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Minimal Displacement of Branching Random Walk

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Summary. Let $\mathfrak{X}$ denote a branching random walk in $\mathbb{R}^1$ with mean particle production m, $m>1$, and with incremental spatial distribution G, with $G(\{0\}) = p$ and $G(\{1\}) = 1-p$. If $mp=1$, then the minimal displacement of $\mathfrak{X}$ behaves asymptotically like $\log\log n/\log 2$. If the condition $G(\{1\})=1-p$ is replaced by $G((0,\infty))=1-p$, we obtain a similar result.

1. Introduction

The Galton-Watson branching process $\{X_n\}$, with $X_0=1$, together with the i.i.d. collection of random variables $\{X(a_1,\dots,a_n)\}$, $a_k\in\mathbb{Z}^+$, $k=1,\dots,n$, defines a *branching random walk* $\mathfrak{X}$ (in $\mathbb{R}^1$), where $S(a_1,\dots,a_n)=\sum_{k=1}^{n} X(a_1,\dots,a_k)$ is interpreted as the spatial position of the a_n^{th} individual of the n^{th} generation with forebears (a_1), $(a_1,a_2),\dots,(a_1,\dots,a_{n-1})$. (See Harris [5], page 122, for greater detail.) If we set $M_n=\min_{a_1,\dots,a_n} S(a_1,\dots,a_n)$ ($=\infty$ if extinction of the process has occurred by time n), then M_n is the position of the individual farthest to the left at time n, also referred to as the *minimal displacement*. Alternatively, if $X(a_1,\dots,a_n)$ is assumed to be a positive random variable, $X(a_1,\dots,a_n)$ may instead be interpreted as the life-span of the individual $(a_1,\dots,a_n)$. In this case, M_n may be thought of as the first death time of a member of the n^{th} generation of the process $\{X_n\}$.

Hammersley [4] demonstrates the existence of a $\gamma_0\in\mathbb{R}$ such that if $F_n(x) = P[M_n\leqq x]$ and q_0 is the extinction probability of $\mathfrak{X}$, then

$$\begin{aligned} F_n(n\gamma)&\to 0 \quad \text{for } \gamma<\gamma_0 \\ &\to 1-q_0 \quad \text{for } \gamma>\gamma_0. \end{aligned}$$

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In the special case where the branching process $\{X_n\}$ is dyadic and $G(\{0\}) = G(\{1\}) = \frac{1}{2}$, where $G(x) = P[X(a_1, \dots, a_n) \leqq x]$, Joffe-Le Cam-Neveu [6] show quite simply that

$$\frac{M_n}{n} \to 0 \text{ w.p. } 1 \quad \text{as} \quad n \to \infty.$$

In Section 2 of this paper, the technique of Joffe-Le Cam-Neveu is extended to demonstrate

Theorem 1. *Assume that* $EX_1^{2+\delta} < \infty$ *for some* $\delta > 0$, $G(\{0\}) = p$, $G(\{1\}) = 1 - p$, *and* $mp = 1$, *where* $EX_1 = m > 1$. *Then, conditioned on the nonextinction of* $\mathfrak{X}$,

$$\lim_{n \to \infty} \left(M_n - \left\lceil \frac{\log \log n - \log(V + o(1))}{\log 2} \right\rceil \right) = 0$$

holds w.p. 1. V *is a random variable which is defined in Proposition* 4, *and* $o(1)$ *is stochastic.*[1]

In Section 3, we generalize the condition $G(\{1\}) = 1 - p$ to $G((0, \infty)) = 1 - p$, and demonstrate

Theorem 2. *Assume that* $EX_1^{2+\delta} < \infty$ *for some* $\delta > 0$, $G(\{0\}) = p$, $G((0, \infty)) = 1 - p$, *and* $mp = 1$, *where* $EX_1 = m > 1$. *Then, conditioned on the nonextinction of* $\mathfrak{X}$, *if*

$$\sum_{k=1}^{\infty} G^{-1}(p + (1-p) \cdot \exp(-\lambda^k)) = \infty$$

for some $\lambda > 1$, *then*

$$\lim_{n \to \infty} \frac{M_n}{\sum_{k=1}^{s(n)} G^{-1}(p + (1-p) \cdot \exp(-2^k))} = 1$$

w.p. 1, *whereas if*

$$\sum_{k=1}^{\infty} G^{-1}(p + (1-p) \cdot \exp(-\lambda^k)) < \infty$$

for some $\lambda > 1$, *then*

$$\lim_{n \to \infty} M_n < \infty$$

w.p. 1. *Here,* $s(n) = \lceil \log \log n / \log 2 \rceil$.[2]

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[1] $\lceil x \rceil$ denotes the least integer $\geqq x$, and $\lfloor x \rfloor$ denotes the greatest integer $\leqq x$. Note that because of the presence of $\lceil\ \rceil$, the presence of $o(1)$ within the equation is not extraneous

[2] By $G^{-1}(y)$, we mean $\inf\{x : G(x) \geqq y\}$

2. Proof of Theorem 1

In this section it will be assumed that $G(\{0\})=p$, $G(\{1\})=1-p$, $EX_1=m>1$, and $mp=1$. The essential idea behind the computation of $\{M_n\}$ in this case is to reinterpret $\mathfrak{X}$ as a collection of branching processes within a branching process by means of an appropriate decomposition based on $\{S(a_1,\dots,a_n)\}$. To do so, we introduce the concept of *dynasty*, where the dynasty of an individual $(a_1,\dots,a_n)$ is given by $S(a_1,\dots,a_n)$.

Intuitively, an individual, $(a_1,\dots,a_n)$, is considered to enter a dynasty m if it reaches m by a move from the left, that is, if $S(a_1,\dots,a_n)=m$ and $S(a_1,\dots,a_{n-1})=m-1$. Since the probability of being stationary is p, the descendents of $(a_1,\dots,a_n)$ which do not move from m form a Galton-Watson branching process, which is easily shown to be critical. We denote such a branching process by $\{Y_k^{(i,m)}\}_{i\in I_m}$; $I_m=\{(a_1,\dots,a_n)$: $S(a_1,\dots,a_n)=m$, $S(a_1,\dots,a_{n-1})=m-1\}$ denotes the set of individuals initiating processes at m. Thus, one can picture $\mathfrak{X}$ as a collection of critical branching processes rooted at different spatial positions and beginning at different times. All of these branching processes have a common generation law due to the random walk structure of the spatial movement and the branching structure of $\mathfrak{X}$; we denote the prototype by $\{Y_k\}$. It should be noted that the subscript k of $\{Y_k^{(i,m)}\}$ does not refer to real time, but rather the number of generations that have elapsed since the individual i first reached m. (If $m=0$, then i is unique, and of course k is also the real time.)

In addition to the branching processes $\{Y_k^{(i,m)}\}_{i\in I_m}$, we also introduce the process $\{Z_m\}$, where $Z_m=|I_m|$, the cardinality of I_m. In other words, Z_m is the number of individuals ever reaching position m by a move from the left, and is thus the number of distinct critical branching processes $\{Y_k^{(i,m)}\}$ emanating from position m. Due to the branching and spatial structure of $\mathfrak{X}$, $\{Z_m\}$ is also a Galton-Watson branching process. Whereas $\{Y_k\}$ is a critical branching process, $\{Z_m\}$ has infinite mean particle production (see Proposition 2).

