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Probability Theory and Related Fields

A New Aspect of L_∞ in the Space of BMO-Martingales

Norihiko Kazamaki
Department of Mathematics, Toyama University, Gofuku, Toyama 930, Japan

Summary. Let $M=(M_t, \mathfrak{F}_t)$ be a continuous BMO-martingale. Then the associated exponential martingale $\mathscr{E}(M)$ satisfies the reverse Hölder inequality

$$(R_p) \qquad E[\mathscr{E}(M)_\infty^p \mid \mathfrak{F}_T] \leqq C_p \mathscr{E}(M)_T^p$$

for some $p>1$, where T is an arbitrary stopping time (see [3, 5]). Our claim is, in a word, that the (R_p) condition bears upon the distance between M and L_∞ in BMO. Especially, we shall prove that M belongs to the BMO-closure of L_∞ if and only if $\mathscr{E}(\lambda M)$ satisfies all (R_p) for every real number λ. Some related problems are also considered.

1. Introduction

Throughout this paper, let $(\Omega, \mathfrak{F}, P)$ be a complete probability space with a filtration $(\mathfrak{F}_t)$ satisfying the usual conditions, and *we suppose that any martingale adapted to this filtration is continuous.*

Let now M be a local martingale with $M_0=0$. Then the associated exponential local martingale $\mathscr{E}(M)$ is given by the formula

$$(1) \qquad \mathscr{E}(M)_t = \exp(M_t - \langle M \rangle_t/2) \qquad (0 \leqq t < \infty),$$

where $\langle M \rangle$ denotes the continuous increasing process such that $M^2 - \langle M \rangle$ is a local martingale. Note that $\mathscr{E}(M)$ is not always a uniformly integrable martingale.

The fundamentally important result in performing our investigation is that the following are equivalent (see [2, 3, 12, 13]).

(a) $M \in \mathrm{BMO}$.

(b) $\mathscr{E}(M)$ is a uniformly integrable martingale which satisfies the reverse Hölder inequality (R_p) for some $p>1$.

(c) $\mathscr{E}(M)$ satisfies the condition

$$(A_p) \qquad \sup_T \|E[\{\mathscr{E}(M)_T/\mathscr{E}(M)_\infty\}^{1/(p-1)} | \mathfrak{F}_T]\|_\infty < \infty$$

for some $p > 1$, where the supremum is taken over all stopping times T.

Originally, these three conditions were introduced in the classical analysis, and the duality between the space of BMO-functions and the Hardy space H_1 is especially known (see [6]). The condition (A_p) is a probabilistic version of the one introduced in [19] by Muckenhoupt. The former is a necessary and sufficient condition for the validity of some weighted norm inequalities for martingales (see [10, 21] for example) as the latter plays an essential role in the problems of weighted norm inequalities for many operators, such as the Hardy-Littlewood maximal operator and the singular integral operators. However, the relation between BMO and the (A_p) condition in the classical analysis does not go on smoothly as in the probability setting. In fact, for any function $w(x)$ satisfying the classical (A_p) condition, $\log w(x)$ is a BMO-function, but the converse fails. For example, the function $\log 1/|x|$ is in BMO, but $1/|x|$ satisfies no (A_p). The great advantage of the investigation from the point of view of martingales consists in settling this trouble.

On the other hand, there exist two important subclasses of BMO, namely, the class L_∞ of all bounded martingales and the class H_∞ of all martingales M such that $\langle M\rangle_\infty$ is bounded. As is easily seen, BMO is neither L_∞ nor H_∞ in general, and further there is not an inclusion relation between L_∞ and H_∞. However, it has recently been verified in [16] that the BMO-closure of L_∞ contains H_∞ in general.

We now explain the contents of this paper. Roughly speaking, our purpose is to establish new relationships between these classes and the conditions (A_p) and (R_p). Section 3 contains a necessary and sufficient condition for a martingale M to be in the BMO-closure of L_∞. Furthermore, in Sect. 4 we show that the reverse Hölder inequality for $\mathscr{E}(M)$ is closely connected with the distance in BMO between M and L_∞. By contrast, the (A_p) condition does not bear upon the distance to L_∞. In Sect. 5 we give a necessary and sufficient condition for that both $\mathscr{E}(M)$ and $\mathscr{E}(-M)$ have all (A_p).

2. Preliminaries

Here we collect several remarks and lemmas which are used in later sections.

First of all, let us assume that the exponential process $\mathscr{E}(M)$ given by (1) is a uniformly integrable martingale. This implies that $d\hat{P} = \mathscr{E}(M)_\infty\, dP$ is a probability measure on Ω. But the martingale property is not invariant under such a change of law. Fortunately, a nice key of settling this trouble was given in [9] by Girsanov. It comes to this that under the absolutely continuous change in probability measure a Brownian motion is transformed into the sum of a Brownian motion and a second process with sample functions which are absolutely continuous with respect to the Lebesgue measure. The following lemma given in [22] by Van Schuppen and Wong is a natural generalization of this result.

Lemma 1. *For any local martingale X, the process $\hat{X}=\langle X, M\rangle - X$ is a local martingale with respect to $d\hat{P}$, where $\langle X, M\rangle = (\langle X+M\rangle - \langle X-M\rangle)/4$. Furthermore, we have $\langle \hat{X}\rangle = \langle X\rangle$ under either probability measure.*

The mapping $\phi: X \to \hat{X}$ is often called the Girsanov transformation. We remark in passing that the generalization to the right continuous local martingales is established by Lenglart in [17]. We deal entirely with continuous BMO-martingales. Recall that a uniformly integrable martingale M is said to be in the class BMO if

$$\|M\|_{\mathrm{BMO}_p} = \sup_T \|E[|M_\infty - M_T|^p \mid \mathfrak{F}_T]^{1/p}\|_\infty < \infty \qquad (1 \leqq p < \infty),$$

where the supremum is taken over all stopping times T. Note that the norms $\|\ \|_{\mathrm{BMO}_p}$ are all equivalent. As the case may be, we denote it by $\|M\|_{\mathrm{BMO}_p(P)}$ to specify the underlying probability measure dP.

