto the first $(d-k)$ coordinates. Let D be a nonempty bounded open subset of $\mathbb{R}^{d-k}$ and $\eta=\inf\{t\geqq 0: \pi X(t)\notin D\}$.* Then $\sup\{E_x(\eta): x\in D X\mathbb{R}^k\}<\infty$. □

A similar lemma is proved for diffusions with boundary conditions in the next section. As the proof of the above lemma is easily obtained from the proof of Lemma 3.1, it is omitted here.

For $x\in\mathbb{R}^d$ such that $\pi(x)\neq 0$, where π is as in the preceding lemma, set

$$\begin{aligned} A(x)&=\sum_{i,j=1}^{d-k} a_{ij}(x)\frac{x_i x_j}{|\pi(x)|^2},\\ B(x)&=\sum_{i=1}^{d-k} a_{ii}(x), \qquad C(x)=2\sum_{i=1}^{d-k} x_i b_i(x). \end{aligned} \tag{2.3}$$

For $r>0$, define

$$\beta(r)=\inf_{|\pi(x)|=r}\frac{B(x)-A(x)+C(x)}{A(x)}. \tag{2.4}$$

Let $c>0$. For $r>0$ define

$$\begin{aligned} I(r;c) &= \int_c^r \frac{1}{u}\beta(u)\,du \\ F(r;c) &= \int_c^r \exp(-I(u;c))\,du. \end{aligned} \tag{2.5}$$

Note that $A(\cdot)$ is bounded away from zero. Hence β is well defined and bounded over $(0, n]$ for every $n>0$; consequently the quantities in (2.5) are well defined. $\left(\text{In (2.5), for } r<c, \int_c^r \dots \text{ is just } -\int_r^c \dots\right)$

Lemma 2.2. *Let conditions (I) hold; let k and π be as in the preceding lemma. Let c, n be real numbers such that $0<c<n$. Then for $x\in\mathbb{R}^d$ such that $c<|\pi(x)|<n$,*

$$\frac{F(|\pi(x)|;c)}{F(n;c)} \leqq P_x(\tau_n<\tau_c) \tag{2.6}$$

where

$$\tau_n=\inf\{t\geqq 0\colon \pi(X(t))\in\partial B_{d-k}(0:n)\}$$

and

$$\tau_c=\inf\{t\geqq 0\colon \pi(X(t))\in\partial B_{d-k}(0:c)\}.$$

Proof. Follows from the preceding lemma and Lemma 2.1 of Ramasubramanian (1983) (see also Lemma 3.2 below). □

The next two results state that the diffusion does not hit a $(d-2)$ dimensional subspace.

Theorem 2.3. *Let conditions (I) hold and let $k\leqq d-2$. Let $M=\{x\in\mathbb{R}^d\colon x_j=0$ for $j=1,2,\dots,d-k\}$. Then*

$$P_x(\{X(t)\in M \text{ for some } t\geqq 0\})=0$$

for all $x\notin M$.

Proof. Let $\pi\colon \mathbb{R}^d\to\mathbb{R}^{d-k}$ be the canonical projection onto the first $(d-k)$ coordinates; note that $\pi(M)=\{0\}$. For proving the theorem it is enough to establish that

$$P_x(\{\pi(X(t))=0 \text{ for some } t\geqq 0\})=0 \tag{2.7}$$

for all $x\notin M$.

Since $k\leqq(d-2)$, note that for any $x\in\mathbb{R}^d$,

$$B(x)-A(x)\geqq \text{sum of } (d-k-1) \text{ eigenvalues of } ((a_{ij}(x)))_{1\leqq i,j\leqq d-k}\geqq (d-k-1)\,\lambda_0>0. \tag{2.8}$$

Consequently

$$\lim_{s\to 0}\frac{1}{s}\beta(s)=\infty. \tag{2.9}$$

Let $x \notin M$ be fixed. Then $r = |\pi(x)| > 0$. Let r_0, n be fixed such that $0 < r_0 < r < n$. It is simple to check that for $c > 0$,

$$\frac{F(r;c)}{F(n;c)} = \frac{\int_c^r \exp(-I(u;r_0))\,du}{\int_c^n \exp(-I(u;r_0))\,du}$$

$$= 1 - \frac{\int_r^n \exp(-I(u;r_0))\,du}{\int_c^{r_0} \exp\left[\int_u^{r_0} \frac{1}{s}\beta(s)\,ds\right] du + \int_{r_0}^n \exp(-I(u;r_0))\,du}.$$

By (2.9) the first integral in the denominator on the right side of the above equation tends to ∞ as $c \downarrow 0$; and the other terms on the right side of the above equation are independent of c. Thus for any $x \notin M$, any $n > |\pi(x)|$ we have

$$\lim_{c \downarrow 0} \frac{F(|\pi(x)|;c)}{F(n;c)} = 1. \tag{2.10}$$

By (2.6) and (2.10) we get

$$\lim_{c \downarrow 0} P_x(\tau_n < \tau_c) = 1 \tag{2.11}$$

for any $x \notin M$ and all $n > |\pi(x)|$. Now (2.7) is an immediate consequence of (2.11). This completes the proof.