Under this interpretation of $\mathfrak{X}$, M_n denotes the earliest dynasty still present at time n. Certainly, the behavior of $\{M_n\}$ and $\{Z_m\}$ will be closely connected. The key point behind the computations that follow is that (conditional on nonextinction of $\mathfrak{X}$) $\{Z_m\}$ will in general increase extremely rapidly as $m\to\infty$ – to such an extent that, because of the simple nature of $\{Y_k^{(i,m)}\}$, knowledge of the asymptotic behavior of $\{Z_m\}$ alone is sufficient for accurate computation of $\{M_n\}$. Proposition 4 describes the asymptotic behavior of $\{Z_m\}$. Together with Corollary 1, which describes the asymptotic behavior of $\{Y_k\}$, this is sufficient to enable us to derive Theorem 1, which characterizes the asymptotic behavior of $\{M_n\}$.

In the following, ϕ_W will denote the generating function of the first generation distribution W_1 of the branching process $\{W_m\}$, and $\phi_W^{(m)}$, the generating function of the m^{th} generation distribution W_m. In addition, ϕ_W will denote the generating function of the distribution W; it will be clear from the context which is meant.

Proposition 1. *If* $\{\hat{Y}_k\}$ *is a critical branching process, i.e.,* $E\hat{Y}_1=1$, *with variance*

$0<\sigma^2<\infty$, *then*

$$P[\hat{Y}_k>0]\sim\frac{2}{k\sigma^2}.$$

Proof. See Athreya-Ney [1], page 19. □

Corollary 1. *Assume that X_1 has variance $\sigma^2<\infty$, and let p be as in the beginning of the section, with $mp=1$. Then,*

$$P[Y_k>0]\sim\frac{2}{k(p^2\sigma^2+1-p)}.$$

Proof. Since $mp=1$ where $EX_1=m$, $\{Y_k\}$ is a critical branching process. A simple computation of $\phi_Y''(1)$, based on the equality

$$\phi_Y(s)=\phi_X(1-p+ps),$$

shows that

$$\sigma_Y^2=p^2\sigma^2+1-p,$$

where σ_Y^2 is the variance of Y_1. Now apply Proposition 1. □

We now proceed to examine the asymptotic behavior of $\{Z_m\}$. Our plan is to first obtain an asymptotic expression for $\phi_Z^{(m)}$ (Proposition 5). We will apply a result of Darling [3] to reduce this to an explicit statement of weak convergence of $\{Z_m\}$ (Corollary 2). Applying a result of Cohn [2], we then sharpen this result to one of pointwise asymptotic behavior of $\{Z_m\}$ (Proposition 4). We commence by examining the behavior of $\phi_Z(s)$ for s close to 1. For ease of notation, we define $k(s)=1-\phi_Z(1-s)$, and therefore examine $k(s)$ for small s.

Proposition 2. *Assume that for some $\delta>0$, X_1 has a finite $(2+\delta)^{\text{th}}$ moment, i.e.*

$$\sum_{j=0}^{\infty}p_j j^{2+\delta}<\infty,$$

where $p_j=P[X_1=j]$. Then,

$$k(s)=a\,s^{1/2}(1+O(s^{\delta/2})),$$

where $a=[2(1-p)/p(p^2\sigma^2+1-p)]^{1/2}$.

Proof. Decomposition of $\mathfrak{X}$ based on the spatial motion of the individual branches yields the functional equation

$$\phi_Z(s)=\phi_X((1-p)s+p\,\phi_Z(s)).$$

Therefore,

$$k(s)=1-\phi_X(1-(1-p)s-p\,k(s)). \tag{1}$$

Since X_1 has finite $(2+\delta)^{\text{th}}$ moment, if we assume that $0<\delta<1$, we may rewrite ϕ_X as

$$\phi_X(s)=1+\frac{1}{p}\cdot(s-1)+\left(\sigma^2+\frac{1-p}{p^2}\right)\cdot\frac{(s-1)^2}{2!}+O(|s-1|^{2+\delta}). \tag{2}$$

(See Loève [7], page 199.) Substituting (2) into (1), we obtain

$$k(s)=\frac{(1-p)}{p}\cdot s+k(s)-\left(\sigma^2+\frac{1-p}{p^2}\right)\frac{[(1-p)s+pk(s)]^2}{2!} + O[(1-p)s+pk(s)]^{2+\delta},$$

and hence

$$\begin{aligned} k(s)&=-\frac{(1-p)}{p}\cdot s+\left[\frac{2((1-p)s+pO[(1-p)s+pk(s)]^{2+\delta})}{p(p^2\sigma^2+1-p)}\right]^{1/2} \\ &=-\frac{(1-p)}{p}\cdot s+a[s+O(s^{2+\delta})+O(k(s)^{2+\delta})]^{1/2}. \end{aligned} \tag{3}$$

Dividing by $k(s)$, (3) becomes

$$1=-\frac{(1-p)}{p}\cdot\frac{s}{k(s)}+a\left[\frac{s}{k^2(s)}+O\left(\frac{s^{2+\delta}}{k^2(s)}\right)+O(k^\delta(s))\right]^{1/2}. \tag{4}$$

Now since $Z_1<\infty$ w.p. 1, $\phi_Z(s)$ is continuous at 1, and therefore

$$k^\delta(s)\to 0 \quad \text{as} \quad s\to 0. \tag{5}$$

Moreover, (4) implies that $s/k^2(s)$ is bounded as $s\to 0$, and therefore

$$s^{2+\delta}/k^2(s)\to 0 \quad \text{as} \quad s\to 0 \tag{6}$$

and

$$s/k(s)\to 0 \quad \text{as} \quad s\to 0. \tag{7}$$

Therefore, if we apply (5), (6), and (7), it follows from (4) that

$$1=\lim_{s\to 0}\frac{a}{k(s)}\cdot s^{1/2},$$

and hence

$$k(s)=aA(s)\,s^{1/2}, \tag{8}$$

where $A(s)\to 1$ as $s\to 0$. Plugging (8) into (3), we obtain

$$\begin{aligned} k(s)&=\frac{-(1-p)}{p}\cdot s+a[s+O(s^{2+\delta})+O(s^{1+\delta/2})]^{1/2} \\ &=a\,s^{1/2}(1+O(s^{\delta/2})). \quad \square \end{aligned}$$

By applying Proposition 2, we will show in Proposition 5 of Section 3 (in somewhat greater generality) that the existence of a $(2+\delta)^{\text{th}}$ moment for X_1 is enough to ensure that

$$\phi_Z^{(m)}(1-\exp(-2^m t)) \to v(t) \tag{9}$$

as $m \to \infty$, for all $0<t<\infty$, where v is a distribution function which is continuous and strictly increasing on $x>0$, with $v(0+)=q_0$. (q_0 is the extinction probability of $\mathfrak{X}$.) By means of a computation involving the Laplace transform, it is possible to reduce (9) to an explicit statement concerning the asymptotic behavior of $\{Z_m\}$. The following result is due to Darling [3].

Proposition 3. *Let $\{\hat{Z}_m\}$ be a sequence of integer valued random variables, and assume that*

$$\phi_{\hat{Z}_m}(1-\exp(-b^m t)) \to \hat{v}(t)$$

as $m \to \infty$, for all $0<t<\infty$, where $b>1$. Then,

$$P[b^{-m}\log(\hat{Z}_m+1) \leqq x] \to \hat{v}(x)$$

as $m \to \infty$, for all $0<x<\infty$.