Lemma 2. *Let $M \in \mathrm{BMO}$ and $d\hat{P} = \mathscr{E}(M)_\infty\, dP$. Then the Girsanov transformation $\phi: X \to \hat{X}$ is an isomorphism of* BMO *onto* BMO$(\hat{P})$.

For the proof, see [14]. The first important result on BMO-functions was the John-Nirenberg Theorem ([11]), which we recall here in the probabilistic setting. It was given in [8] by Garsia for discrete parameter martingales and by Dóleans-Dade and Meyer [2] for general martingales.

Lemma 3. *If $\|M\|_{\mathrm{BMO}_1} < 1/4$, then for every stopping time T we have*

$$E[\exp\{|M_\infty - M_T|\} \mid \mathfrak{F}_T] \leqq (1 - 4\|M\|_{\mathrm{BMO}_1})^{-1}. \tag{2}$$

Furthermore, if $\|M\|_{\mathrm{BMO}_2} < 1$, then

$$E[\exp\{\langle M\rangle_\infty - \langle M\rangle_T\} \mid \mathfrak{F}_T] \leqq (1 - \|M\|^2_{\mathrm{BMO}_2})^{-1}. \tag{3}$$

These inequalities are the main tools to deal with various questions about BMO-martingales.

Let $d_p(\ ,\ )$ be the distance on BMO deduced from the norm $\|\ \|_{\mathrm{BMO}_p}$ by the usual procedure, and let $a(M)$ be the supremum of the set of a for which

$$\sup_T \|E[\exp\{a|M_\infty - M_T|\} \mid \mathfrak{F}_T]\|_\infty < \infty.$$

There is a very beautiful relation between $d_1(M, L_\infty)$ and $a(M)$ as follows:

Lemma 4. *For $M \in \mathrm{BMO}$ we have*

$$\frac{1}{4 d_1(M, L_\infty)} \leqq a(M) \leqq \frac{4}{d_1(M, L_\infty)}. \tag{4}$$

This result had originally been obtained by Garnett and Jones [7] in classical analysis. Its probabilistic analogue was first established by Varopoulos [23] for Brownian martingales and then by Emery [4] for continuous martingales.

For simplicity, we denote by $\bar{L}_\infty$ the closure of L_∞ in BMO. Then from (4) if follows at once that $M \in \bar{L}_\infty$ if and only if $a(M) = \infty$. In this connection, it should be noted that L_∞ is neither closed nor dense in BMO whenever BMO $\neq L_\infty$ (see [1]). Quite recently, it was verified in [15] that the assumption "BMO $\neq L_\infty$" is none other than the reasonable condition "$\mathfrak{F}_t \neq \mathfrak{F}_0$ for some t".

The reader is assumed to be familiar with martingale theory as expounded in [18]. We denote by C a positive constant and by C_β a positive constant depending only on indexed parameter β. Note that the letters C and C_β are not necessarily the same from line to line.

3. A Characterization of the BMO-Closure of L_∞

Garnett and Jones [7] proved implicitly that a locally integrable function f on R^d belongs to the BMO-closure of L_∞ if and only if both e^f and e^{-f} satisfy the Muckenhoupt (A_p) condition for all $p>1$. We can easily establish the probabilistic analogue of this result: a uniformly integrable martingale M belongs to $\bar{L}_\infty$ if and only if both $E[\exp(M_\infty)\,|\,\mathfrak{F}.]$ and $E[\exp(-M_\infty)\,|\,\mathfrak{F}.]$ satisfy the probabilistic (A_p) condition for all $p>1$. But it seems to me that this equivalence is not so interesting for two reasons. First, it is exponential martingales which play an essential role in various questions concerning the absolute continuity of probability laws of stochastic processes, and neither $E[\exp(M_\infty)\,|\,\mathfrak{F}.]$ nor $E[\exp(-M_\infty)\,|\,\mathfrak{F}.]$ are exponential martingales. The aim of this section is to give a new characterization of $\bar{L}_\infty$ in the framework of exponential martingales.

Theorem 1. *$M \in \bar{L}_\infty$ if and only if $\mathscr{E}(\lambda M)$ satisfies all (R_p) for every real number λ.*

Proof. We first show the "only if" part. For that, suppose $M \in \bar{L}_\infty$. Then $a(M) = \infty$ by Lemma 4. On the other hand, for every λ and every stopping time T

$$\mathscr{E}(\lambda M)_\infty / \mathscr{E}(\lambda M)_T \leqq \exp\{|\lambda|\,|M_\infty - M_T|\}$$

by the definition of $\mathscr{E}(\lambda M)$. Therefore, recalling the definition of $a(M)$, we find

$$\begin{aligned} E[\mathscr{E}(\lambda M)_\infty^p \,|\, \mathfrak{F}_T] &= E[\{\mathscr{E}(\lambda M)_\infty / \mathscr{E}(\lambda M)_T\}^p \,|\, \mathfrak{F}_T]\, \mathscr{E}(\lambda M)_T^p \\ &\leqq E[\exp\{|\lambda|\, p\, |M_\infty - M_T|\} \,|\, \mathfrak{F}_T]\, \mathscr{E}(\lambda M)_T^p \\ &\leqq C_{\lambda,p}\, \mathscr{E}(\lambda M)_T^p \end{aligned}$$

for every λ and every $p>1$, with a constant $C_{\lambda,p}>0$ independent of T.

