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Stochastic Differential Equations for Multi-dimensional Domain with Reflecting Boundary

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Summary. In this paper we prove that there exists a unique solution of the Skorohod equation for a domain in $\mathbf{R}^d$ with a reflecting boundary condition. We remove the admissibility condition of the domain which is assumed in the work [4] of Lions and Sznitman. We first consider a deterministic case and then discuss a stochastic case.

Introduction

In this paper we investigate multi-dimensional Skorohod stochastic differential equations (abbreviated: SDE's) on a domain D with a reflecting boundary condition. If w is a given process in $\mathbf{R}^d$, our problem is to find a solution ξ of the equation $\xi(t)=w(t)+\phi(t)$ satisfying certain conditions (see (1.2) and (1.3) of §1) so that ξ is a reflecting process on $\bar{D}$ (the precise definition of the equation will be given in §1). This equation is called a multi-dimensional Skorohod equation on the analogy of the one-dimensional case first discussed by Skorohod [9] (see also [3, 5]). First we consider a deterministic version of the Skorohod equation. This kind of problem has been discussed by Tanaka [10] when D is a convex domain and then by Lions and Sznitman [4] when D is a general domain satisfying Condition (A) and Condition (B) (see §1) together with the admissibility condition. Here the admissibility means roughly that D can be approximated in some sense by smooth domains. One of the purposes of this paper is to prove the existence and uniqueness of solutions of the Skorohod equation $\xi=w+\phi$ only assuming Conditions (A) and (B) (Theorem 4.1). We do not need the admissibility condition. To prove this result we employ the method of Tanaka [10]; more precisely, we approximate w by step functions $\{w_m\}$, consider the problem $\xi_m=w_m+\phi_m$ which is easy to solve and then take a limit in m to obtain a solution of $\xi=w+\phi$.

After completing this work the author came to know the recent work by Frankowska [2] in which a deterministic Skorohod problem was discussed by making use of techniques from viability theory; especially Theorem 3.4 of [2] seems to contain our Theorem 4.1 as a special case. However, our method of proving Theorem 4.1 is elementary and quite different from Frankowska's, involving somewhat detailed estimates of the total variation $|\phi|_t$ and also of the modulus of continuity of ξ in terms of the modulus of continuity of w. These estimates are also used in discussion of our second problem, the Skorohod SDE.

We also consider the Skorohod SDE. Let σ: $\bar{D}\to\mathbf{R}^d\otimes\mathbf{R}^d$ and b: $\bar{D}\to\mathbf{R}^d$ be bounded and Lipschitz continuous. Assuming that D satisfies Conditions (A) and (B) we consider the Skorohod SDE

$$dX(t)=\sigma(X(t))\,dB(t)+b(X(t))\,dt+d\Phi(t),$$

which gives rise to a reflecting diffusion process, where $B(t)$ is a standard d-dimensional Brownian motion. Lions and Sznitman [4] proved the existence and uniqueness of solutions of the above SDE assuming the admissibility condition plus another additional condition (Condition (C) of § 1). Our result is that we can remove both of these conditions to have a unique strong solution of the above SDE.

In §1 we introduce some conditions on D and give the definition of the Skorohod equation. Several remarks related to the conditions and the main results are also stated in this section. In §2 we give some lemmas which will be used in subsequent sections, and in §3 we estimate the total variation $|\phi_m|_t$. Solvability of the Skorohod equation is treated in §4 (deterministic case) and in § 5 (stochastic case).

§ 1. Formulation of the Problem and the Results

Let D be a domain in $\mathbf{R}^d$ and define the set $\mathcal{N}_x$ of inward normal unit vectors at $x\in\partial D$ by

$$\mathcal{N}_x=\bigcup_{r>0}\mathcal{N}_{x,r},$$

$$\mathcal{N}_{x,r}=\{\mathbf{n}\in\mathbf{R}^d\colon\ |\mathbf{n}|=1,\ B(x-r\,\mathbf{n},r)\cap D=\varnothing\},$$

where $B(z,r)=\{y\in\mathbf{R}^d\colon\ |y-z|<r\}$, $z\in\mathbf{R}^d$, $r>0$. In general, it can happen that $\mathcal{N}_x=\varnothing$. Following Lions and Sznitman [4], we introduce two conditions on the domain D.