In our specific case, (9) implies that we obtain

Corollary 2. $P[2^{-m}\log(Z_m+1) \leqq x] \to v(x)$ *as $m \to \infty$, for all $0<x<\infty$.*

Cohn [2] shows that in the case where $\{\hat{Z}_m\}$ is a Galton-Watson branching process, and $\hat{v}(t)$ is a distribution function which is continuous and strictly increasing on $x>0$, weak convergence as in the conclusion of Proposition 3 actually implies a.s. convergence to a random variable $\hat{V}$ having distribution $\hat{v}$. Therefore, Corollary 2 may be strengthened to

Proposition 4. $2^{-m}\log(Z_m+1) \to V$ *w.p.* 1 *as $m \to \infty$, where V is a random variable having distribution v. $v(0+)=q_0$, and therefore $V>0$* w.p. 1 *on the set of nonextinction of $\mathfrak{X}$.*

Corollary 1 and Proposition 4 provide us with precise enough information regarding the asymptotic behavior of $\{Y_k\}$ and $\{Z_m\}$ to analyze $\{M_n\}$.

Theorem 1. *Assume that $\{M_n\}$ is the minimal displacement of the branching random walk $\mathfrak{X}$, where $EX_1^{2+\delta}<\infty$ for some $\delta>0$; $G(\{0\})=p$, $G(\{1\})=1-p$, and $mp=1$, where $EX_1=m>1$. Then, conditioned on the nonextinction of $\mathfrak{X}$,*

$$\lim_{n\to\infty}\left(M_n-\left\lceil\frac{\log\log n-\log(V+o(1))}{\log 2}\right\rceil\right)=0$$

holds w.p. 1. V is defined in Proposition 4, and $o(1)$ is stochastic.

Proof. (a) Since $\sum_{i\in I_m} Y_n^{(i,m)}>0$ implies that $M_n \leqq m$, determination of $\sum_{i\in I_m} Y_n^{(i,m)}$ will give an upper bound for M_n. By Corollary 1, there exists some $C>0$ s.t.

$$P[Y_n>0] \geqq C/n$$

for all $n \geqq 1$. Hence

$$P[\sum_{i \in I_m} Y_n^{(i,m)} > 0 | Z_m] \geqq 1 - (1 - C/n)^{Z_m} \geqq 1 - e^{-CZ_m/n},$$

which is at least

$$1 - e^{-Cm} \quad \text{for } 1 \leqq n \leqq Z_m/m.$$

Let Q be the set such that $\liminf_{m \to \infty} Z_m/m > 1$. Since $\sum_{m=0}^{\infty} e^{-Cm} < \infty$, a simple Borel-Cantelli argument implies that

$$P[\liminf_{m \to \infty} \sum_{i \in I_m} Y_{\lfloor Z_m/m \rfloor}^{(i,m)} > 0 | Q] = 1.$$

Let Q_0 denote the set of nonextinction of $\mathfrak{X}$. Then, Proposition 4 implies that $P[Q] = P[Q_0] = 1 - q_0$, and that $\lfloor Z_m/m \rfloor = \exp[2^m(V + o(1))]$ on Q_0 for some appropriate $o(1)$. Therefore,

$$P[\liminf_{m \to \infty} \sum_{i \in I_m} Y_{e_m}^{(i,m)} > 0 | Q_0] = 1,$$

where $e_m = \exp[2^m(V + o(1))]$. Inverting e_m, it follows that

$$\limsup_{n \to \infty} \left(M_n - \left\lceil \frac{\log \log n - \log(V + o(1))}{\log 2} \right\rceil \right) \leqq 0$$

w.p. 1 on Q_0, the set of nonextinction of $\mathfrak{X}$.

(b) Analysis of $\sum_{i \in I_k} Y_n^{(i,k)}$, for $k = 1, \ldots, m$, also gives a lower bound for M_n. Proceeding in a manner similar to part (a), Corollary 1 implies that

$$P[\sum_{i \in I_k} Y_n^{(i,k)} > 0 | Z_k] \leqq 1 - e^{-C' Z_k/n} \tag{10}$$

for some $C' > 0$ and all k and n. The l.h.s. of (10) is therefore at most $1 - \exp[-C' e^{-m}]$ for $n \geqq e^m Z_k$. Since for large m, $1 - \exp[-C' e^{-m}] \sim C' e^{-m}$, it follows that

$$P\left[\sum_{k=0}^{m} \sum_{i \in I_k} Y_{f_{m,k}}^{(i,k)} > 0\right] \leqq C''(m+1) e^{-m} \tag{11}$$

for some $C'' > 0$, where $f_{m,k} = \lceil e^m Z_k \rceil$. Now, set $g_m = \sum_{k=0}^{m} f_{m,k} = \sum_{k=0}^{m} \lceil e^m Z_k \rceil$. Notice that if $\sum_{k=0}^{m} \sum_{i \in I_k} Y_{f_{m,k}}^{(i,k)} = 0$, then the 0^{th} dynasty ends by time $f_{m,0}$, implying that the first dynasty ends by time $f_{m,0} + f_{m,1}$, and so on, showing that the m^{th} dynasty ends by time g_m. Thus,

$$\{M_{g_m} \leqq m\} \subset \left\{\sum_{k=0}^{m} \sum_{i \in I_k} Y_{f_{m,k}}^{(i,k)} > 0\right\},$$

which, together with (11), implies that

$$P[M_{g_m} \leqq m] \leqq C''(m+1)e^{-m}. \tag{12}$$

Proposition 4 implies that if $V>0$,

$$g_m = \sum_{k=0}^{m} \exp[m + 2^k(V+o(1))].$$

(If $V=0$, g_m is, of course, bounded w.p. 1.) In either case,

$$g_m = \exp[2^m(V+o(1))].$$

Since $\sum_{m=0}^{\infty}(m+1)e^{-m}<\infty$, we may apply Borel-Cantelli, from which it follows that

$$P[\liminf_{m\to\infty}(M_{g_m}-m)\leqq 0]=0.$$

Inverting g_m, we obtain

$$\liminf_{n\to\infty}\left(M_n - \left\lceil\frac{\log\log n - \log(V+o(1))}{\log 2}\right\rceil\right) \geqq 0$$

w.p.1.

The assertion follows from parts (a) and (b). □

3. Proof of Theorem 2

If G is generalized from the two point distribution, where $G(\{0\})=p$, $G(\{1\})=1-p$, and $mp=1$, to a distribution with $G(\{0\})=p$, $G((0,\infty))=1-p$, and $mp=1$, then the basic techniques of Section 2 are still applicable toward computation of $\{M_n\}$. We again apply the concept of dynasty in decomposing the branching random walk $\mathfrak{X}$, but with the modification that the individual $(a_1,\ldots,a_n)$ is now considered a member of the m^{th} dynasty if

$$\sum_{k=1}^{n} \chi_{\{X(a_1,\ldots,a_k)>0\}} = m,$$

where χ is the indicator function. (Denote by $\Gamma(a_1,\ldots,a_n)$ the dynasty of $(a_1,\ldots,a_n)$.) Furthermore, we will say that $(a_1,\ldots,a_n)$ is a first generation member of its dynasty if $X(a_1,\ldots,a_n)>0$.