We are now going to prove the "if" part. By (3) in Lemma 3 there is some positive α_0 such that for every stopping time T

$$E[\exp\{\alpha_0(\langle M\rangle_\infty - \langle M\rangle_T/2)\} \,|\, \mathfrak{F}_T] \leqq C_0,$$

where C_0 is independent of T. Next, let $\lambda>0$, and choose α such that $0<\alpha<\min\{2\lambda, \alpha_0/(2\lambda)\}$. If we set $p=2\lambda/\alpha$, then $1<p<\alpha_0/\alpha^2$ and so

$$\begin{aligned} &\exp\{2\lambda(M_\infty - M_T) - \alpha_0(\langle M\rangle_\infty - \langle M\rangle_T)/2\} \\ &\quad = \exp\{2\lambda(\alpha M_\infty - \alpha M_T)/\alpha - (\alpha_0/\alpha^2)(\langle \alpha M\rangle_\infty - \langle \alpha M\rangle_T)/2\} \\ &\quad \leqq \{\mathscr{E}(\alpha M)_\infty / \mathscr{E}(\alpha M)_T\}^p. \end{aligned}$$

Combining this with the Schwarz inequality, we obtain

$$\begin{aligned}E[\exp\{\lambda(M_\infty-M_T)\}\mid\mathfrak{F}_T]&=E[\exp\{\lambda(M_\infty-M_T)-\alpha_0(\langle M\rangle_\infty-\langle M\rangle_T)/4\}\\&\quad\cdot\exp\{\alpha_0(\langle M\rangle_\infty-\langle M\rangle_T)/4\}\mid\mathfrak{F}_T]\\&\leqq E[\exp\{\mathscr{E}(\alpha M)_\infty/\mathscr{E}(\alpha M)_T\}^p\mid\mathfrak{F}_T]^{1/2}\,E[\exp\{\alpha_0(\langle M\rangle_\infty-\langle M\rangle_T)/2\}\mid\mathfrak{F}_T]^{1/2}.\end{aligned}$$

Since $\mathscr{E}(\alpha M)$ satisfies (R_p) by the assumption, the first conditional expectation in the last expression is dominated by some constant. Furthermore, the second term is smaller than $C_0^{1/2}$ as previously stated. Therefore, we have $E[\exp\{\lambda(M_\infty-M_T)\}\mid\mathfrak{F}_T]\leqq C_\lambda$ for every stopping time T, with a constant $C_\lambda>0$ depending only on λ. The same argument works if M is replaced by $-M$, so that for every $\lambda>0$

$$E[\exp\{\lambda|M_\infty-M_T|\}\mid\mathfrak{F}_T]\leqq C_\lambda,$$

where T is an arbitrary stopping time. This implies $a(M)=\infty$. Then $M\in\bar{L}_\infty$ by Lemma 4. Thus the theorem is established.

From the above proof it follows that if $\mathscr{E}(\lambda M)$ satisfies all (R_p) for sufficiently small λ, then M belongs to $\bar{L}_\infty$. It is natural to ask if $\mathscr{E}(\alpha M)$ has (R_s) for $\alpha<0$ and $s>1$ whenever $\mathscr{E}(\lambda M)$ has all (R_p) for every $\lambda>0$. But the answer in general is no. We give below an example.

Example 1. Let $B=(B_t\cdot\mathfrak{F}_t)$ be a one dimensional Brownian motion with $B_0=0$ defined on a probability space $(\Omega,\mathfrak{F},Q)$ and let $\tau=\inf\{t:|B_t|=1\}$. Then B^τ is clearly a bounded martingale with respect to Q, so that $dP=\exp(B_\tau-\tau/2)\,dQ$ is a probability measure on Ω. Consider now the process $M=2B^\tau-2\langle B^\tau\rangle$, which is a BMO-martingale with respect to dP by Lemma 2. Noticing $\langle M\rangle_t=4(t\wedge\tau)$, we find that

$$\begin{aligned}&E[\{\mathscr{E}(\lambda M)_\infty/\mathscr{E}(\lambda M)_T\}^p\mid\mathfrak{F}_T]\\&\quad=E_Q[\exp\{(B_\tau-B_{T\wedge\tau})-(\tau-T\wedge\tau)/2\}\\&\qquad\cdot\exp\{p\lambda(M_\infty-M_T)-(\langle M\rangle_\infty-\langle M\rangle_T)\,p\lambda^2/2\}\mid\mathfrak{F}_T]\\&\quad=E_Q[\exp\{(1+2p\lambda)(B_\tau-B_{T\wedge\tau})\}\\&\qquad\cdot\exp\{-(4p\lambda^2+4p\lambda+1)(\tau-T\wedge\tau)/2\}\mid\mathfrak{F}_T],\end{aligned}$$

where $E_Q[\ ]$ denotes expectation with respect to dQ.

Thus, if $4p\lambda^2+4p\lambda+1\geqq 0$ (that is, $|\lambda+1/2|\geqq 1/(2\sqrt{q})$ where $p^{-1}+q^{-1}=1$), then we have

$$E[\{\mathscr{E}(\lambda M)_\infty/\mathscr{E}(\lambda M)_T\}^p\mid\mathfrak{F}_T]\leqq\exp\{2(1+2p|\lambda|)\}.$$

This implies that $\mathscr{E}(\lambda M)$ has all (R_p) whenever $\lambda>0$ or $\lambda\leqq -1$. Particularly, both $\mathscr{E}(M)$ and $\mathscr{E}(-M)$ have all (R_p).

On the other hand, if $-1<\lambda<0$, then $\mathscr{E}(\lambda M)$ does not satisfies (R_p) for $p\geqq(1+\pi^2/4)/\{1-(2\lambda+1)^2\}$. To verify it, recall that $E_Q[\exp(\alpha\tau)]=\infty$ for $\alpha\geqq\pi^2/8$

(see Proposition 8.4 in [20]). Since $-(4p\lambda^2+4p\lambda+1)\geqq\pi^2/4$ for such λ and p, we have

$$E[\mathscr{E}(\lambda M)_\infty^p]\geqq\exp\{-(1+2p)\}\,E_Q[\exp\{-(4p\lambda^2+4p\lambda+1)\,\tau/2\}]=\infty,$$

which implies that $\mathscr{E}(\lambda M)$ has not the (R_p) property.

We can estimate $d_1(M, L_\infty)$ as follows. Let $\lambda\geqq 1/4+\pi^2/16$. Then $2\lambda-1/2\geqq\pi^2/8$ clearly, so that

$$\begin{aligned}E[\exp\{-\lambda M_\infty\}]&=E_Q[\exp\{B_\tau-\tau/2\}\exp\{-2\lambda B_\tau+2\lambda\tau\}]\\&=E_Q[\exp\{(1-2\lambda)B_\tau\}\exp\{(2\lambda-1/2)t\}]\\&\geqq\exp\{-(1+2\lambda)\}\,E_Q[\exp\{(2\lambda-1/2)\tau\}]=\infty.\end{aligned}$$

This implies that $a(M)\leqq 1/4+\pi^2/16$. Thus $d_1(M, L_\infty)\geqq 4/(4+\pi^2)$ by Lemma 4. Further, it is easy to see that $d_1(M, L_\infty)\leqq 16$.

