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Probability Theory and Related Fields

On the Exceedance Point Process for a Stationary Sequence*

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Summary. It is known that the exceedance points of a high level by a stationary sequence are asymptotically Poisson as the level increases, under appropriate long range and local dependence conditions. When the local dependence conditions are relaxed, clustering of exceedances may occur, based on Poisson positions for the clusters. In this paper a detailed analysis of the exceedance point process is given, and shows that, under wide conditions, any limiting point process for exceedances is necessarily compound Poisson. More generally the possible random measure limits for normalized exceedance point processes are characterized. Sufficient conditions are also given for the existence of a point process limit. The limiting distributions of extreme order statistics are derived as corollaries.

1. Introduction

Many problems in extremal theory may be most naturally and profitably discussed in terms of certain underlying point processes. Typically one is interested in the limit of a sequence of point processes obtained from extremal considerations, and it is often the case that a Poisson convergence result can be derived. For example, Pickands [13], Resnick [14] and Shorrock [17] all consider point processes involving "record times" in i.i.d. settings – a research direction which was initiated by the works of Dwass [2] and Lamperti [6] on extremal processes. Resnick [15] further noted that many results in this setting can be derived from a "Complete Poisson Convergence Theorem" in two dimensions.

It is known that the i.i.d. assumption can often be relaxed. For example Leadbetter [8] considers the point process of exceedances of a high level u_n by a stationary sequence ξ_i (i.e., points where $\xi_i > u_n$), obtaining Poisson limits under quite weak dependence restrictions. These involve a long range dependence condition "$D(u_n)$" of mixing type, but much weaker than strong mixing,

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and a local dependence condition "$D'(u_n)$". Adler [1] generalizes Resnick's two dimensional result in [15] by assuming the conditions D and D'. In results of this kind, the long range dependence condition (e.g., $D(u_n)$) is used to give asymptotic independence of exceedances whereas the local restriction (e.g., $D'(u_n)$) avoids clustering of exceedances. As a result in the limit, the point process under consideration behaves just like one obtained from an i.i.d. sequence. If the local condition is weakened or omitted, then clustering of exceedances may occur. This clustering does not materially affect the asymptotic distribution of the maximum, but more significantly changes those of all other extreme order statistics. Some such situations have been considered. For example, Rootzén [16] studies the exceedance point process for a class of stable process. Leadbetter [9] considers Poisson results for cluster centers which yield the asymptotic distribution of the sequence maxima but not of other order statistics. Mori [12] characterizes the limit of a sequence of point processes in two dimensions under strong-mixing.

Our aim in this work is to study the detailed structure of the limiting forms of exceedance point processes under broad assumptions – especially when clustering may occur. The results yield, in particular, the asymptotic distributions of extreme order statistics in the more general form required by the presence of high local dependence.

Section 2 of this paper summarizes the notation used regarding random measures and point processes, along with a discussion and some basic results on the dependence condition $\Delta(u_n)$ under which the theory of this paper is developed.

In Sect. 3, the possible distributional limits are characterized, under $\Delta(u_n)$, for normalized point processes of exceedances of arbitrary levels u_n. These limits form a class of infinitely divisible random measures, whereas the limits of non-normalized point processes of exceedances must be compound Poisson.

Section 4 studies the connection between the exceedance point process and a cluster distribution, thereby obtaining a sufficient condition for the convergence of the point process. The case where the levels u_n are normalized (i.e., coordinated with sample size in such a way that the exceedance rate is approximately constant) is considered in Sect. 5.

As noted above (cf. also [9]) the presence of exceedance clustering does not affect the asymptotic distribution of the maximum. It does, however, alter the asymptotic distributions of other order statistics by virtue of the fact that, e.g., the second largest value may now occur in the same cluster as the largest. In Sect. 6 we apply the results of earlier sections to obtain specific forms for the asymptotic distributions of extreme order statistics in terms of the relevant extreme value distributions for the maximum, and the cluster size distribution.

Several suggestions from the editors have led to improved results in this paper, including the use of random measure rather than purely point process limits and a more detailed treatment of non normalized levels. As the Editors have also pointed out, it would be possible to cast a number of the results in terms of sequences of point processes which are not necessarily exceedances, but satisfy appropriate dependence restrictions. While not explicitly stated for clarity of exposition, such generalizations will be evident to the reader.

Finally we note that corresponding multi-level theorems and generalizations of the two-dimensional point process result of [12] may be found in the thesis [3], a separate paper (by T. Hsing) also being planned on this topic.

2. Preliminaries and Framework

Throughout we shall be concerned with point processes and random measures on the space [0, 1], i.e., random elements η of the space M of Borel measures on [0, 1] with the vague topology and Borel σ-field (η being integer valued in the point process case). The notation $L_\eta(f)$ will be used for the Laplace Transform of η, i.e., $L_\eta(f)=\mathscr{E}\exp(-\int_{[0,1]} f\,d\eta)$, defined for non-negative measurable f on [0, 1]. The basic properties of random measures and point processes needed here will be stated as used and may be found in detail in [5].

Throughout, $\xi_1, \xi_2, \ldots$ will be a stationary sequence of random variables. Write $M(I)=\max(\xi_i: i\in I)$ for any set I of integers, and $M_n=\max(\xi_i: 1\leqq i\leqq n)$. For a given sequence of constants $\{u_n\}$, let $\chi_{n,j}$ be the indicator of the event $(\xi_j>u_n)$, $j=1, \ldots, n$ and N_n the point process on [0, 1] with points (j/n: $1\leqq j\leqq n$ for which $\xi_j>u_n$). Thus, N_n is the point process (on [0,. 1]) of *exceedances* of the "level" u_n by the random variables $\xi_1, \ldots, \xi_n$ after "time-normalization" by the factor $1/n$.