Unlike the simpler case in Section 2, it is now no longer sufficient to calculate Z_m, the number of individuals at the start of the m^{th} dynasty, to obtain an asymptotic estimate for $\{M_n\}$. We will, however, be able to obtain asymptotic upper and lower bounds for $\{M_n\}$ by examining certain auxiliary processes of $\mathfrak{X}$.

The computation of the upper bound is messy, that of the lower bound is simple; our methodology is such that Theorem 2, which characterizes $\{M_n\}$, is somewhat weaker than its analogue, Theorem 1.

Computation of an Upper Bound for M_n

To compute an upper bound for $\{M_n\}$, we introduce an auxiliary process $\mathfrak{X}^\gamma$ by "trimming" the tree associated with $\mathfrak{X}$, so that only those individuals of the m^{th} dynasty with spatial movement at most $\gamma(m)$ are preserved. (γ will be assumed to be a decreasing function on $\mathbb{Z}^+$.) That is, an individual $(a_1, \dots, a_n)$ is retained only if

$$X(a_1, \dots, a_k) \leqq \gamma \circ \Gamma(a_1, \dots, a_k) \qquad \text{for } k=1, \dots, n.$$

Note that trimming occurs only among the first generation of each dynasty. The corresponding minimal displacement M_n^γ of $\mathfrak{X}^\gamma$ will be bounded above by $\sum_{k=0}^{m-1} \gamma(k)$ if extinction of the m^{th} dynasty of $\mathfrak{X}^\gamma$ has not yet occurred by time n. This provides us with an upper bound for $\{M_n\}$, M_n being less than M_n^γ.

We will be making use of those γ of the form

$$\gamma^{\alpha_{\varepsilon,j}}(k) = G^{-1}(p + (1-p)\, e^{-\alpha_{\varepsilon,j}(k)}), \tag{13}$$

where

$$\alpha_{\varepsilon,j}(k) = \begin{cases} \alpha^{k-j} & \text{if } k > j \\ \varepsilon & \text{if } k \leqq j, \end{cases}$$

and either $1 < \alpha < 2$ or $\alpha = 0$, with $0 \leqq \varepsilon \leqq \alpha$. In the event that G is a bounded distribution, it will suffice to set $\varepsilon = 0$. When no misunderstanding is possible, we will drop γ, ε, and j from the superscript, and will write $\mathfrak{X}^\alpha$, $\{M_n^\alpha\}$, and $\{Z_m^\alpha\}$ for the auxiliary processes induced by $\gamma^{\alpha_{\varepsilon,j}}$. The key point in the following estimates is that although $\{Z_m^\alpha\}$ is subject to the effects of increasingly vigorous trimming as $m \to \infty$, the doubly exponential growth of $\{Z_m\}$ (see Proposition 4 of Section 2) will offset this trimming so that $\{Z_m^\alpha\}$ will retain the same asymptotic behavior as $\{Z_m\}$. As in Section 2, this behavior of $\{Z_m^\alpha\}$ will enable us to analyze $\{M_n^\alpha\}$. The basic technique employed in Section 2 of using generating functions to examine $\{Z_m\}$ will still be valid. However, $\alpha_{\varepsilon,j}$ will not in general be constant, and therefore $\{Z_m^\alpha\}$ not a Galton-Watson branching process, nor its generating function after m generations the m-fold iterate of some fixed generating function.[3] Therefore, we must introduce some new terminology, and define $\phi_\alpha^{(m+l,m)}(s)$ to be the m-fold composite generating function governing reproduction from the l^{th} to $(l+m)^{\text{th}}$ generations of $\{Z_m^\alpha\}$.[4] That is, $\phi_\alpha^{(m+l,m)}(s)$ is the generating function of Z_{m+l}^α given that $Z_l^\alpha = 1$. Clearly,

$$\phi_\alpha^{(l,0)}(s) = s,$$

[3] Generation will of course have a different meaning depending on whether it is used in the context of $\mathfrak{X}^\alpha$ or of $\{Z_m^\alpha\}$

[4] Again, we suppress subscripts when convenient

whereas we obtain the inductive relations

$$\begin{aligned}\phi_\alpha^{(m+l,m)}(s) &= \phi_\alpha^{(m+l-1,m-1)} \circ \phi_\alpha^{(m+l,1)}(s) \\ &= \phi_\alpha^{(l+1,1)} \circ \phi_\alpha^{(m+l,m-1)}(s) \\ &= \phi_Z \left(1 - \frac{[G(\gamma^{\alpha_\varepsilon,j}(l+1)) - p]}{(1-p)} \right. \\ &\qquad \left. + \frac{[G(\gamma^{\alpha_\varepsilon,j}(l+1)) - p]}{(1-p)} \cdot \phi_\alpha^{(m+l,m-1)}(s)\right).^5 \end{aligned} \tag{14}$$

If G is continuous, the last line is equivalent to

$$\phi_Z(1 - e^{-\alpha_{\varepsilon,j}(l+1)} + e^{-\alpha_{\varepsilon,j}(l+1)} \cdot \phi_\alpha^{(m+l,m-1)}(s)).$$

Note that $\phi_\alpha^{(m,m)}$ is simply the generating function of Z_m^α.

For computational purposes, also define

$$h_\alpha^{(m+l,m)}(s) = -\log[1 - \phi_\alpha^{(m+l,m)}(1 - \exp(-s))], \tag{15}$$

from which it follows that

$$h_\alpha^{(m+l,m)}(s) = h_\alpha^{(l+1,1)} \circ h_\alpha^{(m+l,m-1)}(s). \tag{16}$$

Also set

$$h(s) = -\log[1 - \phi_Z(1 - \exp(-s))], \tag{17}$$

whence

$$\begin{aligned} h(s) &= -\log k(\exp(-s)) \\ &= h_0^{(1,1)}(s) = h_0^{(l,1)}(s), \end{aligned} \tag{18}$$

where $k(s)$ is defined in Section 2. From (14), (15), and (17), it follows that

$$h_\alpha^{(l+1,1)}(s) = h\left(s - \log\left[\frac{G(\gamma^{\alpha_\varepsilon,j}(l+1)) - p}{1-p}\right]\right), \tag{19}$$

which equals $h(s + \alpha_{\varepsilon,j}(l+1))$ if G is continuous.

As in Section 2, we will derive the limiting behavior of $\phi_\alpha^{(m,m)}(1 - \exp(-2^m s))$, and apply Darling's and Cohn's results to obtain analogous properties of $\{Z_m^\alpha\}$, and hence $\{M_n^\alpha\}$. We begin with a pair of lemmas which characterize $h_\alpha^{(m+l,m)}$.

Lemma 1. *Assume that $EX_1^{2+\delta} < \infty$ for some $\delta > 0$, and define*

$$\bar{\alpha}_{\varepsilon,j}(l) = -\log\left[\frac{G(\gamma^{\alpha_\varepsilon,j}(l)) - p}{1-p}\right]. \tag{20}$$

Then,

$$\bar{\alpha}_{\varepsilon,j}(l) \leqq \alpha_{\varepsilon,j}(l),$$

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[5] As in Section 2, ϕ_Z still refers to the generating function of Z_1

and

$$h_\alpha^{(l+1,1)}(s)=h(s+\bar{\alpha}_{\varepsilon,j}(l)). \tag{21}$$

Moreover,

$$h_\alpha^{(m+l,m)}(2^{m+l}s)=2^l s+\sum_{k=1}^{m} 2^{-k}\bar{\alpha}_{\varepsilon,j}(k+l)$$
$$-2(1-2^{-m})\log a+O_{l,m}(\exp(-2^{l-1}\delta s)), \tag{22}$$

where a is defined in Proposition 2. For some fixed u_0, $0<u<u_0$ implies that

$$\sup_{l,m,u}\frac{|O_{l,m}(u)|}{u}<\infty. \tag{23}$$

(Therefore, in future computations involving Lemma 1, the subscripts will be dropped from $O_{l,m}$.)