Remark. If $k = d-1$, then (2.8) fails to hold; consequently (2.10), (2.11) and (2.7) cannot be asserted. In fact, a non degenerate diffusion hits any $(d-1)$ dimensional hyperplane with positive probability.

Theorem 2.4. *Let conditions (I) hold and let $k \leqq d-2$. Let M be as in Theorem* 2.3. *Then M is a polar set for the diffusion $X(t)$, that is,*

$$P_x(\{X(t) \in M \text{ for some } t > 0\}) = 0 \tag{2.12}$$

for all x in $\mathbb{R}^d$.

Proof. In view of Theorem 2.3 it is enough to prove (2.12) for $x \in M$. For $\varepsilon > 0$, let

$$\tau_\varepsilon = \inf\{t \geqq 0 \colon X(t) \notin M X B_{d-k}(0:\varepsilon)\}.$$

Then by Lemma 2.1 $P_x(\tau_\varepsilon < \infty) = 1$ for all $x \in M$. Now by the strong Markov property and Theorem 2.3 we get

$$P_x(\{X(t) \in M \text{ for some } t \geqq \tau_\varepsilon\}) = 0 \tag{2.13}$$

for all $x \in M$ and $\varepsilon > 0$.

Let $P(t, x, \cdot)$ denote the transition probability function of the Markov process $X(t)$; let

$$\tau_0 = \inf\{t > 0 : X(t) \notin M\}.$$

As $k \leqq d-2$, by Lemma 9.2.2 (p. 234) of Stroock and Varadhan (1979) $P(t, x, M) = 0$ for all $t > 0$ and x. Hence by Exercise (3.9) or (3.13) of Chap. II (pp. 82–83) of Blumenthal and Getoor (1968) it follows that every x in $\mathbb{R}^d$ is regular for M^c; that is, $P_x(\tau_0 = 0) = 1$ for all $x \in \mathbb{R}^d$. Therefore by Proposition (10.4) of Chap. I (p. 53) of Blumenthal and Getoor (1968) it follows that

$$P_x(\{\lim_{\varepsilon \to 0} \tau_\varepsilon = 0\}) = 1 \tag{2.14}$$

for all $x \in M$. From (2.13) and (2.14), now (2.12) follows for any $x \in M$. This completes the proof of the theorem.

The next lemma concerns transformation of diffusions under diffeomorphisms.

Lemma 2.5. *Let conditions I hold and let L be given by* (2.1). *Let T: $\mathbb{R}^d \to \mathbb{R}^d$ be a C^2-diffeomorphism given by*

$$(x_1, x_2, \ldots, x_d) \mapsto T(x_1, \ldots, x_d) := (z_1(x), z_2(x), \ldots, z_d(x)).$$

Let $j_T(x) = \left(\left(\frac{\partial z_k(x)}{\partial x_i}\right)\right)$ denote the Jacobian of the diffeomorphism T at x; suppose there exist constants c_1 and c_2 such that $0 < c_1 \leqq \|j_T(x)\| \leqq c_2 < \infty$, for all $x \in \mathbb{R}^d$. Assume further that $\frac{\partial^2 z_k}{\partial x_i \partial x_j}$ are bounded and continuous functions for $1 \leqq i, j, k \leqq d$. For $z \in \mathbb{R}^d$, let

$$\tilde{a}(z) = ((\tilde{a}_{km}(z))) = j_T^*(T^{-1}(z))\, a(T^{-1}(z))\, j_T(T^{-1}(z)), \tag{2.15}$$

$$\begin{aligned}\tilde{b}_k(z) = {} & \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(T^{-1}(z)) \frac{\partial^2 z_k}{\partial x_i \partial x_j}(T^{-1}(z)) \\ & + \sum_{i=1}^{d} b_i(T^{-1}(z)) \frac{\partial z_k}{\partial x_i}(T^{-1}(z)), \qquad k = 1, 2, \ldots, d.\end{aligned} \tag{2.16}$$

Define the operator $\tilde{L}$ by

$$\tilde{L} f(z) = \frac{1}{2} \sum_{k,m=1}^{d} \tilde{a}_{km}(z) \frac{\partial^2 f(z)}{\partial z_k \partial z_m} + \sum_{k=1}^{d} \tilde{b}_k(z) \frac{\partial f(z)}{\partial z_k}. \tag{2.17}$$

Define $\tilde{T}$: $C([0, \infty): \mathbb{R}^d) \to C([0, \infty): \mathbb{R}^d)$ by $(\tilde{T}\omega)(t) = T(\omega(t))$. For any $g \in C_b^2(\mathbb{R}^d)$ define $\hat{g}$ by $\hat{g}(x) = g(T(x))$. Then