The type of long range dependence condition appropriate for the present context is defined as follows. If $\{u_n\}$ is a sequence of constants, for each n, i, j with $1\leqq i\leqq j\leqq n$, define $\mathscr{B}_i^j(u_n)$ to be the σ-field generated by the events $(\xi_s\leqq u_n)$ $i\leqq s\leqq j$. Also for each n and $1\leqq l\leqq n-1$, write

$$\alpha_{n,l}=\max(|P(A\cap B)-P(A)P(B)|: A\in\mathscr{B}_1^k(u_n), B\in\mathscr{B}_{k+l}^n(u_n), 1\leqq k\leqq n-l). \tag{2.1}$$

$\{\xi_j\}$ is said to satisfy the condition $\Delta(u_n)$ if $\alpha_{n,l_n}\to 0$ as $n\to\infty$ for some sequence $\{l_n\}$ with $l_n=o(n)$. The array of constants $\alpha_{n,l}$, $l=1, 2, \ldots, n-1$, will be referred to as the mixing coefficients of the condition $\Delta(u_n)$ whenever there is no danger of causing ambiguity. It is worth noting that the condition $\Delta(u_n)$ is stronger than the distributional mixing condition $D(u_n)$ (cf. [10]), but weaker than strong mixing.

Since there are only a finite number of events involved for each n, the condition $\Delta(u_n)$ can be easily verified in some cases. Indeed, the strong mixing condition is "unnecessarily strong" for most situations in the study of extreme value theory in that it poses restrictions not just on the extremal, but on the overall behavior of the underlying sequence.

The condition $\Delta(u_n)$ can be expressed in terms of random variables as well. The following result is a special case of [18], equation (I').

Lemma 2.1. *For each n and $1\leqq l\leqq n-1$, write $\beta_{n,l}=\sup(|\mathscr{E}YZ-\mathscr{E}Y\cdot\mathscr{E}Z|$: Y and Z measurable with respect to $\mathscr{B}_1^j(u_n)$ and $\mathscr{B}_{j+l}^n(u_n)$ respectively, $0\leqq Y, Z\leqq 1$, $1\leqq j\leqq n-l$). Then $\alpha_{n,l}\leqq\beta_{n,l}\leqq 4\alpha_{n,l}$ where $\alpha_{n,l}$ is the mixing coefficient of the condition $\Delta(u_n)$. In particular, ξ_j satisfies the condition $\Delta(u_n)$ if and only if $\beta_{n,l_n}\to 0$ for some $\{l_n\}$ with $l_n=o(n)$.* □

The following technical result, which is slightly more general than needed, is applied extensively throughout this paper.

Lemma 2.2. *Suppose the condition $\Delta(u_n)$ holds for $\{\xi_j\}$, and that $\{k_n\}$ is a sequence of integers for which there exists a sequence $\{l_n\}$ such that $k_n l_n/n \to 0$ and $k_n \alpha_{n,l_n} \to 0$, where $\alpha_{n,l}$ is the mixing coefficient of the condition $\Delta(u_n)$. For each n, let J_{ni}, $1 \leqq i \leqq k_n$, be k_n disjoint sets of integers in $\{1, \ldots, n\}$ such that $\#\left(\bigcup_{i=1}^{k_n} J_{ni}\right) \sim n$. Then for any sequence of non-negative constants $\{a_n\}$ and any non-negative continuous or step function f on [0, 1],*

$$(2.2)\qquad d_n \overset{\text{def}}{=} \mathscr{E}\exp\left(-a_n \sum_{m=1}^{n} f(m/n)\,\chi_{n,m}\right) - \prod_{i=1}^{k_n} \mathscr{E}\exp\left(-a_n \sum_{m\in J_{ni}} f(m/n)\,\chi_{n,m}\right) \to 0 \quad \text{as } n\to\infty.$$

In particular, (2.2) holds for $J_{ni} = \{m: (i-1)r_n + 1 \leqq m \leqq i r_n\}$, $1 \leqq i \leqq k_n$, where $r_n = [n/k_n]$.

Proof. When f is identically zero the result is trivial, and hence we assume otherwise throughout. For simplicity we assume that each J_{ni} consists of at least l_n integers, and that $\bigcup_{1}^{k_n} J_{ni} = \{1, \ldots, n\}$, the proof being readily extended to the more general cases.

It is sufficient to show that for any subsequence S, there exists a further subsequence S' through which $d_n \to 0$. Let S be any subsequence of integers. Since the set $\left\{c_n \overset{\text{def}}{=} \left\{\mathscr{E}\exp\left(-a_n \sum_{m=1}^{l_n} \chi_{n,m}\right)\right\}^{k_n} : n \in S\right\}$ contains infinitely many numbers in [0, 1], there exists a subsequence S' of S such that $c_n \to$ some c through S'. We shall show that $d_n \to 0$ through S', from which (2.2) follows. In the remaining part of the proof, unless otherwise stated, limits are obtained by letting $n \to \infty$ through S'. Consider separately the following two possibilities:

(a) $c=1$. For $1 \leqq i \leqq k_n$, let I_{ni}^* be the set of the largest l_n elements in J_{ni} and $I_{ni} = J_{ni} \setminus I_{ni}^*$. By the triangle inequality, d_n is bounded in absolute value by

$$(2.3)\qquad \left|\mathscr{E}\exp\left(-a_n \sum_{m=1}^{n} f(m/n)\,\chi_{n,m}\right) - \mathscr{E}\exp\left(-a_n \sum_{i=1}^{k_n} \sum_{m\in I_{ni}} f(m/n)\,\chi_{n,m}\right)\right|$$
$$+ \left|\mathscr{E}\exp\left(-a_n \sum_{i=1}^{k_n} \sum_{m\in I_{ni}} f(m/n)\,\chi_{n,m}\right) - \prod_{i=1}^{k_n} \mathscr{E}\exp\left(-a_n \sum_{m\in I_{ni}} f(m/n)\,\chi_{n,m}\right)\right|$$
$$+ \left|\prod_{i=1}^{k_n} \mathscr{E}\exp\left(-a_n \sum_{m\in I_{ni}} f(n/m)\,\chi_{n,m}\right) - \prod_{i=1}^{k_n} \mathscr{E}\exp\left(-a_n \sum_{m\in J_{ni}} f(m/n)\,\chi_{n,m}\right)\right|.$$