Proof. $\bar{\alpha}_{\varepsilon,j}(l)\leqq\alpha_{\varepsilon,j}(l)$ follows immediately from (13), and $h_\alpha^{(l+1,1)}(s)=h(s+\bar{\alpha}_{\varepsilon,j}(l))$ follows immediately from (19). To demonstrate (22), we apply induction on m simultaneously for all l. Equation (16), together with (22) stated for $m-1$, implies that

$$h_\alpha^{(m+l,m)}(2^{m+l}s)=h_\alpha^{(l+1,1)}\Bigg(2^{l+1}s+\sum_{k=2}^{m}2^{-(k-1)}\bar{\alpha}_{\varepsilon,j}(k+l)$$
$$-2(1-2^{1-m})\cdot\log a+O_{l+1,m-1}(\exp(-2^l\delta s))\Bigg),$$

which by (21), equals

$$h\Bigg(2^{l+1}s+\sum_{k=1}^{m}2^{-(k-1)}\bar{\alpha}_{\varepsilon,j}(k+l)$$
$$-2(1-2^{1-m})\cdot\log a+O_{l+1,m-1}(\exp(-2^l\delta s))\Bigg). \tag{24}$$

Now, Proposition 2 and (18) together imply that

$$h(s)=\frac{s}{2}-\log a-\log\left[1+O\left(\exp\left(-\frac{\delta}{2}s\right)\right)\right]$$
$$=\frac{s}{2}-\log a+O'\left(\exp\left(-\frac{\delta}{2}s\right)\right)$$

(for some O'). Therefore, (24) equals

$$2^l s+\sum_{k=1}^{m}2^{-k}\bar{\alpha}_{\varepsilon,j}(k+l)-2(1-2^{-m})\cdot\log a+\tfrac{1}{2}O_{l+1,m-1}(\exp(-2^l\delta s))$$
$$+O'\left[\exp\left(-\delta\left(2^l s+\sum_{k=1}^{m}2^{-k}\bar{\alpha}_{\varepsilon,j}(k+l)-(1-2^{1-m})\cdot\log a\right.\right.\right.$$
$$\left.\left.\left.+\tfrac{1}{2}O_{l+1,m-1}(\exp(-2^l\delta s))\right)\right)\right]$$
$$=2^l s+\sum_{k=1}^{m}2^{-k}\bar{\alpha}_{\varepsilon,j}(k+l)-2(1-2^{-m})\cdot\log a+O_{l,m}(\exp(-2^{l-1}\delta s)).$$

This demonstrates (22). Now, choose M, t_1, and t_2 so that

$$|O'(\exp(-\delta t))| < M \cdot \exp(-\delta t)$$

and

$$|O_{l+1,m-1}(\exp(-\delta t))| < 2M \cdot \exp(-\delta t)$$

for $0 < t_1 < t$, and

$$t > 2|\log a| + 2M \cdot \exp(-\delta t)$$

for $0 < t_2 < t$. It follows that for $t > t_0$, where $t_0 = t_1 \vee t_2$,

$$|O_{l,m}(\exp(-\delta t))| < 2M \cdot \exp(-\delta t),$$

the bound being independent of l. This implies (23). □

Lemma 2. *$h(s)$ (hence $h_\alpha^{(l,1)}(s)$ and $h_\alpha^{(m+l,l)}(s)$) is norm decreasing. That is, if $s_1 > s_2$, then $s_1 - s_2 > h(s_1) - h(s_2)$.*

Proof. Since $k(t) = 1 - \phi_Z(1-t)$, where ϕ_Z is a strictly convex generating function, $k(t)$ is strictly concave with $k(0) = 0$. Therefore, for $t_2 > t_1 > 0$,

$$\frac{k(t_2)}{k(t_1)} < \frac{t_2}{t_1}.$$

If we set $s_i = -\log t_i$, then (18) implies that

$$\begin{aligned} h(s_1) - h(s_2) &= \log[k(\exp(-s_2))/k(\exp(-s_1))] \\ &= \log[k(t_2)/k(t_1)] \\ &< \log \frac{t_2}{t_1} = s_1 - s_2. \end{aligned}$$

Equations (21) and (16) imply that the same is true for $h_\alpha^{(l,1)}(s)$ and for $h_\alpha^{(m+l,l)}(s)$. □

We now state our main technical result, which was used in Section 2 to characterize the asymptotic behavior of $\{Z_m\}$, and will be similarly used in this section to characterize $\{Z_m^\alpha\}$.

Proposition 5. *Assume that $EX_1^{2+\delta} < \infty$ for some $\delta > 0$. Then,*

$$\phi_{\alpha_{\varepsilon,j}}^{(m,m)}(1 - \exp(-2^m s)) \to v_{\alpha_{\varepsilon,j}}(s)$$

as $m \to \infty$ for $0 < s < \infty$, where $v_{\alpha_{\varepsilon,j}}(s)$ is continuous and strictly increasing, and $v_{\alpha_{\varepsilon,j}}(\infty) = 1$. As $j \uparrow \infty$ and $\varepsilon \downarrow 0$, $v_{\alpha_{\varepsilon,j}}(s) \downarrow v_0(s) = v(s)$, where v is defined in (9). $v(0+) = q_0$.

Proof. Making use of (15), it suffices to demonstrate

Proposition 5'. *Assume that $EX_1^{2+\delta} < \infty$ for some $\delta > 0$. Then,*

$$h_{\alpha_{\varepsilon,j}}^{(m+l,m)}(2^{m+l} s) \to w_{\alpha_{\varepsilon,j};l}(s)$$

as $m\to\infty$ *for* $0<s<\infty$. *If we set* $w_{\alpha_{\varepsilon,j}}(s)=w_{\alpha_{\varepsilon,j};0}(s)$, *then*

$$w_{\alpha_{\varepsilon,j}}(s)=h^{(l,l)}_{\alpha_{\varepsilon,j}}\circ w_{\alpha_{\varepsilon,j};l}(s),$$

and $w_{\alpha_{\varepsilon,j}}$ *is continuous and strictly increasing, with* $w_{\alpha_{\varepsilon,j}}(\infty)=\infty$. *As* $j\uparrow\infty$ *and* $\varepsilon\downarrow 0$, $w_{\alpha_{\varepsilon,j}}(s)\downarrow w(s)$, *which is defined as* $w_0(s)$. $w(0+)=-\log(1-q_0)$.