4. The Distance to L_∞ in BMO

The purpose of this section is to establish a new relation between the reverse Hölder inequality and the distance in BMO to L_∞. Roughly speaking, we show that $\mathscr{E}(M)$ satisfies a stronger reverse Hölder's inequality as M approaches L_∞. For that, we set

$$(5)\qquad \Phi(p)=[1+p^{-2}\log\{(2p-1)/2(p-1)\}]^{1/2}-1\qquad(1<p<\infty).$$

It is clearly a continuous decreasing function such that $\Phi(1+0)=\infty$ and $\Phi(\infty)=0$. The reverse Hölder inequality for $\mathscr{E}(M)$ was first obtained by Doléans-Dade and Meyer [3]. Recently, Emery [5] has given another proof of this result. The following is obtained by examining carefully the proof of Emery.

Lemma 5. *Let* $1<p<\infty$. *If* $\|M\|_{\mathrm{BMO}_2}<\Phi(p)$, then $\mathscr{E}(M)$ satisfies (R_p).

Proof. We exclude the trivial case $\|M\|_{\mathrm{BMO}_2}=0$, and let us set $n(M)=2\|M\|_{\mathrm{BMO}_1}+\|M\|_{\mathrm{BMO}_2}^2$ for convenience' sake.

Suppose now $\|M\|_{\mathrm{BMO}_2}<\Phi(p)$. Then we have

$$n(M)\leqq(\|M\|_{\mathrm{BMO}_2}+1)^2-1<p^{-2}\log\{(2p-1)/2(p-1)\}$$

and so $0<2(p-1)(2p-1)^{-1}\exp\{p^2 n(M)\}<1$. The main point in proving Lemma 5 is to verify that

$$(6)\qquad E[\mathscr{E}(M)_\infty^p]\leqq\frac{2}{[1-2(p-1)(2p-1)^{-1}\exp\{p^2 n(M)\}]}.$$

For simplicity, let $K_{p,M}$ denote the right hand side. For any stopping time T we have $n(M^T)\leqq n(M)$ and so $K_{p,M^T}\leqq K_{p,M}$. Therefore, in order to show

(6), we may assume that $\mathscr{E}(M)$ is bounded. Next, let $\delta=\exp\{-pn(M)\}$, which is smaller than 1. A key to the proof of (6) is to use the following inequality:

$$(7)\qquad E[\mathscr{E}(M)_\infty : \mathscr{E}(M)_\infty > \lambda] \leqq \frac{2p\lambda}{2p-1} P\{\mathscr{E}(M)_\infty > \delta\lambda\} \qquad (\lambda > 1).$$

We are now going to prove this inequality. For that, consider the stopping time $T=\inf\{t: \mathscr{E}(M)_t > \lambda\}$. Noticing $\log 1/\delta = pn(M)$, we find

$$\begin{aligned}
&P\{\mathscr{E}(M)_\infty/\mathscr{E}(M)_T < \delta \mid \mathfrak{F}_T\} \\
&\quad = P\{1/\delta < \mathscr{E}(M)_T/\mathscr{E}(M)_\infty \mid \mathfrak{F}_T\} \\
&\quad = P\{pn(M) < M_T - M_\infty + (\langle M\rangle_\infty - \langle M\rangle_T)/2 \mid \mathfrak{F}_T\} \\
&\quad \leqq \frac{1}{2pn(M)} \{2E[|M_\infty - M_T| \mid \mathfrak{F}_T] + E[\langle M\rangle_\infty - \langle M\rangle_T \mid \mathfrak{F}_T]\} \\
&\quad \leqq \frac{n(M)}{2pn(M)} = \frac{1}{2p},
\end{aligned}$$

so that $P\{\mathscr{E}(M)_\infty/\mathscr{E}(M)_T \geqq \delta \mid \mathfrak{F}_T\} \geqq 1-1/(2p)$. In addition to it, noticing $\mathscr{E}(M)_T=\lambda$ on $\{T<\infty\}$, we can obtain

$$P\{\mathscr{E}(M)_\infty \geqq \delta\lambda \mid \mathfrak{F}_T\} \geqq \frac{2p-1}{2p} I_{\{T<\infty\}}.$$

Therefore, it follows that

$$\begin{aligned}
E[\mathscr{E}(M)_\infty : \mathscr{E}(M)_\infty > \lambda] &\leqq E[\mathscr{E}(M)_\infty : T<\infty] \\
&\leqq E[\mathscr{E}(M)_T : T<\infty] \\
&\leqq \lambda P(T<\infty) \\
&\leqq \frac{2p\lambda}{2p-1} P\{\mathscr{E}(M)_\infty \geqq \delta\lambda\}.
\end{aligned}$$

Then, multiplying both sides of (7) by $(p-1)\lambda^{p-2}$ and integrating with respect to λ on the interval $[1, \infty[$, we find

$$E[\mathscr{E}(M)_\infty^p - \mathscr{E}(M)_\infty : \mathscr{E}(M)_\infty > 1] \leqq \frac{2(p-1)}{2p-1} E[\{\mathscr{E}(M)_\infty/\delta\}^p - 1 : \mathscr{E}(M)_\infty > \delta],$$

that is,

$$\left\{1 - \frac{2(p-1)}{(2p-1)\delta^p}\right\} E[\mathscr{E}(M)_\infty^p : \mathscr{E}(M)_\infty > 1] \leqq 1 + \frac{2(p-1)}{2(p-1)\delta^p}.$$

Obviously, this yields (6).