By the inequality

$$(2.4)\qquad |\prod x_i - \prod y_i| \leqq \sum |x_i - y_i|, \quad 0 \leqq x_i,\, y_i \leqq 1,$$

and the fact that f is bounded by some integer A, the first term in (2.3) is bounded by

$$(2.5)\quad \sum_{i=1}^{k_n} \mathscr{E}(1-\exp(-a_n \sum_{m \in I_{ni}^*} f(m/n)\, \chi_{n,m})) \leqq \sum_{i=1}^{k_n} \mathscr{E}(1-\exp(-a_n A \sum_{m \in I_{ni}^*} \chi_{n,m}))$$

$$= k_n \mathscr{E}\left(1-\exp\left(-a_n A \sum_{m=1}^{l_n} \chi_{n,m}\right)\right) \leqq A k_n \mathscr{E}\left(1-\exp\left(-a_n \sum_{m=1}^{l_n} \chi_{n,m}\right)\right).$$

Thus the first term of (2.3) tends to zero, since $c=1$ is equivalent to $k_n E\left(1 - \exp\left(-a_n \sum_{m=1}^{l_n} \chi_{n,m}\right)\right) \to 0$. The third term in (2.3) tends to zero since it is bounded by (2.5), whereas the second term in (2.3) is bounded by $4 k_n \alpha_{n,l_n}$ (cf. Lemma 2.1) which tends to zero as well. This concludes the case $c=1$.

(b) $c<1$. Since f is nonzero, there is an interval $I \subset [0,1]$, and an $\alpha \in (0,1)$ such that $\inf_{x \in I} f(x) \geqq \alpha$. Write $nI = \{nx : x \in I\}$. For each n, in each $J_{ni} \cap nI$ which contains more than $2 l_n$ integers place θ_{ni} sets of l_n consecutive integers E_{nij}, $1 \leqq j \leqq \theta_{ni}$, where the sets E_{nij}, $1 \leqq j \leqq \theta_{ni}$, $1 \leqq i \leqq k_n$, are separated by at least l_n integers and the θ_{ni} are chosen so that $\theta_n \stackrel{\text{def}}{=} \sum_{i=1}^{k_n} \theta_{ni}$ satisfies

$$(2.6)\qquad \frac{\theta_n}{k_n} \to \infty, \quad \text{and} \quad \theta_n \alpha_{n,l_n} \to 0.$$

This can be done by the choice of k_n and the fact that for some $\lambda > 0$

$$\lim_{n\to\infty} \sum_{i=1}^{k_n} \left[\frac{\#(J_i \cap nI)}{2 l_n}\right] \Big/ k_n \geqq \lambda \lim_{n\to\infty} \frac{n}{l_n k_n} = \infty.$$

By Lemma 2.1 and stationarity,

$$\mathscr{E} \exp\left(-a_n \sum_{m=1}^{n} f(m/n)\, \chi_{n,m}\right) \leqq \mathscr{E} \exp(-a_n \sum_{m \in \bigcup_{ij} E_{nij}} f(m/n)\, x_{n,m})$$

$$\leqq \mathscr{E}^{\theta_n} \exp\left(-a_n \alpha \sum_{m=1}^{l_n} \chi_{n,m}\right) + 4 \theta_n \alpha_{n,l_n}.$$

It follows from Hölder's inequality and (2.4) that the right hand side is bounded by

$$\left\{\mathscr{E} \exp\left(-a_n \sum_{m=1}^{l_n} \chi_{n,m}\right)\right\}^{\alpha \theta_n} + o(1) = c_n^{\alpha \theta_n / k_n} + o(1) \to 0.$$

Similarly by (2.4) we obtain

$$\prod_{i=1}^{k_n} \mathscr{E} \exp(-a_n \sum_{m \in J_{ni}} f(m/n) \chi_{n,m}) \leqq \mathscr{E}^{\theta_n} \exp\left(-a_n \sum_{m=1}^{l_n} \chi_{n,m}\right) + 4\theta_n \alpha_{n,l_n}$$

which tends to zero as previously. Thus both terms in d_n tend to zero, concluding the proof. □

The following similar result can be proved along the same lines as Lemma 2.2 with even a weaker mixing condition $D(u_n)$ replacing $\Delta(u_n)$, thus generalizing Lemma 2.1 of [9]. However we are very grateful to the Editor for pointing out that under the condition $\Delta(u_n)$ it can be obtained simply from Lemma 2.2 as follows.

Lemma 2.3. *Suppose the condition $\Delta(u_n)$ holds for $\{\xi_j\}$, and let k_n, and J_{ni}, $1 \leqq i \leqq k_n$, be as defined in Lemma 2.2. Then*

$$P[M_n \leqq u_n] - \prod_{i=1}^{k_n} P[M(J_{ni}) \leqq u_n] \to 0 \quad \text{as } n \to \infty.$$

Proof. By Lemma 2.2 with $f \equiv 1$, $a_n \to \infty$, and $k_n e^{-a_n} \to 0$,

$$\mathscr{E} e^{-a_n N_n(0,1)} - \prod_{i=1}^{k_n} \mathscr{E} e^{-a_n N_n(J_{ni})} \to 0.$$

But

$$\mathscr{E} e^{-a_n N_n(0,1)} - P\{N_n(0,1)=0\} = \sum_1^\infty e^{-a_n r} P\{N_n(0,1)=r\} \leqq e^{-a_n} \to 0$$

and

$$\left|\prod_1^k \mathscr{E} e^{-a_n N_n(J_{ni})} - \prod_1^{k_n} P\{N_n(J_{ni})=0\}\right| \leqq k_n e^{-a_n} \to 0,$$

giving the desired result. □

3. Characterization of Limits

A main question of interest is the determination of the class of point processes N which may occur as limits in distribution of the exceedance point process N_n. Here we first explore the more general question to identify the class of random measures N which may appear as distributional limits of the random measures $a_n N_n$ for a sequence of non-negative constants $\{a_n\}$.

Lemma 3.1. *Let N_n be the exceedance point process corresponding to the level u_n, for the stationary sequence $\{\xi_j\}$. Suppose $\Delta(u_n)$ holds for $\{\xi_j\}$ and that $a_n N_n \xrightarrow{d} N$ for some sequence $\{a_n\}$ of non-negative constants and some random measure N. Then*

(i) *N is stationary, and in particular,* $N([a, b]) \stackrel{d}{=} N([a+\tau, b+\tau])$ *for all* $\tau > 0$, a, b, *such that* $0 \leqq a \leqq b \leqq 1-\tau$.