Condition (*A*) (uniform exterior sphere condition). There exists a constant $r_0>0$ such that

$$\mathcal{N}_x=\mathcal{N}_{x,r_0}\neq\varnothing \qquad \text{for any } x\in\partial D.$$

Condition (*B*). There exist constants $\delta>0$ and $\beta\in[1,\infty)$ with the following property: for any $x\in\partial D$ there exists a unit vector $\boldsymbol{l}_x$ such that

$$\langle \boldsymbol{l}_x,\mathbf{n}\rangle\geqq 1/\beta \qquad \text{for any } \mathbf{n}\in\bigcup_{y\in B(x,\delta)\cap\partial D}\mathcal{N}_y.$$

In what follows $\langle\cdot,\cdot\rangle$ denotes the usual inner product in $\mathbf{R}^d$.

Remark 1.1. It is easy to see that the following two statements for a unit vector $\mathbf{n}$ are equivalent (cf. [4], Remark 1.2).

(i) $\mathbf{n} \in \mathcal{N}_{x,r}$

(ii) $\langle y-x, \mathbf{n}\rangle + \frac{1}{2r}|y-x|^2 \geqq 0$ for any $y \in \bar{D}$.

Remark 1.2. D satisfies Condition (B) if it satisfies the following condition.

Condition (*B'*) (uniform interior cone condition). There exist $\delta > 0$ and $\alpha \in [0,1)$ with the following property: for any $x \in \partial D$ there exists a unit vector $\boldsymbol{l}_x$ such that

$$C(y, \boldsymbol{l}_x, \alpha) \cap B(x, \delta) \subset \bar{D}, \qquad \forall\, y \in B(x, \delta) \cap \partial D;$$

where $C(y, \boldsymbol{l}_x, \alpha)$ is the convex cone with vertex y, defined by

$$C(y, \boldsymbol{l}_x, \alpha) = \{z \in \mathbf{R}^d\colon\ \langle z-y, \boldsymbol{l}_x\rangle \geqq \alpha |z-y|\}.$$

Remark 1.3. Under Condition (A), there exists a unique $\bar{x} \in \bar{D}$ such that $|x-\bar{x}| = \operatorname{dist}(x, \bar{D})$ for any $x \in \mathbf{R}^d$ with $\operatorname{dist}(x, \bar{D}) < r_0$, and $(\bar{x}-x)/|\bar{x}-x| \in \mathcal{N}_{\bar{x}}$ if $x \notin \bar{D}$. The notation $\bar{x}$ is used in this sense throughout the paper (there will be no confusion with the closure $\bar{D}$).

The deterministic problem is stated as follows. Denote by $\mathbf{W}(\mathbf{R}^d)$ (resp. $\mathbf{W}(\bar{D})$) the space of continuous paths in $\mathbf{R}^d$ (resp. $\bar{D}$) and consider an equation

$$\xi(t) = w(t) + \phi(t), \tag{1.1}$$

where $w \in \mathbf{W}(\mathbf{R}^d)$ is given and satisfies $w(0) \in \bar{D}$; a solution of (1.1) is a pair of ξ and ϕ which should be found under the following two conditions (we often call ξ a solution of (1.1)).

(1.2) $\xi \in \mathbf{W}(\bar{D})$.

(1.3) ϕ is an $\mathbf{R}^d$-valued continuous function with bounded variation on each finite interval satisfying $\phi(0) = 0$ and

$$\phi(t) = \int_0^t \mathbf{n}(s)\, d|\phi|_s,$$

$$|\phi|_t = \int_0^t \mathbb{1}_{\partial D}(\xi(s))\, d|\phi|_s,$$

where

$$\mathbf{n}(s) \in \mathcal{N}_{\xi(s)} \quad \text{if } \xi(s) \in \partial D,$$

$$|\phi|_t = \text{the total variation of } \phi \text{ on } [0,t] = \sup \sum_{k=1}^n |\phi(t_k) - \phi(t_{k-1})|,$$

the supremum being taken over all partitions $0 = t_0 < t_1 < \cdots < t_n = t$.