(i) $L\hat{g}(x)=\tilde{L}g(z)$, *where* $z=T(x)$;

(ii) $\tilde{a}$ *and* $\tilde{b}$ *also satisfy Conditions (I); and* $\tilde{L}$ *is a uniformly elliptic operator with ellipticity constant* $\lambda_0 c_1^2$; *and*

(iii) $\{P_x\tilde{T}^{-1}: x\in\mathbb{R}^d\}$ *is the diffusion corresponding to* $\tilde{L}$, *where* $\{P_x: x\in\mathbb{R}^d\}$ *is the diffusion corresponding to L.*

Proof. An elementary differentiation yields

$$\begin{aligned} L\hat{g}(x)=&\frac{1}{2}\sum_{k,m=1}^{d}\left[\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial z_m(x)}{\partial x_j}\frac{\partial z_k(x)}{\partial x_i}\right]\frac{\partial^2 g(z(x))}{\partial z_m\,\partial z_k}\\ &+\sum_{k=1}^{d}\left[\frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2 z_k(x)}{\partial x_i\,\partial x_j}+\sum_{i=1}^{d} b_i(x)\frac{\partial z_k(x)}{\partial x_i}\right]\frac{\partial g(z(x))}{\partial z_k}\\ =&\frac{1}{2}\sum_{k,m=1}^{d}\tilde{a}_{km}(z(x))\frac{\partial^2 g(z(x))}{\partial z_m\,\partial z_k}+\sum_{k=1}^{d}\tilde{b}_k(z(x))\frac{\partial g(z(x))}{\partial z_k}\end{aligned}$$

whence the first assertion of the lemma follows. Continuity and boundedness of $\tilde{a}$ and $\tilde{b}$ are clear; that $\tilde{a}(z)$ is a $(d\times d)$ real symmetric matrix for each z is also easy to see. For z, $y\in\mathbb{R}^d$, note that

$$\langle\tilde{a}(z)\,y,\,y\rangle\geqq\lambda_0\,\|j_T(T^{-1}(z))\,y\|^2.$$

As $j_T(T^{-1}(z))$ is nonsingular it now follows that $\tilde{a}(z)$ is positive definite for each z. Next, any $z\in\mathbb{R}^d$ note that

$$\begin{aligned}\|\tilde{a}(z)^{-1}\|&=\|j_T^{-1}(T^{-1}(z))\,a^{-1}(T^{-1}(z))\,j_T^{*\,-1}(T^{-1}(z))\|\\ &\leqq\lambda_0^{-1}c_1^{-2}<\infty,\end{aligned}$$

from which the second assertion of the lemma follows.

To establish the third assertion, let $\tilde{P}_z=P_x\tilde{T}^{-1}$ where $z=T(x)$. Let $0\leqq t_1<t_2<\infty$ and let $E\in\mathscr{B}_{t_1}$. Note that $\tilde{T}^{-1}(E)\in\mathscr{B}_{t_1}$. Let $f\in C_b^2(\mathbb{R}^d)$. Using the first assertion of the lemma and (2.2) it is easy to see that

$$\begin{aligned}&\int_E\left[f(X(t_2,\omega))-f(X(0,\omega))-\int_0^{t_2}\tilde{L}f(X(s,\omega))\,ds\right]d\tilde{P}_z(\omega)\\ &=\int_{\tilde{T}^{-1}(E)}\left[\hat{f}(X(t_2,\omega))-\hat{f}(X(0,\omega))-\int_0^{t_2}L\hat{f}(X(s,\omega))\,ds\right]dP_x(\omega)\\ &=\int_{\tilde{T}^{-1}(E)}\left[\hat{f}(X(t_1,\omega))-\hat{f}(X(0,\omega))-\int_0^{t_1}L\hat{f}(X(s,\omega))\,ds\right]dP_x(\omega)\\ &=\int_E\left[f(X(t_1,\omega))-f(X(0,\omega))-\int_0^{t_1}\tilde{L}f(X(s,\omega))\,ds\right]d\tilde{P}_z(\omega)\end{aligned}$$

whence the third assertion of the lemma follows. This completes the proof. □

Lemma 2.6. *Let K be a compact subset of $\mathbb{R}^m$ such that there is a C^2-diffeomorphism $\varphi\colon K \to \overline{B_m(0:1)}$. Then there exist a C^2-diffeomorphism T of $\mathbb{R}^m$ and a compact set $\hat{K}$ of $\mathbb{R}^m$ such that*

(i) *$T(y)=\varphi(y)$ for all $y \in K$;*
(ii) *$T(y)=y$ for all $y \notin \hat{K}$;*
(iii) *there exist constants c_1 and c_2 such that*

$$0 < c_1 \leqq \|j_T(y)\| \leqq c_2 < \infty \qquad \text{for all } y \in \mathbb{R}^m,$$

where j_T is the Jacobian of the transformation T; and

(iv) *all the second order partial derivatives of the transformation T are bounded and continuous functions on $\mathbb{R}^m$.*