(ii) *N has no fixed atoms, i.e.* $N(\{x\}) = 0$ *a.s. for each* $x \in [0, 1]$.

(iii) *N has independent increments.*

Proof. (i) follows by routine calculation from stationarity of $a_n N_n$ and its convergence in distribution to N, and (ii) follows from (i) and the fact that N has at most countably many fixed atoms. Finally since for random measures $\{\eta_n\}$, η, $\eta_n \xrightarrow{d} \eta$ *if and only if* $L_{\eta_n}(f) \to L_\eta(f)$ for all non-negative measurable f on $[0, 1]$ whose set of discontinuities have zero η-measure a.s., it follows by (ii) that

$$(a_n N_n(I_1) \ldots a_n N_n(I_m)) \xrightarrow{d} (N(I_1) \ldots N(I_m))$$

for intervals $I_1 \ldots I_m$. If $I_1 \ldots I_m$ are mutually disjoint Lemma 2.2 then readily shows that $N(I_1) \ldots N(I_m)$ are independent, giving (iii). □

As a stationary random measure with independent increments on $[0, 1]$, N is thus also infinitely divisible with a (possible) degenerate component which is a constant multiple of Lebesgue measure, and canonical measure λ concentrated on the degenerate measures $y\delta_x$, $y > 0$, $x \in [0, 1]$ (where δ_x denotes unit mass at x). This leads in a standard way to the Lévy-type representation given in the following theorem.

Theorem 3.2. Let N_n be the exceedance point process corresponding to the level u_n, for the stationary sequence $\{\xi_j\}$. Suppose $\Delta(u_n)$ holds and $a_n N_n \xrightarrow{d} N$ for some sequence of constants $a_n \geqq 0$, and some random measure N. Then N has Laplace Transform given by

$$-\log L_N(f) = \alpha \int_0^1 f\,dx + \int_0^1 \int_0^\infty (1 - e^{-yf(x)})\,d\nu(y)\,dx \tag{3.1}$$

where $\alpha \geqq 0$ and the (Lévy) measure ν on $(0, \infty)$ satisfies $\int_0^\infty (1 - e^{-y})\,d\nu(y) < \infty$.

Consequently N has the cluster representation

$$N(\cdot) \stackrel{d}{=} \alpha m(\cdot) + \int_{x=0}^{1} \int_{y=0}^{\infty} y\delta_x(\cdot)\,d\eta(x, y) \tag{3.2}$$

where $\alpha \geqq 0$, m denotes Lebesgue measure, and η is a Poisson process on $[0, 1] \times (0, \infty)$ with intensity measure $m \times \nu$. □

Thus any random measure limit N for the (normalized) exceedance point process has a uniform mass on $[0, 1]$ together with a sequence of point masses y_i at points x_i where (x_i, y_i) are the points of a Poisson process in $[0, 1] \times (0, \infty)$ with intensity $m \times \nu$. If ν is finite, the x_i form a stationary Poisson process on the line with intensity parameter $\nu(0, \infty)$. In any case the points x_i for which $y_i > a$ form a Poisson process with intensity parameter $\nu(a, \infty)$ for any $a > 0$.

Note also that the Laplace Transform of $N[0, 1]$ is (from (3.1)) given by

$$-\log \mathscr{E} e^{-sN[0,1]} = \alpha s + \int_0^\infty (1-e^{-sy})\, d\nu(y)$$

and hence $P\{N[0,1]=0\} = \lim_{s\to\infty} \mathscr{E} \exp(-sN[0,1]) = 0$ if $\alpha>0$, and $\exp(-\nu(0,\infty))$ if $\alpha=0$. Thus $P\{N[0,1]=0\}>0$ if and only if $\alpha=0$ and $\nu(0,\infty)<\infty$. That is, if $\alpha\neq 0$ or $\nu(0,\infty)=\infty$, the interval $[0, 1]$ (and likewise every interval) contains mass with probability one.

If $\alpha=0$ and $\nu(0,\infty)<\infty$ write π for the probability distribution $\nu(\cdot)/\nu(0,\infty)$ and ϕ for its Laplace Transform, obtaining, from (3.1)

$$-\log L_N(f) = \nu \int_0^1 [1-\phi(f(x))]\, dx \tag{3.3}$$

($\nu=\nu(0,\infty)$), which is the standard representation for a compound Poisson process occurring with a Poisson rate ν and (not necessarily integer-valued) multiplicities with distribution π.

We now discuss briefly how the behavior of a_n (at ∞) relates to the form of N. Assume the conditions of Lemma 3.1 and consider the following situations:

(a) Suppose there exists a subsequence $\{a_{n'}\}$ of $\{a_n\}$ tending to some a in $(0,\infty)$. Then obviously N_n converges without normalization, and the limit is necessarily a point process since the distributional limit of point processes must be a point process (cf. [5], 15.7.4). Thus, in the representation (3.1) of the limit, $\alpha=0$ and $\nu(\cdot)$ is finite and is concentrated on the positive integers. In particular, then, the following result holds.

Corollary 3.3. *Let N_n be the exceedance point process corresponding to the level u_n in the stationary sequence $\{\xi_j\}$. Suppose that $\Delta(u_n)$ holds for $\{\xi_j\}$ and that $N_n \xrightarrow{d} N$, for some point process N. Then N is necessarily a compound Poisson point process with Laplace Transform given by* (3.3), *Poisson rate ν, and $\phi(s) = \sum_{j=1}^\infty \pi(j)\, e^{-sj}$, $(\pi(j), j=1, 2, \ldots)$ being the distribution of multiplicities.* □

(b) Suppose there exists a subsequence $\{a_{n'}\}$ of $\{a_n\}$ tending to infinity. Then for each $c>0$ $[c a_{n'}] N_{n'} \xrightarrow{d} cN$ since $c a_{n'} \sim [c a_{n'}]$. As a limit of point processes, cN is a point process for each $c>0$, showing that N is the null measure with probability one.