Proof. Throughout the proof, we suppress the subscripts ε and j when convenient. We first demonstrate existence of the limit $w_{\alpha;l}$. For m, m' given, with $m<m'$, Lemma 2 implies that for $m_0\leqq m$,

$$\begin{aligned}&|h^{(m+l,m)}_{\alpha}(2^{m+l}s)-h^{(m'+l,m')}_{\alpha}(2^{m'+l}s)|\\&\quad\leqq|h^{(m+l,m-m_0)}_{\alpha}(2^{m+l}s)-h^{(m'+l,m'-m_0)}_{\alpha}(2^{m'+l}s)|.\end{aligned}$$

By Lemma 1, this is at most

$$\sum_{k=m-m_0+1}^{\infty}2^{-k}\bar{\alpha}(k+l+m_0)+2^{1+m_0-m}\cdot\log a+O(\exp(-2^{l+m_0-1}\delta s)).\tag{25}$$

If $m\to\infty$, then we may choose $m_0\to\infty$ in a manner so that

$$\begin{aligned}\sum_{k=m-m_0+1}^{\infty}2^{-k}\bar{\alpha}(k+l+m_0)&\leqq\sum_{k=m-m_0+1}^{\infty}2^{-k}\alpha(k+l+m_0)\\&=\alpha^{l-j+m_0}\cdot\sum_{k=m-m_0+1}^{\infty}\left(\frac{\alpha}{2}\right)^k\to 0.\end{aligned}$$

Therefore, (25) implies that $\{h^{(m+l,m)}_{\alpha}(2^{m+l}s)\}$ is Cauchy in m, and hence $w_{\alpha;l}(s)$ exists.

$w_{\alpha}(s)=h^{(l,l)}_{\alpha}\circ w_{\alpha;l}(s)$ follows immediately from the construction of $w_{\alpha;l}(s)$.

Continuity of w_{α} is demonstrated in a manner analogous to the demonstration of the existence of $w_{\alpha;l}$. From Lemmas 1 and 2, it follows that for s, $s_1>0$,

$$\begin{aligned}&|h^{(m+l,m+l)}_{\alpha}(2^{m+l}s_1)-h^{(m+l,m+l)}_{\alpha}(2^{m+l}s)|\\&\quad\leqq 2^l|s_1-s|+O(2\cdot\exp(-2^{l-1}\delta(s_1\wedge s))).\end{aligned}\tag{26}$$

If we let $m\to\infty$, then (26) implies that

$$|w_{\alpha}(s_1)-w_{\alpha}(s)|\leqq 2^l|s_1-s|+O(2\cdot\exp(-2^{l-1}\delta(s_1\wedge s))).\tag{27}$$

Choosing l appropriately, we see that the r.h.s. of (27) approaches 0 as s approaches s_1, and hence w_{α} is continuous.

w_{α} is strictly increasing: For $s_1>s_2$, if we choose l so that

$$O(2\cdot\exp(-2^{l-1}\delta s_2))<2^{l-1}(s_1-s_2),$$

Lemma 1 implies that

$$h^{(m+l,m)}_{\alpha}(2^{m+l}s_1)-h^{(m+l,m)}_{\alpha}(2^{m+l}s_2)\geqq 2^{l-1}(s_1-s_2).$$

Letting $m \to \infty$, we obtain

$$w_{\alpha;l}(s_1) - w_{\alpha;l}(s_2) \geqq 2^{l-1}(s_1 - s_2).$$

Therefore,

$$w_\alpha(s_1) - w_\alpha(s_2) = h_\alpha^{(l,l)} \circ w_{\alpha;l}(s_1) - h_\alpha^{(l,l)} \circ w_{\alpha;l}(s_2) > 0,$$

since $h_\alpha^{(l,l)}$ is strictly increasing.

$w_\alpha(\infty) = \infty$ follows trivially from Lemma 1 by setting $l=0$, and letting $s \to \infty$, $m \to \infty$ in (22).

$w_{\alpha_{\varepsilon,j}}(s) \downarrow w(s)$ as $j \uparrow \infty$, $\varepsilon \downarrow 0$: Lemma 1 implies that

$$h_{\alpha_{\varepsilon,j}}^{(m+l,m)}(2^{m+l} s) - h_0^{(m+l,m)}(2^{m+l} s) \leqq \sum_{k=1}^{m} 2^{-k} \bar{\alpha}_{\varepsilon,j}(k+l) + O(2 \cdot \exp(-2^{l-1} \delta s)).$$

By (13) and (20), this is at most

$$\varepsilon + 2^{(l-j) \wedge 0} \cdot \sum_{k=1}^{\infty} \left(\frac{\alpha}{2}\right)^k + O(2 \cdot \exp(-2^{l-1} \delta s)).$$

Letting $m \to \infty$ and applying Lemma 2, we obtain

$$\begin{aligned} h_{\alpha_{\varepsilon,j}}^{(l,l)} \circ w_{\alpha_{\varepsilon,j};l}(s) - h_{\alpha_{\varepsilon,j}}^{(l,l)} \circ w_{0;l}(s) &\leqq w_{\alpha_{\varepsilon,j};l}(s) - w_{0;l}(s) \\ &\leqq \varepsilon + 2^{(l-j) \wedge 0} \cdot \sum_{k=1}^{\infty} \left(\frac{\alpha}{2}\right)^k + O(2 \cdot \exp(-2^{l-1} \delta s)). \end{aligned} \tag{28}$$

Now, for $j \geqq l$, as $\varepsilon \to 0$,

$$h_{\alpha_{\varepsilon,j}}^{(l,l)} \circ w_{0;l}(s) - h^{(l)} \circ w_{0;l}(s) \to 0.$$

This, together with (28), implies that

$$w_{\alpha_{\varepsilon,j}}(s) - w(s) = h_{\alpha_{\varepsilon,j}}^{(l,l)} \circ w_{\alpha_{\varepsilon,j};l} - h_{\alpha_{\varepsilon,j}}^{(l,l)} \circ w_{0;l}(s) + h_{\alpha_{\varepsilon,j}}^{(l,l)} \circ w_{0;l}(s) - h^{(l)} \circ w_{0;l}(s) \to 0$$

as $\varepsilon \to 0$ and $j \to \infty$, which we see by choosing l so that $l \to \infty$ and $j - l \to \infty$. Convergence is clearly monotone in ε and j.

$w(0+) = -\log(1 - q_0)$: For $s > 0$,

$$\begin{aligned} w(0+) &= \lim_{l \to \infty} w(2^{-l} s) = \lim_{l \to \infty} \lim_{m \to \infty} h^{(m)}(2^{m-l} s) \\ &= \lim_{l \to \infty} h^l(\lim_{m \to \infty} h^m(2^m s)) \\ &= \lim_{l \to \infty} h^l(w(s)). \end{aligned}$$

By (17), this equals

$$\lim_{l \to \infty} [-\log(1 - \phi_Z^{(l)}(v(s)))] = -\log(1 - \lim_{l \to \infty} \phi_Z^{(l)}(v(s))).$$

Since $0 < v(s) < 1$,

$$\lim_{l \to \infty} \phi_Z^{(l)}(v(s)) = q_0,$$

q_0 being the extinction probability of $\{Z_m\}$, and hence of $\mathfrak{X}$. (See Athreya-Ney [1], page 4.) Therefore,

$$w(0+) = -\log(1-q_0). \quad \square$$

Proposition 5 shows that $\{Z_m^\alpha\}$ satisfies the conditions of Proposition 3. Therefore, applying Proposition 3, we conclude that $\{Z_m^\alpha\}$ has the same basic asymptotic behavior as $\{Z_m\}$, which is expressed by the following weak convergence result.