Proof. Define the functions ψ_0 and ψ_1 from $\overline{B_m(0:1)}$ into $\mathbb{R}^m$ by $\psi_0(y)=y$ and $\psi_1(y)=\varphi^{-1}(y)$ respectively. Note that $\psi_1(\overline{B_m(0:1)})=K$. Since K is the diffeomorphic image of $\overline{B_m(0:1)}$, K is orientable, and hence we may assume without loss of generality that both ψ_0 and ψ_1 preserve orientation. Then by Theorem 8.3.1 of Hirsch ((1976), p. 185) ψ_0 and ψ_1 are isotopic. As ψ_0 extends to the identity map on the whole of $\mathbb{R}^m$, by Theorem 8.1.3 (p. 180) and Exercise 4 on p. 182 of Hirsch (1976), there exists a C^2-diffeomorphism $\hat{\psi}_1\colon \mathbb{R}^m \to \mathbb{R}^m$ and a compact set $\hat{K}$ of $\mathbb{R}^m$ such that $\hat{\psi}_1(y)=\psi_1(y)$ for $y \in \overline{B_m(0:1)}$ and $\hat{\psi}_1(y)=y$ for $y \notin \hat{K}$. It is now clear that $y \mapsto j(y)$ is a C^1-map, that $0<\hat{c}_1 \leqq \|j(y)\| \leqq \hat{c}_2 < \infty$, and that all the second order partial derivatives of $\hat{\psi}_1$ are bounded continuous functions, where j is the Jacobian of the transformation $\hat{\psi}_1$ and $\hat{c}_1$, $\hat{c}_2$ are suitable constants. Take $T=\hat{\psi}_1^{-1}$; then T has the desired properties. □

Let M be a C^2-submanifold of $\mathbb{R}^d$ of dimension k. That is, M is a subset of $\mathbb{R}^d$ with the induced topology, and for each $x \in M$ there is an open set $U \subseteq \mathbb{R}^d$ and a C^2-diffeomorphism $\varphi\colon U \to \mathbb{R}^d$ such that $x \in U$, $\varphi(x)=0$ and $M \cap U = \varphi^{-1}(\mathbb{R}^k)$, where $\mathbb{R}^k = \{y \in \mathbb{R}^d\colon y_i=0,\ 1 \leqq i \leqq d-k\}$.

We can now state the main theorem of this section.

Theorem 2.7. *Let Conditions I hold and let L be given by* (2.1); *let $\{P_x\colon x \in \mathbb{R}^d\}$ be the diffusion corresponding to L. Let M be a C^2-submanifold of $\mathbb{R}^d$ of dimension k, where $k \leqq d-2$. Then*

$$P_x(\{X(t) \in M \text{ for some } t>0\})=0 \tag{2.18}$$

for all x in $\mathbb{R}^d$; that is M is a polar set for the diffusion.

Proof. Note that M is separable; let $\{x_1, x_2, \ldots\}$ be a countable dense set in M. Then there exist K_i, φ_i such that

(i) K_i is a compact neighbourhood of x_i in $\mathbb{R}^d$;

(ii) $\varphi_i\colon K_i \to \overline{B_d(0:1)}$ is a C^2-diffeomorphism with $\varphi_i(x_i)=0$;

(iii) $K_i \cap M = \varphi_i^{-1}(\overline{B_k(0:1)})$; that is, $y \in K_i \cap M$ if and only if $\pi(\varphi_i(y))=0$ and $\|\varphi_i(y)\| \leqq 1$, where $\pi\colon \mathbb{R}^d \to \mathbb{R}^{d-k}$ is the canonical projection onto the first $(d-k)$ coordinates.

Note that $M=\bigcup_i (K_i \cap M)$; therefore to prove the theorem it is enough to prove that

$$P_x(\{X(t)\in K_i \cap M \text{ for some } t>0\})=0 \tag{2.19}$$

for $i=1, 2, \ldots$ and any x.

Let i be fixed. Set $K=K_i$. Then by Lemma 2.6, there exists a C^2-diffeomorphism T of $\mathbb{R}^d$ (which is an extension of φ_i) such that

$$T(K\cap M)=\overline{B_k(0:1)}=\{z\in\mathbb{R}^d\colon z_j=0,\ 1\leqq j\leqq d-k,\ \|z\|\leqq 1\}$$

and satisfying the hypothesis of Lemma 2.5. Let $\tilde{a}$, $\tilde{b}$, $\tilde{L}$, $\tilde{T}$ be as in Lemma 2.5. Now by Theorem 2.4 and Lemma 2.5, for any $x\in\mathbb{R}^d$, we get

$$\begin{aligned}
&P_x(\{\omega\colon X(t,\omega)\in K\cap M \text{ for some } t>0\})\\
&\quad=P_x(\{\omega\colon X(t,\tilde{T}\omega)\in \overline{B_k(0:1)} \text{ for some } t>0\})\\
&\quad=P_x\tilde{T}^{-1}(\{\omega'\colon X(t,\omega')\in \overline{B_k(0:1)} \text{ for some } t>0\})\\
&\quad=0.
\end{aligned}$$