(c) Suppose $\{a_{n'}\}$ is a subsequence which tends to zero. It is not clear what forms N may have. However, if the mixing rate α_{n,l_n} tends to zero sufficiently fast, a weak law of large numbers can be proved, showing that $N=\alpha m(\cdot)$ for some α.

The remaining part of this paper will be exclusively confined to the setting in (a).

4. Clustering of Exceedances and a Sufficient Condition for Convergence of N_n

The multiplicity distribution π of the point process limit N of N_n has an intuitively appealing interpretation as a limiting distribution of a cluster of exceedances of u_n, by $\{\xi_j\}$, as we shall now see.

First separate $\{\xi_j\}$ into successive groups $(\xi_1, \ldots, \xi_{r_n})$, $(\xi_{r_n+1}, \ldots, \xi_{2r_n})$, ... of r_n consecutive terms (for appropriately chosen r_n). Then all exceedances of u_n within a group are regarded as forming a cluster (coalescing to a multiple point in the limiting point process if $r_n = o(n)$). The distribution π_n of cluster sizes is then naturally defined on $\{1, 2, 3, \ldots\}$ by

$$\pi_n(j) = P\left\{\sum_{i=1}^{r_n} \chi_{n,i} = j \,\middle|\, \sum_{i=1}^{r_n} \chi_{n,j} > 0\right\}, \quad j = 1, 2, \ldots. \tag{4.1}$$

We show now that the multiplicity distribution π is just the limit of the cluster size distribution π_n under the conditions of Corollary 3.3.

Theorem 4.1. *Assume the conditions of Corollary* 3.3 *and use the notation there. The parameter v satisfies $\lim_{n\to\infty} P\{M_n \leq u_n\} = e^{-v}$, and if $v \neq 0$, the probability distribution π satisfies*

$$\pi(j) = \lim_{n\to\infty} \pi_n(j), \quad j = 1, 2, \ldots \tag{4.2}$$

where π_n is the cluster distribution given by (4.1) *for $r_n = \left[\frac{n}{k_n}\right]$ with k_n tending to infinity and satisfying the conditions in Lemma 2.2.*

Proof. That $P\{M_n \leq u_n\} \to e^{-v}$ follows from the discussion after Theorem 3.2. To show (4.2), note first that $\mathscr{E} \exp(-sN_n[0, 1])$ converges and hence, by Lemma 2.2, so does $\left(\mathscr{E} \exp\left(-s \sum_{j=1}^{r_n} \chi_{n,j}\right)\right)^{k_n}$. By Lemma 2.3

$$\begin{aligned}\mathscr{E} \exp\left(-s \sum_{j=1}^{r_n} \chi_{n,j}\right) &= 1 - P\{M_{r_n} \leq u_n\}(1 - \sum e^{-sj} \pi_n(j)) \\ &= 1 - \frac{v}{k_n}(1 - \sum e^{-sj} \pi_n(j))(1 + o(1)).\end{aligned}$$

Writing $\alpha_n = v(1 - \sum e^{-sj} \pi_n(j))$, it follows that $(1 - \alpha_n/k_n)^{k_n}$ converges from which it follows that α_n converges. Thus if $v \neq 0$, $\sum e^{-sj} \pi_n(j)$ converges as $n \to \infty$ for each $s \geq 0$. This is equivalent to the existence of a measure π' on $\{1, 2, 3, \ldots\}$ such that $\pi'(j) = \lim_{n\to\infty} \pi_n(j)$, $j = 1, 2, \ldots$, and in this case

$$\lim_{n\to\infty} \sum e^{-sj} \pi_n(j) = \sum e^{-sj} \pi'(j),$$

and hence by the above calculations $\mathscr{E} \exp(-sN_n[0, 1]) \to \exp(-v(1 - \sum e^{-sj} \pi'(j)))$. But $\mathscr{E} \exp(-sN_n[0, 1]) \to \mathscr{E} \exp(-sN[0, 1]) = \exp(-v(1 - \sum e^{-sj} \pi(j)))$ so that $\pi \equiv \pi'$, giving (4.2). □

It is also possible to give a constructive result for convergence of N_n based on the convergence of $P\{M_n \leq u_n\}$ and π_n, as follows.

Theorem 4.2. *Assume that the stationary sequence $\{\xi_j\}$ satisfies the condition $\Delta(u_n)$ and that $\lim_{n\to\infty} P\{M_n \leq u_n\} = e^{-v}$ for some $v \in (0, \infty)$. Suppose there exists a* *probability distribution π on $\{1, 2, 3, \ldots\}$ and a sequence $\{k_n\}$ which tends to infinity and satisfies the conditions in Lemma 2.2 and for which $\pi(j) = \lim_{n\to\infty} \pi_n(j)$ for each $j = 1, 2, \ldots$, where π_n is defined by (4.1) with $r_n = [n/k_n]$. Then N_n converges in distribution to a compound Poisson process with Laplace Transform $\exp\left(-\nu \int_0^1 \left(1 - \sum_{j=1}^{\infty} e^{-f(t)j} \pi(j)\right) dt\right)$.*

To obtain Theorem 4.2, we first prove a technical lemma.

Lemma 4.3. *Suppose the assumptions of Theorem 4.2 hold. For a fixed step function f on $[0, 1]$, define a function R_n on $[0, 1]$ by*

$$R_n(t) = \begin{cases} 1 - \mathscr{E} \exp\left(-\sum_{j=(i-1)r_n+1}^{ir_n} f(j/n)\, \chi_{n,j}\right), & \frac{(i-1)r_n}{n} < t \leqq \frac{ir_n}{n},\ 1 \leqq i \leqq k_n \\ 0, & t = 0 \quad \text{or} \quad \frac{k_n r_n}{n} < t \leqq 1. \end{cases}$$

Then as $n \to \infty$,

(i) *$\frac{n}{r_n} R_n(t)$ is uniformly bounded, and*

(ii) *$\frac{n}{r_n} \int_0^1 R_n(t)\, dt \to \nu \int_0^1 (1 - \sum e^{-f(t)j} \pi(j))\, dt.$*