Remark 1.4. For $w \in \mathbf{D}(\mathbf{R}_+ \to \mathbf{R}^d)$ with $w(0) \in \bar{D}$, the problem (1.1) can be also posed if we replace $\mathbf{W}(\cdot)$ in the additional conditions after (1.1) by $\mathbf{D}(\mathbf{R}_+ \to \cdot)$, where $\mathbf{D}(\mathbf{R}_+ \to \mathbf{R}^d)$ (resp. $\mathbf{D}(\mathbf{R}_+ \to \bar{D})$) is the space of right continuous paths in $\mathbf{R}^d$ (resp. $\bar{D}$) with left-hand limits. For instance, when w is a right continuous step function with small jumps, a solution of (1.1) can be constructed as follows: let $w(t) = w(t_k)$, $t_k \leqq t < t_{k+1}$, $k = 0, 1, \cdots$, where $0 = t_0 < t_1 < \cdots$ and $\lim_{n \to \infty} t_n = \infty$ and assume that $\sup_{k \geqq 1} |w(t_k) - w(t_{k-1})| < r_0$. Put

$$\xi(t) = \begin{cases} w(t), & 0 \leqq t < t_1, \\ \overline{\xi(t_{k-1}) + w(t_k) - w(t_{k-1})}, & t_k \leqq t < t_{k+1} \quad (k \geqq 1), \end{cases}$$

$$\phi(t) = \begin{cases} 0, & 0 \leqq t < t_1, \\ \phi(t_{k-1}) + \overline{\xi(t_{k-1}) + w(t_k) - w(t_{k-1})} & \\ \quad - \xi(t_{k-1}) - w(t_k) + w(t_{k-1}), & t_k \leqq t < t_{k+1} \quad (k \geqq 1). \end{cases}$$

Then if $t_n \leqq t < t_{n+1}$, we have

$$|\phi|_t = \sum_{k=1}^{n} |\overline{\{\xi(t_{k-1}) + w(t_k) - w(t_{k-1})\}} - \xi(t_{k-1}) - w(t_k) + w(t_{k-1})|.$$

Since $\xi(t_k) \in D$ implies $\phi(t_k) - \phi(t_k -) = 0$, we have

$$|\phi|_t = \int_{(0,t]} \mathbb{1}_{\partial D}(\xi(s))\, d\,|\phi|_s,$$

where $\phi(t_k -) = \lim_{t \uparrow t_k} \phi(t)$. Therefore by Remark 1.3, we have

$$\phi(t) = \int_{(0,t]} \mathbf{n}(s)\, d\,|\phi|_s,$$

and (ξ, ϕ) is a solution of (1.1) for w.

One of the purposes of this paper is to prove the following theorem.

Theorem 4.1. *If the domain D satisfies Conditions (A) and (B), then there exists a unique solution of* (1.1) *for any given $w \in \mathbf{W}(\mathbf{R}^d)$ with $w(0) \in \bar{D}$.*

The above theorem was proved by Lions and Sznitman [4] under the additional condition that D is admissible. We must also remark that Theorem 3.4 of the recent paper [2] by Frankowska contains the above theorem as a special case as can be verified by careful checks of the assumptions of Theorem 3.4 of [2]. Here we give a direct and elementary proof of the above theorem.

Next given

$$\sigma \colon \bar{D} \to \mathbf{R}^d \otimes \mathbf{R}^d, \qquad b \colon \bar{D} \to \mathbf{R}^d,$$

we consider a Skorohod SDE on a probability space $(\Omega, \mathscr{F}, P)$:

$$dX(t) = \sigma(X(t))\, dB(t) + b(X(t))\, dt + d\Phi(t), \tag{1.4}$$

where the initial value $X(0) \in \bar{D}$ is assumed to be an $\mathscr{F}_0$-measurable random variable and $B(t)$ is a d-dimensional $\mathscr{F}_t$-Brownian motion with $B(0) = 0$. Here

$\{\mathscr{F}_t\}$ is a filtration such that $\mathscr{F}_0$ contains all P-negligible sets and $\mathscr{F}_t = \bigcap_{\varepsilon > 0} \mathscr{F}_{t+\varepsilon}$. A solution $(X(t), \Phi(t))$ should be found under the following conditions (1.5)–(1.7).