Since i is arbitrary, (2.19) is thus established. Hence the theorem is proved. □

Remark 2.8. Let M be a C^2-submanifold with boundary; let k be the dimension of M where $k\leqq(d-2)$. Note that the boundary ∂M is a $(k-1)$ dimensional C^2-submanifold, and that $M\backslash\partial M$ is a k-dimensional C^2-submanifold. Therefore (2.18) holds also for k dimensional C^2-submanifolds with boundary, where $k\leqq d-2$. In fact it is now easily seen that (2.18) holds for any subset M of $\mathbb{R}^d$ which is contained in a countable union of $(d-2)$ or lower dimensional C^2-submanifolds; consequently any such set M is a polar set for the diffusion $\{P_x\}$.

3. Diffusions with Reflecting Boundary Conditions

Let $G=\{x=(x_1,x_2,\ldots,x_d)\colon x_1>0\}$, $\partial G=\{x\in\mathbb{R}^d\colon x_1=0\}$ and $\bar{G}=G\cup\partial G$, where $d\geqq 2$. We now consider diffusions in the closed half space $\bar{G}$ with reflecting boundary conditions. We have the coefficients a, b defined on $\bar{G}$, and γ defined on ∂G satisfying the following conditions.

Conditions. (II). (II 1): Same as (I 1) with $\mathbb{R}^d$ replaced by $\bar{G}$;

(II 2): $b(\cdot)=(b_1(\cdot),\ldots,b_d(\cdot))$ is a bounded and continuous $\mathbb{R}^d$-valued function on $\bar{G}$;

(II 3): $\gamma(\cdot)=(\gamma_2(\cdot),\ldots,\gamma_d(\cdot))$ is an $\mathbb{R}^{d-1}$-valued function on ∂G; $\gamma_j\in C_b^2(\partial G)$ for $j=2,\ldots,d$.

Let the elliptic operator L be given by (2.1). Define the boundary operator J by

$$J f(x)=\frac{\partial f(x)}{\partial x_1}+\sum_{i=2}^{d} \gamma_i(x) \frac{\partial f(x)}{\partial x_i}. \tag{3.1}$$

Let Ω, $\mathscr{B}_t$, $X(t)$ be as in Sect. 2. Under conditions less restrictive than the set of Conditions (II), Stroock and Varadhan (1971) have established the existence of a unique solution to the submartingale problem corresponding to the coefficients a, b, γ. Thus, when Conditions (II) hold, for each $x\in\bar{G}$ there exists a unique probability measure $\hat{P}_x$ on Ω such that

(i) $\hat{P}_x(\{X(t)\in\bar{G}$ for all $t\geqq 0$ and $X(0)=x\})=1$;

(ii) $f(X(t))-\int_0^t [I_G\cdot(Lf)](X(s))\,ds$

is a $\hat{P}_x$-submartingale for any $f\in C_0^2(\mathbb{R}^d)$ satisfying $Jf\geqq 0$ on ∂G. Also the process $X(t)$ is strong Markov and Feller continuous, under $\{\hat{P}_x\}$. Moreover, there exists a continuous, non-decreasing, non anticipating process $\xi(t)$ on Ω such that

$$
\begin{aligned}
&\text{(i)}\quad \xi(t)=\int_0^t I_{\partial G}(X(s))\,d\xi(s),\\
&\text{(ii)}\quad \int_0^t I_{\partial G}(X(s))\,ds=0, \quad \text{a.s. } \hat{P}_x\\
&\text{(iii)}\quad f(X(t))-\int_0^t Lf(X(s))\,ds-\int_0^t Jf(X(s))\,d\xi(s)\\
&\qquad \text{is a } \hat{P}_x\text{-martingale for every } f\in C_b^2(\mathbb{R}^d).
\end{aligned}
\tag{3.2}
$$

We shall call the family $\{\hat{P}_x\colon x\in\bar{G}\}$ the diffusion corresponding to (L, J).