Secondly, let T be any fixed stopping time. For an arbitrary element A of $\mathfrak{F}_T$ such that $P(A)>0$, we set

$$dP' = I_A dP/P(A), \qquad \mathfrak{F}'_t = \mathfrak{F}_{T+t}, \qquad M'_t = M_{T+t} - M_T \qquad (0 \leqq t < \infty).$$

Clearly dP' is a probability measure and the process $M'=(M'_t, \mathfrak{F}'_t)$ is a martingale with respect to dP'. Note that $\mathscr{E}(M')_t = \mathscr{E}(M)_{T+t}/\mathscr{E}(M)_T$. An elementary calculation shows that $\|M'\|_{\mathrm{BMO}_r(P')} \leqq \|M\|_{\mathrm{BMO}_r(P)}$ for every $r>1$. Thus $\|M'\|_{\mathrm{BMO}_2(P')} < \Phi(p)$. Then, repeating the same argument as above, we get

$$E'[\mathscr{E}(M')^p_\infty] \leqq K_{p,M'},$$

where $E'[\ ]$ denotes expectation over Ω with respect to dP' and $K_{p,M'}$ is the constant corresponding to $K_{p,M}$ in (6). Namely, we have

$$E[\{\mathscr{E}(M)_\infty/\mathscr{E}(M)_T\}^p : A] \leqq K_{p,M'} P(A).$$

However, since $\|M'\|_{\mathrm{BMO}_r(P')} \leqq \|M\|_{\mathrm{BMO}_r(P)}$, we have $n(M') \leqq n(M)$ and so $K_{p,M'} \leqq K_{p,M}$. Thus the inequality

$$E[\mathscr{E}(M)^p_\infty : A] \leqq K_{p,M}\, \mathscr{E}(M)^p_T P(A)$$

is valid for every $A \in \mathfrak{F}_T$. This yields the reverse Hölder inequality (R_p) for $\mathscr{E}(M)$. Hence the lemma is established.

In the following theorem, L^∞_K denotes the class of all martingales bounded by a positive constant K.

Theorem 2. *Let* $1<p<\infty$. *If* $d_2(M, L^\infty_K) < e^{-K}\Phi(p)$, *then* $\mathscr{E}(M)$ *has* (R_p).

Proof. By the assumption, $\|M-N\|_{\mathrm{BMO}_2} < e^{-K}\Phi(p)$ for some $N \in L^\infty_K$. Let now $d\hat{P} = \exp(N_\infty - \langle N\rangle_\infty/2)\, dP$, which is obviously a probability measure. We set $X = N - M$. According to Lemma 1, $\hat{X} = M - N - \langle M-N, N\rangle$ is a martingale with respect to $d\hat{P}$ such that $\langle \hat{X}\rangle = \langle X\rangle$, and by the definition of the conditional expectation we have

$$\begin{aligned}&\hat{E}[\langle \hat{X}\rangle_\infty - \langle X\rangle_T \mid \mathfrak{F}_T]\\ &\quad = E[(\langle X\rangle_\infty - \langle X\rangle_T)\exp\{(N_\infty - N_T) - (\langle N\rangle_\infty - \langle N\rangle_T)/2\} \mid \mathfrak{F}_T]\\ &\quad \leqq e^{2K}\|X\|^2_{\mathrm{BMO}_2} < \Phi(p)^2.\end{aligned}$$

That is, $\|\hat{X}\|_{\mathrm{BMO}_2(\hat{P})} < \Phi(p)$. Then, according to Lemma 5, the exponential martingale $\mathscr{E}(\hat{X})$ satisfies the reverse Hölder inequality

$$\hat{E}[\mathscr{E}(\hat{X})^p_\infty \mid \mathfrak{F}_T] \leqq K_{p,\hat{X}}\, \mathscr{E}(\hat{X})^p_T.$$

On the other hand, since $\langle M\rangle = \langle M-N\rangle + 2\langle M-N, N\rangle + \langle N\rangle$, we have

$$\begin{aligned}\mathscr{E}(M) &= \exp\{(M-N-\langle M-N, N\rangle) - \langle M-N\rangle/2\}\exp(N - \langle N\rangle/2)\\ &= \mathscr{E}(\hat{X})\exp(N - \langle N\rangle/2).\end{aligned}$$

Therefore, for every stopping time T, we find

$$\begin{aligned}E[\mathscr{E}(M)^p_\infty \mid \mathfrak{F}_T] &= E[\{\mathscr{E}(M)_\infty/\mathscr{E}(M)_T\}^p \mid \mathfrak{F}_T]\, \mathscr{E}(M)^p_T\\ &\leqq \exp\{2(p-1)K\}\, \hat{E}[\{\mathscr{E}(\hat{X})_\infty/\mathscr{E}(\hat{X})_T\}^p \mid \mathfrak{F}_T]\, \mathscr{E}(M)^p_T\\ &\leqq \exp\{2(p-1)K\}\, K_{p,\hat{X}}\, \mathscr{E}(M)^p_T,\end{aligned}$$

which completes the proof.

Example 1 shows that the converse statement in the Theorem 2 fails. Now we give a variant of Theorem 2.

Theorem 3. *Let* $1<p<\infty$. *If there is* $N \in \bar{L}_\infty$ *such that* $\langle M-N, N\rangle = 0$ *and* $\|M-N\|_{\mathrm{BMO}_2} < \Phi(p)$, *then* $\mathscr{E}(M)$ satisfies (R_p).

Proof. The function Φ being continuous, $\|M-N\|_{\mathrm{BMO}_2}<\Phi(u)$ for some $u>p$. Then the exponential martingale $\mathscr{E}(M-N)$ has (R_u) by Lemma 5, and the another exponential martingale $\mathscr{E}(N)$ satisfies (R_r) for all $r>1$ by Theorem 1. Furthermore, we have $\mathscr{E}(M)=\mathscr{E}(M-N)\,\mathscr{E}(N)$ since $\langle M-N, N\rangle=0$. Hence, applying the Hölder inequality with the exponents $\alpha=u/p$ and $\beta=\alpha/(\alpha-1)$, we find

$$\begin{aligned} E[\mathscr{E}(M)_\infty^p\,|\,\mathfrak{F}_T] &= E[\{\mathscr{E}(M)_\infty/\mathscr{E}(M)_T\}^p\,|\,\mathfrak{F}_T]\,\mathscr{E}(M)_T^p \\ &\leqq E[\{\mathscr{E}(M-N)_\infty/\mathscr{E}(M-N)_T\}^u\,|\,\mathfrak{F}_T]^{1/\alpha} \\ &\quad\cdot E[\{\mathscr{E}(N)_\infty/\mathscr{E}(N)_T\}^{p\beta}\,|\,\mathfrak{F}_T]^{1/\beta}\,\mathscr{E}(M)_T^p \\ &\leqq C_{p,M,N}\,\mathscr{E}(M)_T^p. \end{aligned}$$

This completes the proof.