(1.5) $X(t)$ is a $\bar{D}$-valued $\mathscr{F}_t$-adapted continuous process.

(1.6) $\Phi(t)$ is an $\mathbf{R}^d$-valued $\mathscr{F}_t$-adapted continuous process with bounded variation on each finite interval such that $\Phi(0) = 0$,

$$\Phi(t) = \int_0^t \mathbf{n}(s)\, d\,|\Phi|_s,$$

$$|\Phi|_t = \int_0^t \mathbb{1}_{\partial D}(X(s))\, d\,|\Phi|_s.$$

(1.7) $\mathbf{n}(s) \in \mathscr{N}_{X(s)}$ if $X(s) \in \partial D$.

Another main result of this paper is the following theorem.

Theorem 5.1. *Let D be a domain satisfying Conditions (A), (B) and assume that σ and b are bounded and Lipschitz continuous. Then there exists a unique strong solution of* (1.4).

The meaning of the existence of a unique strong solution is the same as in [3], p. 149.

The above theorem was proved by Lions and Sznitman ([4], Theorem 3.1) under the admissibility condition and the following Condition (C). See also Menaldi [6], Menaldi and Robin [7] for related problems.

Condition (*C*). There exists a function f in $\mathbf{C}^2(\mathbf{R}^d)$ which is bounded together with its first and second partial derivatives such that $\exists \gamma > 0$, $\forall x \in \partial D$, $\forall y \in \bar{D}$, $\forall \mathbf{n} \in \mathscr{N}_x$

$$\langle y - x, \mathbf{n} \rangle + \frac{1}{\gamma} \langle \nabla f(x), \mathbf{n} \rangle |y - x|^2 \geqq 0. \tag{1.8}$$

We can prove that if a domain D satisfies Conditions (A) and (B), then D satisfies the following condition.

Condition (*C'*). There exist positive numbers δ' and γ such that for each $x_0 \in \partial D$ we can find a function f in $\mathbf{C}^2(\mathbf{R}^d)$ satisfying (1.8) for any $x \in B(x_0, \delta') \cap \partial D$, $y \in B(x_0, \delta') \cap \bar{D}$ and $\mathbf{n} \in \mathscr{N}_x$.

The above condition means roughly that D satisfies (C) *locally*. Theorem 5.1 asserts that this local condition (C'), which is automatically satisfied under (A) and (B), is enough to have the existence and uniqueness of solutions of (1.4).

§ 2. Lemmas

In this section we prepare some lemmas which will be used in the following sections. The proof of the first lemma is found in [1], pp. 167–171.

Lemma 2.1 ([1]). (i) *Let* $f \in \mathbf{C}^1(\mathbf{R})$ *and* $\theta = (\theta_t)_{t \geqq 0}$ *be a function in* $\mathbf{D}(\mathbf{R}_+ \to \mathbf{R})$ *with bounded variation. Then we have*

$$df(\theta_t) = f'(\theta_{t-})\,d\theta_t + \Delta f(\theta_t) - f'(\theta_{t-})\,\Delta\theta_t, \tag{2.1}$$

where $\Delta f(\theta_t) = f(\theta_t) - f(\theta_{t-})$, $\Delta\theta_t = \theta_t - \theta_{t-}$, $\theta_{t-} = \lim_{s \uparrow t} \theta_s$.