Proof. $\frac{n}{r_n} R_n(t)$ is obviously uniformly bounded as $n \to \infty$, since it is either zero or for some $i = 1, 2, \ldots, k_n$

$$\begin{aligned} \frac{n}{r_n} R_n(t) &= \frac{n}{r_n} P\{M_{r_n} > u_n\} \\ &\quad \cdot \left(1 - \mathscr{E}\left(\exp\left(-\sum_{j=(i-1)r_n+1}^{ir_n} f(j/n)\, \chi_{n,j}\right) \Bigg| \sum_{j=(i-1)r_n+1}^{ir_n} \chi_{n,j} > 0\right)\right) \\ &\leqq \frac{n}{r_n} P\{M_{r_n} > u_n\} \to \nu \end{aligned}$$

by Lemma 2.3. To show (ii) first define $\tilde{R}_n(t)$ on $[0, 1]$ by

$$\tilde{R}_n(t) = \begin{cases} 1 - \mathscr{E} \exp\left(-f(t) \sum_{j=(i-1)r_n+1}^{ir_n} \chi_{n,j}\right), & \frac{(i-1)r_n}{n} < t \leqq \frac{ir_n}{n},\ 1 \leqq i \leqq k_n \\ 0, & t = 0 \quad \text{or} \quad \frac{k_n r_n}{n} < t \leqq 1. \end{cases}$$

By stationarity $\tilde{R}_n(t) = 1 - \mathscr{E} \exp\left(-f(t) \sum_{j=1}^{r_n} \chi_{n,j}\right)$ for $0 < t \leqq \frac{k_n r_n}{n}$. Thus, using again the fact $\frac{n}{r_n} P\{M_{r_n} > u_n\} \to \nu$, (ii) readily follows if R_n is replaced by $\tilde{R}_n$. Suppose

f has m jumps. Then R_n and $\tilde{R}_n$ differ on at most m intervals among $\left(\frac{(i-1)r_n}{n}, \frac{ir_n}{n}\right]$, $1 \leq i \leq k_n$. Thus

$$\lim_{n\to\infty} \frac{n}{r_n} \int_0^1 |R_n(t) - \tilde{R}_n(t)|\, dt \leq \lim_{n\to\infty} \frac{n}{r_n} \frac{mr_n}{n} P\{M_{r_n} > u_n\} = 0,$$

which concludes the proof. □

Proof of Theorem 4.2. It suffices to show that $L_{N_n}(f)$ converges to the Laplace Transform in the theorem for each non-negative step function f on [0, 1] (cf. [5, Theorem 4.2]). Using the notation of Lemma 4.3,

$$\begin{aligned}
&\sum_{i=1}^{k_n} \log \mathscr{E} \exp\left(-\sum_{j=(i-1)r_n+1}^{ir_n} f(j/n)\,\chi_{n,j}\right)\\
&= \frac{n}{r_n} \sum_{i=1}^{k_n} \frac{r_n}{n} \log\left\{1 - \left[1 - \mathscr{E}\exp\left(-\sum_{j=(i-1)r_n+1}^{ir_n} f(j/n)\,\chi_{n,j}\right)\right]\right\}\\
&= \frac{n}{r_n} \int_0^1 \log[1 - R_n(t)]\, dt.
\end{aligned}$$

Write $\psi(x) = -\log(1-x) - x$, $x \in [0, 1)$, so that $\psi(x) \sim x^2/2$ as $x \to 0$. Hence for large n, $|\psi(R_n(t))| \leq R_n^2(t)$ for all $t \in [0, 1]$, since clearly $R_n(t) \to 0$ uniformly in t by Lemma 4.3, showing that

$$\frac{n}{r_n} \int_0^1 |\psi(R_n(t))|\, dt \leq \frac{r_n}{n} \int_0^1 \left(\frac{n}{r_n} R_n(t)\right)^2 dt \to 0$$

since $(n/r_n) R_n(t)$ is uniformly bounded and $r_n/n \to 0$. It thus follows from Lemma 2.2 and Lemma 4.3 that

$$\begin{aligned}
\lim_{n\to\infty} \mathscr{E} \exp\left(-\int f\, dN_n\right) &= \lim_{n\to\infty} \prod_{i=1}^{k_n} \mathscr{E} \exp\left(-\sum_{j=(i-1)r_n+1}^{ir_n} f(j/n)\,\chi_{n,j}\right)\\
&= \lim_{n\to\infty} \exp\left(-\frac{n}{r_n} \int_0^1 R_n(t)\, dt\right)\\
&= \exp\left(-\nu \int_0^1 \left(1 - \sum e^{-f(t)j} \pi(j)\right) dt\right)
\end{aligned}$$

as required. □

5. Normalized Levels

An important case where the exceedance point processes often have useful point process limits occurs when the levels u_n are "normalized" to be approximately the $(1-\tau/n)$-percentile of the underlying d.f. F for each ξ_i. More specifically for $0<\tau<\infty$

$$n[1-F(u_n^{(\tau)})]\to\tau \quad \text{as } n\to\infty. \tag{5.1}$$

The existence of such a family $u_n^{(\tau)}$ is guaranteed for any d.f. F such that $(1-F(x))/(1-F(x-))\to 1$ as $x\to x_F \stackrel{\text{def}}{=} \sup(u\colon F(u)<1)$ (cf. [10], Theorem 1.1.13), which we assume in what follows without further comment. We write $N_n^{(\tau)}$ for the exceedance point process corresponding to the level $u_n^{(\tau)}$. Note that if $u_{n,1}^{(\tau)}$ and $u_{n,2}^{(\tau)}$ are two different sequences satisfying (5.1) and $N_{n,1}^{(\tau)}$, $N_{n,2}^{(\tau)}$ are the corresponding exceedance point processes, then

$$P\{N_{n,1}^{(\tau)} \neq N_{n,2}^{(\tau)}\} \leqq n|F(u_{n,1}^{(\tau)})-F(u_{n,2}^{(\tau)})|\to 0 \quad \text{as } n\to\infty$$

by (5.1). Since we are only interested in weak convergence results, the choice of $\{u_n^{(\tau)}\}$ thus need not be specific, and indeed we can use any convenient $\{u_n^{(\tau)}\}$ satisfying (5.1) for our purposes.