Corollary 3. *$P[2^{-m}\log(Z_m^\alpha+1) \leq x] \to v_\alpha(x)$ as $m\to\infty$, for all $0<x<\infty$, where v_α is defined in Proposition 5.*

We continue to follow the same format as in the prologue to Theorem 1. Having shown weak convergence of $\{Z_m^\alpha\}$ under appropriate renormalization, we desire to convert this statement into one of a.s. convergence. Now, we have shown in Proposition 5 that $v_\alpha(x)$ is a distribution function which is continuous and strictly increasing on $x>0$. Once again we wish to apply Cohn's result to conclude that weak convergence as in Corollary 3 is sufficient to imply a.s. convergence. Although Cohn's assertion is made only for Galton-Watson branching processes, this restriction is stronger than necessary, and his proof carries over for $\{Z_m^\alpha\}$. ($\{Z_m^\alpha\}$ can be thought of as a branching process with varying environment.) Therefore, we obtain

Corollary 4. *$2^{-m}\log(Z_m^\alpha+1) \to V_\alpha$ w.p. 1 as $m\to\infty$, where V_α is a random variable having distribution v_α.*

We now have the same tools available for an investigation of the minimal displacement $\{M_n^\alpha\}$ of the process $\mathfrak{X}^\alpha$ as we had for the branching random walk $\mathfrak{X}$ in Section 2. Corollary 4 assumes the role of Proposition 4 in expressing the asymptotic behavior of $\{Z_m^\alpha\}$. On the other hand, the law governing the termination of a dynasty is the same for both $\mathfrak{X}$ and $\mathfrak{X}^\alpha$, since $\gamma(m)$ induces trimming only among the first generation of each dynasty. Therefore, the branching processes $\{Y_n^{(i,m)}\}$ within the m^{th} dynasty satisfy the same law as their analogues in Section 2, and hence Corollary 1 still holds in our new setting. A duplication of the reasoning of the first part of Theorem 1 will thus yield the same results as before regarding the minimal displacement of the process. The only difference is that an m^{th} dynasty individual, $(a_1,\dots,a_n)$, will now have position $S(a_1,\dots,a_n)$ at most $\sum_{k=0}^{m-1}\gamma(k)$, rather than having $S(a_1,\dots,a_n)=m$ as in Section 2. We therefore obtain the following analogue of Theorem 1(a).

Proposition 6. *Let $\{M_n^\alpha\}$ be the minimal displacement of the "trimmed" branching random walk $\mathfrak{X}^\alpha$, where $EX_1^{2+\delta}<\infty$ for some $\delta>0$; $G(\{0\})=p$, $G((0,\infty))=1-p$, and $mp=1$. Then,*

$$\limsup_{n\to\infty}\left(M_n^\alpha - \sum_{k=1}^{r(n)} \gamma^{\alpha_\varepsilon, j}(k)\right) \leq 0$$

w.p. 1 *on* $\{V_\alpha > 0\}$, *where* V_α *is defined in Corollary* 4,

$$r(n) = \lceil (\log\log n - \log(V_\alpha + o(1)))/\log 2 \rceil,$$

$o(1)$ *is stochastic, and* $1 < \alpha < 2$.

Up to now, we have chosen ε and j somewhat arbitrarily in defining $\gamma^{\alpha_{\varepsilon,j}}$. Proposition 5 states that $v_{\alpha_{\varepsilon,j}}(s) \downarrow v(s)$ as $j \uparrow \infty$ and $\varepsilon \downarrow 0$. This is fairly clear, since if we start trimming at the j^{th} dynasty for j large, Z_j will already be very large, and the trimming will have little effect on the magnitude of Z_m, in the sense of Corollary 4, as $m \to \infty$. Since $v(0+) = q_0$, it follows that $v_{\alpha_{\varepsilon,j}}(0+) \downarrow q_0$ as $j \uparrow \infty$ and $\varepsilon \downarrow 0$. Because $M_n \leqq M_n^{\alpha_{\varepsilon,j}}$, we can modify Proposition 6 into a statement on $\{M_n\}$.

Proposition 6′. *Let* $\{M_n\}$ *be the minimal displacement of the branching random walk* $\mathfrak{X}$, *where* $EX_1^{2+\delta} < \infty$ *for some* $\delta > 0$; $G(\{0\}) = p$, $G((0, \infty)) = 1 - p$, *and* $mp = 1$. *Then,*

$$\limsup_{N} \limsup_{n \to \infty} \left(M_n - \sum_{k=1}^{r'(n)} \gamma^{\alpha_{\varepsilon,j}}(k) \right) \leqq 0$$

w.p. 1 *on* $\{V > 0\}$. V *is defined in Proposition* 4,

$$N = \{(\varepsilon, j) \colon V_{\alpha_{\varepsilon,j}} > 0;\ \varepsilon = 1, 1/2, 1/3, \ldots;\ j \in \mathbb{Z}^+\},$$

$$r'(n) = \lceil (\log\log n - \log(V_{\alpha_{\varepsilon,j}} + o^{\alpha_{\varepsilon,j}}(1)))/\log 2 \rceil,$$

$o^{\alpha_{\varepsilon,j}}(1)$ *is stochastic, and* α *is fixed, with* $1 < \alpha < 2$. ($o^{\alpha_{\varepsilon,j}}(1)$ *denotes dependence of* $o(1)$ *on* $\alpha_{\varepsilon,j}$.)

Computation of a Lower Bound for M_n

During our investigation of an upper bound for $\{M_n\}$, we defined $\gamma^\alpha(m)$ in such a manner that the trimming of $\mathfrak{X}$ thus induced was insufficient to hamper the rapid growth of $\{Z_m^\alpha\}$. We will presently show, however, that if $\beta > 2$, and γ^β is defined so that

$$\gamma^\beta(m) = G^{-1}(p + (1-p) \cdot \exp(-\beta^m)),$$

then the trimming induced by γ^β "kills" the process in a strong enough sense so as to allow a simple computation of a lower bound for $\{M_n\}$.

We will be examining those sets A_m of trees, which possess at least one first generation member within the m^{th} dynasty with spatial movement less than $\gamma(m)$. That is,

$$A_m = \{\omega \colon \min_{a_1, \ldots, a_n} X(a_1, \ldots, a_n) < \gamma^\beta(m) \text{ for those }$$
$$(a_1, \ldots, a_n) \text{ with } \Gamma(a_1, \ldots, a_n) = m \text{ and } X(a_1, \ldots, a_n) > 0\}.$$

Let V be as defined in Proposition 4. $X(a_1, \ldots, a_n)$ conditioned on $X(a_1, \ldots, a_n) > 0$ is independent of $\{Z_m\}$. Therefore,

$$P[A_m | Z_m < \exp(2^{m+1} V)] \leqq 1 - (1 - \exp(-\beta^m))^{\exp(2^{m+1} V)}$$
$$\leqq C \cdot \exp(2^{m+1} V - \beta^m) \tag{29}$$

for some constant $C>0$. Now, Proposition 4 implies that

$$\limsup_{m\to\infty}(Z_m - \exp(2^{m+1}V)) < 0$$

w.p. 1. Since

$$\sum_{m=1}^{\infty} \exp(2^{m+1}V - \beta^m) < \infty$$

w.p. 1, we may therefore apply a Borel-Cantelli argument to (29) to conclude that

$$P[\omega \in A_m \text{ infinitely often}] = 0. \tag{30}$$

Equation (30) states that for almost every tree of our branching random walk $\mathfrak{X}$, for large enough m, each first generation member of the m^{th} dynasty will have spatial movement at least $\gamma^\beta(m)$. Therefore,

$$\inf_m \left\{ S(a_1, \ldots, a_n) - \sum_{k=1}^{m-1} \gamma^\beta(k) : n \in \mathbb{Z}^+, \Gamma(a_1, \ldots, a_n) = m \right\} > -\infty$$

w.p. 1. Actually, we will only need the following weaker assertion:

$$\inf_n \left\{ M_n - \sum_{k=1}^{m-1} \gamma^\beta(k) : m = \min_{a_1, \ldots, a_n} \Gamma(a_1, \ldots, a_n) \right\} > -\infty \quad \text{w.p. 1.} \tag{31}$$