(ii) *Let* g, h *be functions in* $\mathbf{D}(\mathbf{R}_+ \to \mathbf{R}^d)$ *with bounded variations. Then we have*

$$\begin{aligned} \langle g(t), h(t)\rangle - \langle g(0), h(0)\rangle = & \int_{(0,t]} \langle g(s), dh(s)\rangle \\ & + \int_{(0,t]} \langle h(s), dg(s)\rangle - \sum_{s \leqq t} \langle \Delta g(s), \Delta h(s)\rangle. \end{aligned} \tag{2.2}$$

Lemma 2.2.[1] *Let* $\psi \in \mathbf{D}(\mathbf{R}_+ \to \mathbf{R})$ *be a non-decreasing function with* $\psi(0) = 0$ *and* f *be a non-negative Borel measurable function on* $\mathbf{R}_+$. *If*

$$f(t) \leqq K_1 + K_2 \int_{(0,t)} f(s)\,d\psi(s) < \infty, \qquad t \geqq 0 \tag{2.3}$$

holds with some non-negative constants K_1 *and* K_2, *then we have*

$$f(t) \leqq K_1 \exp\{K_2 \psi(t)\}, \qquad t \geqq 0. \tag{2.4}$$

Proof. It is enough to show (2.4) in the case $K_1 > 0$. By (2.3) we have

$$\frac{K_2 f(t)}{K_1 + K_2 \int_{(0,t)} f(s)\,d\psi(s)} \leqq K_2, \qquad t \geqq 0.$$

Integrating both sides with respect to $d\psi$ over $(0,t]$, we have

$$\int_{(0,t]} \frac{K_2 f(u)}{K_1 + K_2 \int_{(0,u)} f(s)\,d\psi(s)}\,d\psi(u) \leqq K_2\,\psi(t), \qquad t \geqq 0. \tag{2.5}$$

If we set

$$F(x) = \log(x/K_1), \qquad x > 0,$$
$$\theta_t = K_1 + K_2 \int_{(0,t]} f(s)\,d\psi(s),$$

then an application of (2.1) yields

$$\begin{aligned} dF(\theta_u) &= F'(\theta_{u-})\,d\theta_u + \Delta F(\theta_u) - F'(\theta_{u-})\,\Delta\theta_u \\ &= \frac{K_2 f(u)\,d\psi(u)}{K_1 + K_2 \int_{(0,u)} f(s)\,d\psi(s)} + \log\left\{1 + \frac{K_2 f(u)\,\Delta\psi(u)}{K_1 + K_2 \int_{(0,u)} f(s)\,d\psi(s)}\right\} \\ &\quad - \frac{K_2 f(u)\,\Delta\psi(u)}{K_1 + K_2 \int_{(0,u)} f(s)\,d\psi(s)} \leqq \frac{K_2 f(u)\,d\psi(u)}{K_1 + K_2 \int_{(0,u)} f(s)\,d\psi(s)}, \end{aligned}$$

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[1] When ψ is continuous, this is well-known as Gronwall's lemma; notice that the interval of the integration (2.3) should be *open* in case ψ has jumps

because $\log(1+x)-x \leqq 0$, $x \geqq 0$. Therefore we have

$$\log(\theta_t/K_1)=F(\theta_t) \leqq \int_{(0,t]} \frac{K_2 f(u)}{K_1+K_2 \int_{(0,u)} f(s)\,d\psi(s)}\,d\psi(u) \leqq K_2\,\psi(t), \qquad t>0$$

by (2.5), that is, $\theta_t \leqq K_1 \exp\{K_2\,\psi(t)\}$. Thus by (2.3) we have

$$f(t) \leqq \theta_t \leqq K_1 \exp\{K_2\,\psi(t)\}, \qquad t \geqq 0,$$

completing the proof.

Lemma 2.3. *Suppose D satisfies Condition* (*A*). (i) *Let w, $w' \in \mathbf{D}(\mathbf{R}_+ \to \mathbf{R}^d)$ and (ξ, ϕ), (ξ', ϕ') be solutions of the Skorohod equations $\xi = w + \phi$ and $\xi' = w' + \phi'$, respectively. Then we have*

$$(2.6) \qquad |\xi(t)-\xi'(t)|^2 \leqq |w(t)-w'(t)|^2 + \frac{1}{r_0} \int_{(0,t]} |\xi(s)-\xi'(s)|^2\, d(|\phi|_s + |\phi'|_s) + 2\int_{(0,t]} \langle w(t)-w(s)-w'(t)+w'(s), d\phi(s)-d\phi'(s)\rangle.$$

(ii) *Let w, ξ, ϕ be as in* (i). *Then we have for $0 \leqq s \leqq t$,*

$$(2.7) \qquad |\xi(t)-\xi(s)|^2 \leqq |w(t)-w(s)|^2 + \frac{1}{r_0} \int_{(s,t]} |\xi(u)-\xi(s)|^2\, d\,|\phi|_u + 2\int_{(s,t]} \langle w(t)-w(u), d\phi(u)\rangle.$$

Remark 2.1. In the case of a convex domain D the inequalities (2.6) and (2.7) are reduced, respectively, to the inequalities in (i) and (ii) of Lemma 2.2 of [10].