The following result shows that if $N_n^{(\tau)}$ has a limit for one τ it has a limit for all τ and the compound Poisson limits obtained are very simply related.

Theorem 5.1. *Suppose that for each $\tau>0$ the stationary sequence $\{\xi_j\}$ satisfies the condition $\Delta(u_n^{(\tau)})$ and that for some $\tau_1>0$, $N_n^{(\tau_1)}$ converges in distribution to a point process $N^{(\tau_1)}$. Then $N_n^{(\tau)}$ converges to a compound Poisson process $N^{(\tau)}$ for all $\tau>0$, with Laplace Transform given by*

$$L_{N^{(\tau)}}(f)=\exp\left\{-\theta\tau\int_0^1[1-\phi(f(t))]\,dt\right\} \tag{5.2}$$

where $0\leqq\theta\leqq 1$ and $\phi(s)=\sum_{j=1}^{\infty}e^{-sj}\pi(j)$ is the Laplace Transform of a probability distribution π on $\{1,2,\ldots\}$, θ and π being independent of τ.

Proof. Assume without loss of generality that $\tau_1=1$. By Corollary 3.3,

$$L_{N^{(1)}}(f)=\exp\left\{-\theta\int_0^1[1-\phi(f(t))]\,dt\right\}$$

so that (5.2) holds for $\tau=1$, where $\theta=-\lim_{n\to\infty}\log P\{M_n\leqq u_n^{(1)}\}$. To show that (5.2) holds for each $\tau>0$, it suffices to show the following:

(*) For each $\tau>0$, there exists a $\delta>0$ such that for each interval $I\subset[0,1]$ with $m(I)<\delta$, $N_n^{(\tau)}(I)$ converges in distribution to a compound Poisson random variable with Laplace Transform $\exp\{-\theta\tau m(I)(1-\phi(s))\}$.

To see that (*) is sufficient, observe that any finite number of disjoint intervals I_i, $1\leqq i\leqq k$, in $[0,1]$ can be decomposed into finer disjoint intervals I_{ij}, $1\leqq j\leqq n_i$, $1\leqq i\leqq k$, each of which has length less than δ, and thus Lemma 2.2 together with (*) imply

$$\lim_{n\to\infty} \mathscr{E}\exp\left(-\sum_{i=1}^{k} s_i N_n^{(\tau)}(I_i)\right)=\lim_{n\to\infty}\prod_{i=1}^{k}\prod_{j=1}^{n_i}\mathscr{E}\exp(-s_i N_n^{(\tau)}(I_{ij}))$$
$$=\exp\left\{-\theta\tau\sum_{i=1}^{k} m(I_i)(1-\phi(s_i))\right\},$$

showing that the finite dimensional distributions of $N_n^{(\tau)}$ converge to those of a compound Poisson process having Laplace Transform given by (5.2) from which the theorem follows. To show (*), first let $\tau<1$ and assume for convenience that $u_n^{(\tau)}=u_{n'}^{(1)}$, where $n'=[n/\tau]$. Since $\left|\#\left\{\frac{j}{n}\in I\right\}-\#\left\{\frac{j}{n'}\in[0,\tau\varepsilon]\right\}\right|\leqq 4$ for each interval I in $[0, 1]$ with length ε, it follows that

$$|\mathscr{E}\exp(-sN_n^{(\tau)}(I))-\mathscr{E}\exp(-sN_{n'}^{(1)}[0,\tau\varepsilon])|\leqq 4[1-F(u_{n'}^{(1)})]\to 0$$

and thus that

$$\lim_{n\to\infty}\mathscr{E}\exp(-sN_n^{(\tau)}(I))=\lim_{n\to\infty}\mathscr{E}\exp(-sN_{n'}^{(1)}[0,\tau\varepsilon])=\exp\{-\theta\tau\varepsilon(1-\phi(s))\}.$$

This proves (*) for $\tau<1$ with $\delta=1$. For $\tau>1$, the proof is identical except that δ must be $1/\tau$. □

The parameter θ is linked to the asymptotic distributional properties of the maximum M_n through the relationship $P\{M_n\leqq u_n^{(\tau)}\}=P\{N_n^{(\tau)}[0,1]=0\}\to e^{-\theta\tau}$ (as is readily seen from (5.2)) under the conditions of the theorem. On this account, θ has been called the extremal index of the sequence $\{\xi_j\}$. Since $N_n^{(\tau)}[0,1]\xrightarrow{d}N^{(\tau)}[0,1]$ it follows (e.g., by Skorohod's representation and Fatou's Lemma) that

$$\tau=\lim_{n\to\infty} n[1-F(u_n^{(\tau)})]=\lim_{n\to\infty}\mathscr{E}N_n^{(\tau)}[0,1]\geqq\mathscr{E}N^{(\tau)}[0,1]=\theta\tau\sum_{j=1}^{\infty} j\pi(j)$$

since the intensity of Poisson events is $\theta\tau$ and π is the multiplicity distribution. Here $\theta\leqq(\sum j\pi(j))^{-1}\leqq 1$ since π is a distribution on $\{1, 2, \ldots\}$. Clearly $\theta\geqq 0$ so that $0\leqq\theta\leqq 1$. If $\theta=1$ it also follows that $\sum j\pi(j)=1$ and hence $\pi(j)=1$ or 0 according as $j=1$ or $j>1$ so that the compound Poisson process reduces to an ordinary Poisson process with intensity τ. Many common cases (and in particular i.i.d. sequences) have extremal index $\theta=1$. A value $\theta<1$ indicates clustering of exceedances of $u_n^{(\tau)}$, giving rise to multiplicities in the limit.

A question that arises naturally from the above discussion is, when $\theta\neq 1$, whether $\theta=(\sum j\pi(j))^{-1}$, or equivalently whether $\lim_{n\to\infty}\sum j\pi_n(j)=\sum j\pi(j)$. Counter-examples using regenerative sequences have been constructed by H. Rootzén and by R.L. Smith. Note, however, that the equality is ensured by, for example, uniform integrability of π_n (or $N_n[0, 1]$), and in turn by rapidly decreasing mixing rate of $\{\xi_j\}$. For example, the equality holds when $\{\xi_j\}$ is m-dependent.