As is well known, $M^T\in L_\infty$ for some stopping time T. Let $\mathscr{T}(M)$ be the class of these kind of martingales. Then we get the following.

Corollary. *Let* $1<p<\infty$. *If* $d_2(M,\mathscr{T}(M))<\Phi(p)$, *then* $\mathscr{E}(M)$ *has* (R_p).

Proof. Since $\langle M-M^T, M^T\rangle=0$ for any stopping time T, the conclusion follows at once from Theorem 3.

5. A Subclass of BMO Related to the (A_p) Condition

Unlike the (R_p) condition, the (A_p) condition is remotely related to the distance between M and L_∞. We first give an example which substantiates this view.

Example 2. Let $M=B^\tau$ where $\tau=\inf\{t: |B_t|=1\}$. Then $M\in L_\infty$ clearly, and so, according to Theorem 1, the exponential martingale $\mathscr{E}(M)$ satisfies (R_p) for all $p>1$. However, it does not satisfy (A_p) for p with $1<p<1+4/\pi^2$. In fact, since $E[\exp(\alpha\tau)]=\infty$ for $\alpha\geqq\pi^2/8$ and $1/\{2(p-1)\}\geqq\pi^2/8$ for $1<p\leqq 1+4/\pi^2$, we find

$$E[\{1/\mathscr{E}(M)\}_\infty^{1/(p-1)}]\geqq\exp\{-1/(p-1)\}\,E[\exp(\tau/\{2(p-1)\})]=\infty.$$

This implies that for $1<p\leqq 1+4/\pi^2$ the (A_p) condition fails.

To discuss the (A_p) condition, we shall consider the class

$$H^\#=\{M\in\mathrm{BMO}: E[\langle M\rangle_\infty\,|\,\mathfrak{F}_.]\in\bar{L}_\infty\}$$

in place of L_∞. It is easy to see that $\|E[\langle M\rangle_\infty\,|\,\mathfrak{F}_.]\|_{\mathrm{BMO}_1}\leqq 2\|M\|_{\mathrm{BMO}_2}^2$. Clearly, $M\in H_\infty$ if and only if $E[\langle M\rangle_\infty\,|\,\mathfrak{F}_.]\in L_\infty$, so that $H_\infty\subset H^\#$. The aim of this section is to claim that $H^\#$ is closely connected with the (A_p) condition. To verify it, we need the following elementary result.

Lemma 6. *If* $M\in\mathrm{BMO}$, *then for every* $\lambda>0$ *and every stopping time* T *we have*

(i) $E[\exp\{\lambda(\langle M\rangle_\infty-\langle M\rangle_T)\}\,|\,\mathfrak{F}_T]$
$\qquad\leqq\exp(\lambda\|M\|_{\mathrm{BMO}_2}^2)\,E[\exp\{\lambda\,|\langle M\rangle_\infty-E[\langle M\rangle_\infty\,|\,\mathfrak{F}_T]|\}\,|\,\mathfrak{F}_T]$

(ii) $E[\exp\{\lambda\,|\langle M\rangle_\infty-E[\langle M\rangle_\infty\,|\,\mathfrak{F}_T]|\}\,|\,\mathfrak{F}_T]$
$\qquad\leqq\exp(\lambda\|M\|_{\mathrm{BMO}_2}^2)\,E[\exp\{\lambda(\langle M\rangle_\infty-\langle M\rangle_T)\}\,|\,\mathfrak{F}_T]$.

Proof. This lemma follows immediately from the definition of the norm $\|M\|_{\mathrm{BMO}_2}$. In fact, for every $\lambda>0$ we have

$$\begin{aligned}&E[\exp\{\lambda(\langle M\rangle_\infty-\langle M\rangle_T)\}\,|\,\mathfrak{F}_T]\\&\quad\leqq E[\exp\{\lambda|\langle M\rangle_\infty-E[\langle M\rangle_\infty\,|\,\mathfrak{F}_T]|\}\exp\{\lambda E[\langle M\rangle_\infty-\langle M\rangle_T\,|\,\mathfrak{F}_T]\}\,|\,\mathfrak{F}_T]\\&\quad\leqq \exp(\lambda\|M\|^2_{\mathrm{BMO}_2})\,E[\exp\{\lambda|\langle M\rangle_\infty-E[\langle M\rangle_\infty\,|\,\mathfrak{F}_T]\}|\,|\,\mathfrak{F}_T].\end{aligned}$$

Thus (i) holds. The proof of (ii) is similar, and so we omit it.

Theorem 4. *For $1<p<\infty$ we have the following:*

(i) *If $d_1(E[\langle M\rangle_\infty\,|\,\mathfrak{F}_.], L_\infty)<(\sqrt{p}-1)^2/2$, then both $\mathscr{E}(M)$ and $\mathscr{E}(-M)$ satisfy (A_p).*

(ii) *Conversely, if both $\mathscr{E}(M)$ and $\mathscr{E}(-M)$ satisfy (A_p), then $d_1(E[\langle M\rangle_\infty\,|\,\mathfrak{F}_.], L_\infty)<8(p-1)$.*

Proof. We first show (i). Suppose $d_1(E[\langle M\rangle_\infty\,|\,\mathfrak{F}_.], L_\infty)<(\sqrt{p}-1)^2/2$. Then, according to the left-hand side inequality in Lemma 4, we have

$$\frac{1}{2(\sqrt{p}-1)^2}<a(E[\langle M\rangle_\infty\,|\,\mathfrak{F}_.]),$$

so that by (i) in Lemma 6

$$E\left[\exp\left\{\frac{1}{2(\sqrt{p}-1)^2}(\langle M\rangle_\infty-\langle M\rangle_T)\right\}\middle|\,\mathfrak{F}_T\right]\leqq C_p.$$