Proof of Lemma 2.3. The proof is similar to that of Lemma 2.2 of [10]. (i) By (2.2) we have

$$\begin{aligned} |\xi(t)-\xi'(t)|^2 &= |w(t)-w'(t)|^2 + |\phi(t)-\phi'(t)|^2 + 2\langle w(t)-w'(t), \phi(t)-\phi'(t)\rangle \\ &= |w(t)-w'(t)|^2 + 2\int_{(0,t]} \langle \phi(s)-\phi'(s), d\phi(s)-d\phi'(s)\rangle \\ &\quad - \sum_{s \leqq t} |\Delta\phi(s)-\Delta\phi'(s)|^2 + 2\langle w(t)-w'(t), \phi(t)-\phi'(t)\rangle, \end{aligned}$$

$$\begin{aligned} \langle w(t)-w'(t), \phi(t)-\phi'(t)\rangle &= \int_{(0,t]} \langle w(t)-w'(t), d\phi(s)-d\phi'(s)\rangle \\ &= \int_{(0,t]} \langle w(t)-w(s)-w'(t)+w'(s), d\phi(s)-d\phi'(s)\rangle \\ &\quad + \int_{(0,t]} \langle w(s)-w'(s), d\phi(s)-d\phi'(s)\rangle. \end{aligned}$$

Therefore

$$(2.8)\quad |\xi(t)-\xi'(t)|^2 \leqq |w(t)-w'(t)|^2 + 2\int_{(0,t]} \langle \xi(s)-\xi'(s), d\phi(s)-d\phi'(s)\rangle + 2\int_{(0,t]} \langle w(t)-w(s)-w'(t)+w'(s), d\phi(s)-d\phi'(s)\rangle.$$

By Condition (A), (1.3) and Remark 1.1, we have

$$\int_{(0,t]} \langle \xi(s)-\xi'(s), d\phi(s)-d\phi'(s)\rangle \leqq \frac{1}{2r_0}\int_{(0,t]} |\xi(s)-\xi'(s)|^2\, d(|\phi|_s+|\phi'|_s)$$

and hence we obtain (2.6).

(ii) By a method similar to (i), we have

$$\begin{aligned}|\xi(t)-\xi(s)|^2 &= |w(t)-w(s)|^2 + |\phi(t)-\phi(s)|^2 + 2\langle w(t)-w(s), \phi(t)-\phi(s)\rangle \\ &= |w(t)-w(s)|^2 + |\phi(t)-\phi(s)|^2 + 2\int_{(s,t]} \langle w(t)-w(s), d\phi(u)\rangle.\end{aligned}$$

Using (2.2),

$$\begin{aligned}|\phi(t)-\phi(s)|^2 &= 2\int_{(s,t]} \langle \phi(u)-\phi(s), d\phi(u)\rangle - \sum_{s<u\leqq t} |\varDelta\phi(u)|^2 \\ &\leqq 2\int_{(s,t]} \langle \phi(u)-\phi(s), d\phi(u)\rangle,\end{aligned}$$

$$\begin{aligned}|\xi(t)-\xi(s)|^2 &\leqq |w(t)-w(s)|^2 + 2\int_{(s,t]} \langle \phi(u)-\phi(s), d\phi(u)\rangle + 2\int_{(s,t]} \langle w(t)-w(s), d\phi(u)\rangle \\ &= |w(t)-w(s)|^2 + 2\int_{(s,t]} \langle \phi(u)-\phi(s), d\phi(u)\rangle \\ &\quad + 2\int_{(s,t]} \langle w(u)-w(s), d\phi(u)\rangle + 2\int_{(s,t]} \langle w(t)-w(u), d\phi(u)\rangle \\ &= |w(t)-w(s)|^2 + 2\int_{(s,t]} \langle \xi(u)-\xi(s), d\phi(u)\rangle + 2\int_{(s,t]} \langle w(t)-w(u), d\phi(u)\rangle.\end{aligned}$$