6. Applications and Examples

First we apply our convergence results to problems that are of concern in the more traditional theory. Let $M_n^{(k)}$ be the k^{th} largest among $\xi_1, \xi_2, \ldots, \xi_n$. It is obvious that $(M_n^{(k)} \leqq u_n^{(\tau)})$ is the same event as $(N_n^{(\tau)} \leqq k-1)$. Using this fact, one can derive asymptotic distributions for properly normalized $M_n^{(k)}$.

Theorem 6.1. *Suppose $\Delta(u_n^{(\tau)})$ holds for $\{\xi_j\}$ for each $\tau > 0$, and that $N_n^{(\tau)}$ converges in distribution to some non-trivial point process $N^{(\tau)}$ for some $\tau > 0$. Assume that $a_n > 0$, b_n are constants such that*

$$P\{a_n(M_n - b_n) \leqq x\} \to G(x) \tag{6.1}$$

for some non-degenerate distribution function G (necessarily of extreme value type). Then for each $k = 1, 2, \ldots,$

$$\lim_{n\to\infty} P\{a_n(M_n^{(k)} - b_n) \leqq x\} = G(x)\left[1 + \sum_{j=1}^{k-1} \sum_{i=j}^{k-1} \frac{(-\log G(x))^j}{j!} \pi^{*j}(i)\right] \tag{6.2}$$

(where $G(x) > 0$, and zero where $G(x) = 0$), where for each j, π^{*j} is the j-fold convolution of the probability distribution π obtained as in Theorem 4.1 by letting u_n there be $u_n^{(\tau)}$ for any $\tau > 0$.

Proof. By Theorem 5.1, $N_n^{(\tau)}$ converges in distribution to $N^{(\tau)}$ for each $\tau > 0$, where the Laplace Transform of $N^{(\tau)}$ is given by (5.2), and θ and ϕ in the transform are independent of τ and can be obtained as in Theorem 4.1 by letting u_n there be $u_n^{(\tau)}$ for any $\tau > 0$. Since the limit is assumed non-trivial we have $\theta > 0$. It follows from (6.1) and Theorem 2.5 of [9] that G is one of the three extreme value type distributions, and $\lim_{n\to\infty} P\{a_n(\hat{M}_n - b_n) \leqq x\} = G^{1/\theta}(x)$ where $\hat{M}_n$ is the maximum of n independent random variables all having the same distribution as ξ_1. Thus

$$\lim_{n\to\infty} P\{\hat{M}_n \leqq a_n^{-1} G^{-1}(e^{-\theta\tau}) + b_n\} = G^{1/\theta}(G^{-1}(e^{-\theta\tau})) = e^{-\tau},$$

which shows by Theorem 1.5.1 of [10] that

$$1 - F(a_n^{-1} G^{-1}(e^{-\theta\tau}) + b_n) \sim \tau/n \quad \text{as } n \to \infty.$$

Writing $\tau(x) = -\log G^{1/\theta}(x)$, we thus have

$$1 - F(a_n^{-1} x + b_n) \sim \tau(x)/n. \tag{6.3}$$

Now it follows from (5.2), (6.3) and the fact that $N_n^{(\tau)}([0,1]) \xrightarrow{d} N^{(\tau)}([0,1])$ that

$$\begin{aligned}\lim_{n\to\infty} P\{a_n(M_n^{(k)} - b_n) \leqq x\} &= \lim_{n\to\infty} P\{M_n^{(k)} \leqq u_n^{(\tau(x))}\} \\ &= \lim_{n\to\infty} P\{N_n^{(\tau(x))}([0,1]) \leqq k-1\}\end{aligned}$$

$$=P\{N^{(\tau(x))}([0,1])\leqq k-1\}$$

$$=e^{-\theta\tau(x)}\left[1+\sum_{j=1}^{k-1}\frac{(\theta\tau(x))^j}{j!}\sum_{i=j}^{k-1}\pi^{*j}(i)\right]$$

which gives (6.2) since $e^{-\theta\tau(x)}=G(x)$. □

We end with two examples which illustrate the theory.

Examples 6.2. A trivial example of a case where clustering occurs is given by $\xi_j=\max(\eta_j,\eta_{j+1})$ where $\{\eta_j\}$ is an i.i.d. sequence. In this case $\theta=1/2$, clusters have size 2 (in the limit) and the limiting distribution (6.2) for $M_n^{(k)}$ becomes

$$\lim P\{a_n(M_n^{(k)}-b_n)\leqq x\}=G(x)\left[1+\sum_{j=1}^{[(k-1)/2]}\frac{(-\log G(x))^j}{j!}\right]$$

where G, a_n, b_n are as in (6.1). This is an obvious modification of the classical result and simply reflects the fact that exceedances occur (predominantly) in pairs.

This example may be extended to include stochastic cluster sizes by defining $\xi_j=\max(\eta_j,\eta_{j+1},\ldots,\eta_{j+\beta_j})$ for i.i.d. positive integer valued β_j independent of the η's. Another example with stochastic cluster sizes is the following.

Example 6.3. Consider the sequence

$$\xi_j=\max_{k\geqq 0}\rho^k Z_{j-k}$$

where $0<\rho<1$ and $\{Z_j\}$ is an i.i.d. sequence with common d.f. $\exp(-1/x)$, $x>0$. This example was due to L. De Haan who showed that $\{\xi_j\}$ has extremal index $\theta=1-\rho$ ((cf. [9]), which can be any value between zero and one. It can be shown by some calculation (cf. [3], Chapter 5) that the limits (4.2) exist and are given by $\pi\{i\}=\rho^{i-1}(1-\rho)$. It then follows from Theorem 4.2 that $N_n^{(\tau)}$ converges in distribution to a Compound Poisson Process with Laplace Transform

$$\exp\left\{-(1-\rho)\tau\int_0^1\left(1-\sum_{j=1}^{\infty}\pi\{j\}e^{-jf(t)}\right)dt\right\}.$$

In particular the limiting cluster sizes follow a geometric distribution.

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