Lemma 3.1. *Let Conditions (II) hold. Let $k\leqq d$; let $\pi\colon\mathbb{R}^d\to\mathbb{R}^{d-k}$ be the canonical projection onto the first $(d-k)$ coordinates. Let D be a nonempty bounded open subset of $\mathbb{R}^{d-k}$ and*

$$\eta=\inf\{t\geqq 0\colon \pi X(t)\notin D\}.$$

Then

$$\sup\{\hat{E}_x(\eta)\colon x\in\bar{G}\cap\pi^{-1}(\bar{D})\}<\infty,$$

and

$$\sup\{\hat{E}_x(\xi(\eta))\colon x\in\bar{G}\cap\pi^{-1}(\bar{D})\}<\infty.$$

Proof. Without loss of generality we may take $D=B_{d-k}(0:N)$. Choose $h\in C_b^2(\mathbb{R}^d)$ such that $h(x)=e^{qx_1}$ for $x\in\pi^{-1}(\bar{D})$, where q is a positive constant so chosen that $Lh(x)>1$ for $x\in\pi^{-1}(\bar{D})$. For $x\in(\partial G)\cap\pi^{-1}(\bar{D})$, note that $Jh(x)=q>0$. Now by (3.2) and the optional sampling theorem we get

$$\hat{E}_x(\eta\wedge T)+q\,\hat{E}_x(\xi(\eta\wedge T))\leqq 2\,e^{qN}$$

for $x\in\bar{G}\cap\pi^{-1}(\bar{D})$ and $T>0$. Letting $T\uparrow\infty$ we obtain the lemma. □

For $x \in \bar{G}$ such that $\pi(x) \neq 0$ (where π is as in the preceding Lemma) let $A(x)$, $B(x)$, $C(x)$ be defined as in (2.3). For $r>0$, let $\beta(r)$ be defined as in (2.4) with infimum taken over $x \in \bar{G}$ such that $|\pi(x)|=r$; for $c>0$, $r>0$ let, $I(r; c)$ and $F(r; c)$ be defined as in (2.5).

Lemma 3.2. *Let Conditions (II) hold; let k and π be as in the preceding lemma. Let c, n be real numbers such that $0<c<n$. Then for $x \in \bar{G}$ such that $c<|\pi(x)|<n$,*

$$\frac{F(|\pi(x)|; c)}{F(n; c)} + \frac{1}{F(n; c)} \hat{E}_x \int_0^{\tau_n \wedge \tau_c} J f_c(X(u))\, d\xi(u) \leqq \hat{P}_x(\tau_n < \tau_c) \tag{3.3}$$

where

$$\tau_n = \inf\{t \geqq 0 : \pi(X(t)) \in \partial B_{d-k}(0:n)\},$$

$$\tau_c = \inf\{t \geqq 0 : \pi(X(t)) \in \partial B_{d-k}(0:c)\},$$

and

$$f_c(y) = F(|\pi(y)|; c).$$

Proof. Write $f(\cdot)=f_c(\cdot)$ and $F(\cdot)=F(\cdot\,; c)$ for the course of the proof. It is easily seen that

$$Lf(x) = \frac{1}{2} A(x)\, F''(|\pi(x)|) + \frac{1}{2} \frac{F'(|\pi(x)|)}{|\pi(x)|} (B(x) - A(x) + C(x)) \tag{3.4}$$

for $x \neq 0$; also for $r>c$ note that

$$F''(r) + \frac{F'(r)}{r} \beta(r) = 0. \tag{3.5}$$

By the preceding lemma $\tau_n < \infty$ a.s. $\hat{P}_x$ for all x such that $c<|\pi(x)|<n$. Now by (3.2), (3.4), (3.5) and the optional sampling theorem we get

$$\hat{E}_x[f(X(\tau_n \wedge \tau_c \wedge T))] \geqq F(|\pi(x)|) + \hat{E}_x\left[\int_0^{\tau_n \wedge \tau_c \wedge T} Jf(X(s))\, d\xi(s)\right]$$

for $c<|\pi(x)|<n$. As $\sup\{f(y): c \leqq |\pi(y)| \leqq n\} < \infty$, we get from the above

$$F(|\pi(x)|) + \hat{E}_x\left[\int_0^{\tau_n \wedge \tau_c} Jf(X(s))\, d\xi(s)\right] \leqq F(n) \cdot \hat{P}_x(\tau_n < \tau_c)$$

which is just (3.3). □

The next result is an analogue of Theorem 2.3.

Theorem 3.3. *Let Conditions (II) hold and let $k \leqq (d-2)$. Let*

$$M = \{x \in \partial G : x_j = 0, j = 1, 2, \ldots, d-k\}.$$

Then

$$\hat{P}_x(\{X(t) \in M \textit{ for some } t \geqq 0\}) = 0$$

for all $x \notin M$.

Proof. Let $\pi: \mathbb{R}^d \to \mathbb{R}^{d-k}$ be the canonical projection onto the first $(d-k)$ coordinates; note that $\pi(M)=0$. For proving the theorem it is enough to establish that

$$\hat{P}_x(\{\pi X(t)=0 \text{ for some } t \geqq 0\})=0 \tag{3.6}$$

for all $x \notin M$. In view of Theorem 1 of Ramasubramanian (1986) it is sufficient to consider the case $J=\frac{\partial}{\partial x_1}$. In that case, for any $c>0$ note that $Jf_c \equiv 0$ on ∂G where f_c is as in (3.3). Hence (3.6) can be established in the same manner as (2.7) was established in Theorem 2.3. This completes the proof. □

The following is an analogue of Theorem 2.4.