Let now $r=\sqrt{p}+1$. The exponent conjugate to r is $s=(\sqrt{p}+1)/\sqrt{p}$. Since $\{s(\sqrt{p}-1)^2\}^{-1}-r(p-1)^{-2}=(p-1)^{-1}$, we have

$$\begin{aligned}\{\mathscr{E}(M)_T/\mathscr{E}(M)_\infty\}^{1/(p-1)}&=\exp\left\{-\frac{1}{p-1}(M_\infty-M_T)-\frac{r}{2(p-1)^2}(\langle M\rangle_\infty-\langle M\rangle_T)\right\}\\&\quad\cdot\exp\left\{\frac{1}{2s(\sqrt{p}-1)^2}(\langle M\rangle_\infty-\langle M\rangle_T)\right\}.\end{aligned}$$

We apply Hölder's inequality with the exponents r and s:

$$\begin{aligned}E[\{\mathscr{E}(M)_T/\mathscr{E}(M)_\infty\}^{1/(p-1)}\,|\,\mathfrak{F}_T]&\leqq E\left[\mathscr{E}\left(-\frac{r}{p-1}M\right)_\infty\Big/\mathscr{E}\left(-\frac{r}{p-1}M\right)_T\middle|\,\mathfrak{F}_T\right]^{1/r}\\&\quad\cdot E\left[\exp\left\{\frac{1}{2(\sqrt{p}-1)^2}(\langle M\rangle_\infty-\langle M\rangle_T)\right\}\middle|\,\mathfrak{F}_T\right]^{1/s}.\end{aligned}$$

The first conditional expectation on the right hand side equals 1, since $\mathscr{E}(-rM/(p-1))$ is a uniformly integrable martingale. The second term is dominated by some constant C_p as stated above. In the same way we can verify that $\mathscr{E}(-M)$ satisfies the (A_p) condition.

To show (ii), we use the inequality

$$E\left[\exp\left\{\frac{1}{2(p-1)}(\langle M\rangle_\infty-\langle M\rangle_T)\right\}\middle|\,\mathfrak{F}_T\right]\leqq E[\{\mathscr{E}(M)_T/\mathscr{E}(M)_\infty\}^{1/(p-1)}\,|\,\mathfrak{F}_T]^{1/2}$$
$$\cdot E[\{\mathscr{E}(-M)_T/\mathscr{E}(-M)_\infty\}^{1/(p-1)}\,|\,\mathfrak{F}_T]^{1/2},$$

which follows from the Schwarz inequality. This inequality implies that

$$E\left[\exp\left\{\frac{1}{2(p-1)}(\langle M\rangle_\infty-\langle M\rangle_T)\right\}\middle|\,\mathfrak{F}_T\right]\leqq C_p$$

whenever both $\mathscr{E}(M)$ and $\mathscr{E}(-M)$ have (A_p). Then $a(E[\langle M\rangle_\infty\,|\,\mathfrak{F}_.])\geqq 1/\{2(p-1)\}$ by (ii) in Lemma 6. On the other hand, if $\mathscr{E}(M)$ has (A_p), then it satisfies $(A_{p-\varepsilon})$ for some ε with $0<\varepsilon<p-1$ (see [3, p. 323]). Thus $a(E[\langle M\rangle_\infty\,|\,\mathfrak{F}_.])>1/\{2(p-1)\}$ in fact, and just then we have $d_1(E[\langle M\rangle_\infty\,|\,\mathfrak{F}_.], L_\infty)<8(p-1)$ by the right-hand side inequality in Lemma 4. This completes the proof.

As a corollary, we can obtain the following.

Theorem 5. *In order that both $\mathscr{E}(M)$ and $\mathscr{E}(-M)$ satisfy all (A_p), a necessary and sufficient condition is that M belongs to the class $H^\#$.*

We now remark that there exists a relation between $H^\#$ and L_∞. An application of the Schwartz inequality yields that for every $\lambda>0$

$$\begin{aligned}&E[\exp\{\lambda(M_\infty-M_T)\}\,|\,\mathfrak{F}_T]\\&\quad=E[\{\mathscr{E}(2\lambda M)_\infty/\mathscr{E}(2\lambda M)_T\}^{1/2}\exp\{\lambda^2(\langle M\rangle_\infty-\langle M\rangle_T)\}\,|\,\mathfrak{F}_T]\\&\quad\leqq E[\mathscr{E}(2\lambda M)_\infty/\mathscr{E}(2\lambda M)_T\,|\,\mathfrak{F}_T]^{1/2}\,E[\exp\{2\lambda^2(\langle M\rangle_\infty-\langle M\rangle_T)\}\,|\,\mathfrak{F}_T]^{1/2}\\&\quad\leqq E[\exp\{2\lambda^2(\langle M\rangle_\infty-\langle M\rangle_T)\}\,|\,\mathfrak{F}_T]^{1/2}.\end{aligned}$$

The same argument works if M is replaced by $-M$. Then for all $\lambda>0$

$$E[\exp\{\lambda|M_\infty-M_T|\}\,|\,\mathfrak{F}_T]\leqq 2E[\exp\{2\lambda^2(\langle M\rangle_\infty-\langle M\rangle_T)\}\,|\,\mathfrak{F}_T]^{1/2}.$$

Therefore, from Lemmas 4 and 6 it follows that $H^\#\subset\bar{L}_\infty$.

Finally, we give a remark on H_∞.

Theorem 6. *Let $1<p<\infty$ and $p^{-1}+q^{-1}=1$. If $d_2(M, H_\infty)<\Phi(p)$, then $\mathscr{E}(M)$ satisfies both (R_p) and (A_q).*

Proof. By the assumption, $\|M-N\|_{\mathrm{BMO}_2}<\Phi(p)$ for some $N\in H_\infty$. Consider now the new probability measure $dQ=\mathscr{E}(M-N)_\infty\,dP$ and set $\tilde{N}=N-\langle M-N, N\rangle$. Then $\tilde{N}\in\mathrm{BMO}(Q)$ and $\langle\tilde{N}\rangle=\langle N\rangle$ by Lemma 2. Further we have

$$\mathscr{E}(M)=\mathscr{E}(M-N)\,\mathscr{E}(\tilde{N}), \tag{8}$$

where $\mathscr{E}(\tilde{N})$ is the exponential martingale under dQ.