Theorem 1(b), together with (31), will now yield a lower bound for the minimal displacement of the branching random walk $\mathfrak{X}$, with $G(\{0\}) = p$ and $G((0,\infty)) = 1-p$. We observe that $S'(a_1, \ldots, a_n) = \Gamma(a_1, \ldots, a_n)$ induces a branching random walk $\mathfrak{X}'$ of the type studied in Section 2, where $G(\{0\}) = p$ and $G(\{1\}) = 1-p$. Therefore, Theorem 1(b) implies that the minimal displacement of $\mathfrak{X}'$, $M_n' = \min_{a_1,\ldots,a_n} \Gamma(a_1, \ldots, a_n)$, satisfies

$$\inf_n (M_n' - \lceil \log\log n / \log 2 \rceil) > -\infty \quad \text{w.p. 1.}$$

This, together with (31), yields the following analogue of Theorem 1(b) for $\mathfrak{X}$.

Proposition 7. *Let $\{M_n\}$ be the minimal displacement of the branching random walk $\mathfrak{X}$, where $EX_1^{2+\delta} < \infty$; $G(\{0\}) = p$, $G((0,\infty)) = 1-p$, and $mp = 1$. Then,*

$$\inf_n \left(M_n - \sum_{k=1}^{s(n)} \gamma^\beta(k) \right) > -\infty$$

w.p. 1, where

$$s(n) = \lceil \log\log n / \log 2 \rceil$$

and $\beta > 2$.

The Theorem

Proposition 6′ provides us with an upper bound for M_n, and Proposition 7 provides us with a lower bound. With the aid of these results, we obtain the following weaker analogue of Theorem 1.[6]

Theorem 2. *Assume that M_n is the minimal displacement of the branching random walk $\mathfrak{X}$, where $EX_1^{2+\delta}<\infty$ for some $\delta>0$; $G(\{0\})=p$, $G((0,\infty))=1-p$, and $mp=1$, where $EX_1=m>1$. Then, conditioned on the nonextinction of $\mathfrak{X}$, if*

$$\sum_{k=1}^{\infty} G^{-1}(p+(1-p)\cdot\exp(-\lambda^k))=\infty \tag{32}$$

for some $\lambda>1$, then

$$\lim_{n\to\infty} \frac{M_n}{\sum_{k=1}^{s(n)} G^{-1}(p+(1-p)\cdot\exp(-2^k))}=1 \tag{33}$$

w.p. 1, whereas if

$$\sum_{k=1}^{\infty} G^{-1}(p+(1-p)\cdot\exp(-\lambda^k))<\infty \tag{34}$$

for some $\lambda>1$, then

$$\lim_{n\to\infty} M_n<\infty \tag{35}$$

w.p. 1. Here, $s(n)=\lceil \log\log n/\log 2\rceil$.

Proof. First observe that for $\lambda_2>\lambda_1>1$,

$$\sum_{k=0}^{m}\gamma^{\lambda_1}(k)\geqq\sum_{k=0}^{m}\gamma^{\lambda_2}(k),$$

whereas if we set $t(m)=\lfloor m\cdot\log\lambda_2/\log\lambda_1\rfloor$,

$$\begin{aligned}\sum_{k=0}^{m}\gamma^{\lambda_2}(k)&=\sum_{k=0}^{m} G^{-1}(p+(1-p)\cdot\exp(-\lambda_2^k))\\&=\sum_{k=0}^{m} G^{-1}(p+(1-p)\cdot\exp(-\lambda_1^{k\cdot\log\lambda_2/\log\lambda_1}))\\&\geqq\frac{\log\lambda_1}{\log\lambda_2}\cdot\sum_{k=0}^{t(m)} G^{-1}(p+(1-p)\cdot\exp(-\lambda_1^k))\\&=\frac{\log\lambda_1}{\log\lambda_2}\cdot\sum_{k=0}^{t(m)}\gamma^{\lambda_1}(k)\end{aligned} \tag{36}$$

[6] To avoid confusion, we explicitly include the subscripts j and ε of $\alpha_{\varepsilon,j}$ in our notation in Theorem 2. $\gamma^{\alpha}(m)$ will mean $G^{-1}(p+(1-p)\cdot\exp(-\alpha^m))$

by the monotonicity of G^{-1}. In particular,

$$\sum_{k=1}^{\infty} G^{-1}(p+(1-p)\cdot\exp(-\lambda_1^k))=\infty$$

iff

$$\sum_{k=1}^{\infty} G^{-1}(p+(1-p)\cdot\exp(-\lambda_2^k))=\infty.$$

By (13),

$$\sum_{k=j+1}^{\infty} \gamma^{\alpha_\varepsilon, j}(k)=\sum_{k=j+1}^{\infty} G^{-1}(p+(1-p)\cdot\exp(-\alpha^{k-j})),$$

where $1<\alpha<2$. Therefore, if $\varepsilon>0$, (34) implies that

$$\sum_{k=1}^{\infty} \gamma^{\alpha_\varepsilon, j}(k)<\infty,$$

and (35) follows from Proposition 6′.

Now, assume that (32) holds. Proposition 6′ implies that for $1<\alpha<2$,

$$\limsup_{n\to\infty}\left(\frac{M_n}{\sum_{k=1}^{s(n)}\gamma^2(k)}-\frac{\sum_{k=1}^{s(n)}\gamma^\alpha(k)}{\sum_{k=1}^{s(n)}\gamma^2(k)}\right)\leqq 0 \tag{37}$$

w.p. 1 on $\{V>0\}$. On the other hand, Proposition 7 implies that for $\beta>2$,

$$\liminf_{n\to\infty}\left(\frac{M_n}{\sum_{k=1}^{s(n)}\gamma^2(k)}-\frac{\sum_{k=1}^{s(n)}\gamma^\beta(k)}{\sum_{k=1}^{s(n)}\gamma^2(k)}\right)\geqq 0 \tag{38}$$

w.p. 1. Together with (36), (37) and (38) imply that

$$\frac{\log 2}{\log\beta}\leqq\liminf_{n\to\infty}\frac{M_n}{\sum_{k=1}^{s(n)}\gamma^2(k)}\leqq\limsup_{n\to\infty}\frac{M_n}{\sum_{k=1}^{s(n)}\gamma^2(k)}\leqq\frac{\log 2}{\log\alpha}$$

on $\{V>0\}$. If we let $\alpha\uparrow 2$, $\beta\downarrow 2$, it follows that

$$\lim_{n\to\infty}\frac{M_n}{\sum_{k=1}^{s(n)} G^{-1}(p+(1-p)\cdot\exp(-2^k))}=\lim_{n\to\infty}\frac{M_n}{\sum_{k=1}^{s(n)}\gamma^2(k)}=1$$

w.p. 1 on $\{V>0\}$, the set of nonextinction of $\mathfrak{X}$. This demonstrates (33), and hence the proof is complete. □

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