Theorem 3.4. *Let Conditions (II) hold and let $k \leq d-2$; let M be as in Theorem 3.3. Then*

$$\hat{P}_x(\{X(t) \in M \text{ for some } t>0\})=0 \tag{3.7}$$

for all x in $\bar{G}$.

Proof. Let $\hat{P}(t, x, \cdot)$ denote the transition probability function of the Markov process $X(t)$. Let $x \in \bar{G}$ and $t>0$ be fixed. Then by (3.2(ii)),

$$\begin{aligned} 0 &= \int_{\Omega} \int_0^t I_{\partial G}(X(u, \omega))\, du\, d\hat{P}_x(\omega) \\ &= \int_0^t \hat{P}(u, x, \partial G)\, du. \end{aligned}$$

Consequently

$$\hat{P}(u, x, \partial G)=0 \quad \text{for a.a. } u \in [0, t]. \tag{3.8}$$

Since t and x are arbitrary it follows from (3.8) that ∂G is a set of potential zero. Consequently M is of potential zero. Now, in view of Exercise (3.9) of Chap. II (p. 82) of Blumenthal and Getoor (1968), Lemma 3.1 and Theorem 3.3, one can derive (3.7) in a manner analogous to the proof of Theorem 2.4. This completes the proof of the theorem. □

Our goal is to extend the above results to submanifolds. First, we need a lemma.

Lemma 3.5. *Let Conditions (II) hold. Let S: $\partial G(=\mathbb{R}^{d-1}) \to \partial G$ be a C^2-diffeomorphism given by*

$$(x_2, \ldots, x_d) \mapsto S(x_2, \ldots, x_d) := (z_2(x), \ldots, z_d(x)).$$

Let there exist constants c_1 and c_2 such that $0<c_1 \leqq \|j_S(x)\| \leqq c_2 < \infty$ for all $x \in \partial G$, where j_S is the Jacobian of the transformation S; also, let all the partial derivatives of second order be bounded continuous functions. Define T: $\mathbb{R}^d \to \mathbb{R}^d$ by

$$T(x_1, x_2, \ldots, x_d)=(z_1(x), z_2(x), \ldots, z_d(x))=(x_1, S(x_2, \ldots, x_d)).$$

Then T is a C^2-diffeomorphism on $\mathbb{R}^d$ with $T(\bar{G})=\bar{G}$, $T(\partial G)=\partial G$ and satisfying the hypothesis of Lemma 2.5. Let L, $\tilde{a}$, $\tilde{b}$, $\tilde{L}$, $\tilde{T}$ be as in Lemma 2.5; let J be given by (3.1). For $z\in\partial G$, let

$$\tilde{\gamma}_k(z)=\sum_{i=2}^{d}\gamma_i(T^{-1}(z))\frac{\partial z_k}{\partial x_i}(T^{-1}(z)),\qquad k=2,\dots,d \tag{3.9}$$

and define the operator $\tilde{J}$ by

$$\tilde{J}f(z)=\frac{\partial f(z)}{\partial z_1}+\sum_{k=2}^{d}\tilde{\gamma}_k(z)\frac{\partial f(z)}{\partial z_k}. \tag{3.10}$$

For any $g\in C_b^2(\mathbb{R}^d)$ define $\hat{g}$ by $\hat{g}(x)=g(T(x))$.
Then

(i) *$L\hat{g}(x)=\tilde{L}g(z)$, $J\hat{g}(x)=\tilde{J}g(z)$ where $z=T(x)$;*

(ii) *$\tilde{a}$, $\tilde{b}$, $\tilde{\gamma}$ also satisfy Conditions (II); and $\tilde{L}$ is a uniformly elliptic operator with ellipticity constant $\lambda_0 c_1^2$;*

(iii) *$\{\hat{P}_x\tilde{T}^{-1}: x\in\bar{G}\}$ is the diffusion corresponding to $(\tilde{L},\tilde{J})$, where $\{\hat{P}_x: x\in\bar{G}\}$ is the diffusion corresponding to (L,J); and*

(iv) *if the (L,J)-diffusion has the strong Feller property then so does the $(\tilde{L},\tilde{J})$-diffusion.*

Proof. It is easily seen that T is a C^2-diffeomorphism with the required properties. Assertions (i) and (ii) are established in a manner analogous to the corresponding assertions of Lemma 2.5. To establish (iii) let $f\in C_b^2(\mathbb{R}^d)$ be such that $\tilde{J}f\geqq 0$ on ∂G; then by (i) above $J\hat{f}\geqq 0$ on ∂G. Since $\{\hat{P}_x\}$ solves the submartingale problem for (L,J), by a change of variables argument it is simple to check that

$$f(X(t))-\int_0^t [I_G\cdot(\tilde{L}f)](X(s))\,ds$$

is a $\hat{P}_x\tilde{T}^{-1}$ submartingale. Thus (iii) is proved. Finally, for any bounded measurable function f and $t>0$, note that

$$\int f(X(t,\omega))\,d\hat{P}_x\tilde{T}^{-1}(\omega)=\int\hat{f}(X(t,\omega))\,d\hat{P}_x(\omega),$$

where $\hat{f}(\cdot)=f(T(\cdot))$; now the last assertion follows immediately, completing the proof. □