We first verify that $\mathscr{E}(M)$ satisfies (R_p). By the definition of H_∞, $\langle N\rangle$ is bounded, so that $\tilde{N}\in H_\infty(dQ)\subset\overline{L_\infty(dQ)}$. Therefore, it follows from Theorem 1 that $\mathscr{E}(\tilde{N})$ satisfies all (R_r) under dQ. Then, noticing (8) and applying Hölder's inequality with exponents r and $s=r/(r-1)$, we find

$$\begin{aligned} E[\{\mathscr{E}(M)_\infty/\mathscr{E}(M)_T\}^p \mid \mathfrak{F}_T] &= E[\{\mathscr{E}(M-N)_\infty/\mathscr{E}(M-N)_T\}^{p-1/r} \\ &\quad \cdot \{\mathscr{E}(M-N)_\infty/\mathscr{E}(M-N)_T\}^{1/r}\{\mathscr{E}(\tilde{N})_\infty/\mathscr{E}(\tilde{N})_T\}^p \mid \mathfrak{F}_T] \\ &\leqq E[\{\mathscr{E}(M-N)_\infty/\mathscr{E}(M-N)_T\}^{(p-1/r)s} \mid \mathfrak{F}_T]^{1/s} \\ &\quad \cdot E_Q[\{\mathscr{E}(\tilde{N})_\infty/\mathscr{E}(\tilde{N})_T\}^{pr} \mid \mathfrak{F}_T]^{1/r} \\ &\leqq C_{p,r}\, E[\{\mathscr{E}(M-N)_\infty/\mathscr{E}(M-N)_T\}^{(p-1/r)s} \mid \mathfrak{F}_T]^{1/s}, \end{aligned}$$

where $E_Q[\ ]$ denotes expectation with respect to dQ.

Since $p<(p-1/r)s=(pr-1)/(r-1)\to p$ as $r\to\infty$, there is some $r>1$ such that $\|M-N\|_{\mathrm{BMO}_2}<\Phi((p-1/r)s)$. Then, according to Lemma 5, $\mathscr{E}(M-N)$ satisfies the $(R_{(p-1/r)s})$ condition. Thus $\mathscr{E}(M)$ has (R_p).

We are now going to prove that $\mathscr{E}(M)$ satisfies (A_q). An elementary calculation shows that $\Phi(p)<\sqrt{q}-1$, so that $\|M-N\|_{\mathrm{BMO}_2}<\sqrt{q}-1$ for some $N\in H_\infty$. Note that $\langle M\rangle_t-\langle M\rangle_s\leqq 2\{(\langle M-N\rangle_t-\langle M-N\rangle_s)+(\langle N\rangle_t-\langle N\rangle_s)\}$ for $s<t$. Recalling the boundedness of $\langle N\rangle$ and applying (3) in Lemma 3 we find

$$\begin{aligned} &E\left[\exp\left\{\frac{1}{2(\sqrt{q}-1)^2}(\langle M\rangle_\infty-\langle M\rangle_T)\right\}\middle|\mathfrak{F}_T\right] \\ &\leqq C_q E\left[\exp\left\{\frac{1}{(\sqrt{q}-1)^2}(\langle M-N\rangle_\infty-\langle M-N\rangle_T)\right\}\middle|\mathfrak{F}_T\right] \\ &\leqq C_q\left\{1-\frac{1}{(\sqrt{q}-1)^2}\|M-N\|^2_{\mathrm{BMO}_2}\right\}^{-1}. \end{aligned}$$

Let now $r=\sqrt{q}+1$ and $s=(\sqrt{q}+1)/\sqrt{q}$. Then, an application of the Hölder inequality yields

$$\begin{aligned} &E[\{\mathscr{E}(M)_T/\mathscr{E}(M)_\infty\}^{1/(q-1)} \mid \mathfrak{F}_T] \\ &= E\left[\exp\left\{-\frac{1}{q-1}(M_\infty-M_T)-\frac{r}{2(q-1)^2}(\langle M\rangle_\infty-\langle M\rangle_T)\right\}\right. \\ &\quad \left.\cdot \exp\left\{\frac{1}{2s(\sqrt{q}-1)^2}(\langle M\rangle_\infty-\langle M\rangle_T)\right\}\middle|\mathfrak{F}_T\right] \\ &\leqq E\left[\mathscr{E}\left(-\frac{r}{q-1}M\right)_\infty\Big/\mathscr{E}\left(-\frac{r}{q-1}M\right)_T\middle|\mathfrak{F}_T\right]^{1/r} \\ &\quad \cdot E\left[\exp\left\{\frac{1}{2(\sqrt{q}-1)^2}(\langle M\rangle_\infty-\langle M\rangle_T)\right\}\middle|\mathfrak{F}_T\right]^{1/s}. \end{aligned}$$

The first conditional expectation in the last expression equals to 1, and the second one is bounded by some constant as remarked above. Thus the proof is complete.

Combining Theorems 5 and 6, it follows immediately that $\bar{H}_\infty \subset H^\#$. However, it is probable that the class $H^\#$ is nothing but $\bar{H}_\infty$. A key to verify it is to establish the following inequality:

$$b(M) \leqq \frac{C}{d_2(M, H_\infty)}$$

where $b(M)$ denotes the supremum of the set of b for which

$$\sup_T \|E[\exp\{b^2(\langle M\rangle_\infty - \langle M\rangle_T)\} \mid \mathfrak{F}_T]\|_\infty < \infty.$$

But this remains to be proved. We remark in passing that the inequality

$$\frac{1}{\sqrt{2}\, d_2(M, H_\infty)} \leqq b(M)$$

is valid for every $M \in \mathrm{BMO}$ (see [16]).

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