Theorem 3.6. *Let Conditions (II) hold. Let L and J be given respectively by (2.1) and (3.1), and let $\{\hat{P}_x: x\in\bar{G}\}$ be the diffusion corresponding to (L,J). Let M be a C^2-submanifold of ∂G of dimension k, where $k\leqq d-2$. Then*

$$\hat{P}_x(\{X(t)\in M \text{ for some } t>0\})=0 \tag{3.11}$$

for all x in $\bar{G}$.

Proof. Note that there exist points x_i in M, sets K_i in ∂G and maps φ_i such that

(i) K_i is a compact neighbourhood of x_i in $\partial G(=\mathbb{R}^{d-1})$,

(ii) φ_i: $K_i \to \overline{B_{d-1}(0:1)}$ is a C^2-diffeomorphism with $\varphi_i(x_i)=0$;

(iii) $K_i \cap M = \varphi_i^{-1}(\overline{B_k(0:1)})$; that is, $y \in K_i \cap M$ if and only if $\pi(\varphi_i(y))=0$ and $\|\varphi_i(y)\| \leqq 1$, where π is the canonical projection onto the first $(d-k)$ coordinates;

(iv) $M = \bigcup_{i=1}^{\infty} (K_i \cap M)$.

Therefore to prove (3.11) it is enough to prove that

$$\hat{P}_x(\{X(t) \in K_i \cap M \text{ for some } t>0\})=0$$

for $i=1, 2, \ldots$, and any x in $\bar{G}$.

Let i be fixed. Set $K=K_i$. Then by Lemma 2.6 there exists a C^2-diffeomorphism S of ∂G (which is an extension of φ_i) satisfying the hypothesis of Lemma 3.5. Now by Lemma 3.5 there exists a C^2-diffeomorphism T of $\mathbb{R}^d$ (which is an extension of S) satisfying analogous properties. In view the Theorem 3.4 and Lemma 3.5, the theorem can now be proved in a manner similar to Theorem 2.7. □

The following is the main theorem of this section.

Theorem 3.7. *Let Conditions (II) hold; let L, J, $\{\hat{P}_x\}$ be as in the preceding theorem. Let $M \subset \bar{G}$ be a subset such that*

(i) $(M \cap \partial G)$ is contained in a countable union of $(d-2)$ or lower dimensional C^2-submanifolds of ∂G, and

(ii) $M \cap G$ is contained in a countable union of $(d-2)$ or lower dimensional C^2-submanifolds of G.

Then (3.11) *holds for all x in $\bar{G}$; that is, M is a polar set for the diffusion $\{\hat{P}_x\}$.*

Proof. By Theorem 3.6 it follows that $M \cap \partial G$ is a polar set for the diffusion. For $r>0$, let $G_r=\{x \in G: x_1>r\}$. Now to establish the theorem it is enough to show that

$$\hat{P}_x(\{X(t) \in M \cap G_r \text{ for some } t>0\})=0 \tag{3.12}$$

for any $r>0$ and any x in $\bar{G}$.

Let $r>0$ and $x \in \bar{G}$ be arbitrary but fixed. Set

$$\tau_1 = \inf\left\{t \geqq 0: X_1(t) = \frac{r}{2}\right\},$$

$$\tau_{2n} = \inf\{t > \tau_{2n-1}: X_1(t)=0\},$$

$$\tau_{2n+1} = \inf\left\{t > \tau_{2n}: X_1(t) = \frac{r}{2}\right\}, \qquad n=1, 2, \ldots.$$

By Lemma 3.1, note that $\tau_{2n+1}<\infty$ a.s. on the set $\{\tau_{2n}<\infty\}$. It is also clear that $\hat{P}_x(\lim_n \tau_n=\infty)=1$.

Since the diffusion in $\bar{G}$ starting from an interior point behaves like a diffusion in $\mathbb{R}^d$ with generator L upto the time of hitting ∂G, it follows by Theorem 2.7 and Remark 2.8 that

$$\hat{P}_y(\{X(t)\in M\cap G_r \text{ for some } 0\leqq t\leqq\eta\})=0$$

for any $y\in G$ such that $y_1=\frac{1}{2}r$, where η is the time of hitting ∂G. Now a repeated application of the strong Markov property yields (3.12). This completes the proof of the theorem. $\square$

Remark. Note that Theorem 3.4 and hence Theorems 3.6 and 3.7 do not hold if the boundary is assumed to be sticky.

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