By Condition (A), (1.3) and Remark 1.1 we have

$$\int_{(s,t]} \langle \xi(u)-\xi(s), d\phi(u)\rangle \leqq \frac{1}{2r_0}\int_{(s,t]} |\xi(u)-\xi(s)|^2\, d|\phi|_u$$

and hence we obtain (2.7). The proof of Lemma 2.3 is finished.

§ 3. Estimate of $|\phi_m|_t$

Let D be a domain in $\mathbf{R}^d$ (not necessarily bounded) satisfying Conditions (A) and (B). For $w \in \mathbf{W}(\mathbf{R}^d)$ define $w_m \in \mathbf{D}(\mathbf{R}_+ \to \mathbf{R}^d)$, $m \geqq 1$, by

$$(3.1)\qquad w_m(t) = w(k2^{-m}), \qquad k2^{-m} \leqq t < (k+1)2^{-m}, \quad k \geqq 0$$

and consider the following Skorohod equation for w_m:

$$\xi_m(t)=w_m(t)+\phi_m(t), \qquad t\geqq 0. \tag{3.2}$$

For simplicity we assume that w is uniformly continuous in $[0,\infty)$ for the time being (this assumption is not essential since we shall consider (3.2) for $0\leqq t\leqq T$, T being arbitrary but fixed). Since w_m is a step function, a solution (ξ_m,ϕ_m) of (3.2) can be constructed explicitly as follows provided that m is sufficiently large (see Remark 1.4).

$$\xi_m(t)=\begin{cases} w_m(0), & 0\leqq t<2^{-m},\\ \overline{\xi_m((k-1)2^{-m})+\Delta w_m(k2^{-m})}, & k2^{-m}\leqq t<(k+1)2^{-m},\end{cases} \tag{3.3}$$

$$\phi_m(t)=\begin{cases} 0, & 0\leqq t<2^{-m},\\ \phi_m((k-1)2^{-m})+\overline{\xi_m((k-1)2^{-m})+\Delta w_m(k2^{-m})} & \\ \quad -\xi_m((k-1)2^{-m})-\Delta w_m(k2^{-m}), & k2^{-m}\leqq t<(k+1)2^{-m}.\end{cases} \tag{3.4}$$

Here $\Delta w_m(t)=w_m(t)-w_m(t-)$, $w_m(t-)=\lim_{s\uparrow t} w_m(s)$. Notice that $|\Delta w_m(t)|<r_0$ for sufficiently large m because of the uniform continuity of w and hence

$$\overline{\xi_m((k-1)2^{-m})+\Delta w_m(k2^{-m})}$$

is uniquely defined (see Remark 1.3).

Remark 3.1. Under Condition (A), it is easy to see that for large m (3.4) implies

$$\Delta|\phi_m|_t=|\Delta\phi_m(t)|\leqq|\Delta w_m(t)|.$$

Throughout the paper we use the following notation: for $w,\phi\in\mathbf{D}(\mathbf{R}_+\to\mathbf{R}^d)$ and s,t with $0\leqq s<t$ we set

$$\Delta_{s,t}(w)=\sup_{s\leqq t_1<t_2\leqq t}|w(t_2)-w(t_1)|,$$

$$\Delta_{s,t,h}(w)=\sup\{|w(t_2)-w(t_1)|:\ s\leqq t_1<t_2\leqq t,\ |t_2-t_1|\leqq h\},$$

$$\|w\|_t=\sup_{0\leqq u\leqq t}|w(u)|,$$

$$|\phi|_t^s=|\phi|_t-|\phi|_s \text{ (the total variation of } \phi \text{ on } (s,t]).$$

The purpose of this section is to prove the following proposition which plays an essential role in the proof of Theorem 4.1 in §4.

Proposition 3.1. *Let $T>0$ be any fixed time. If the domain D satisfies Conditions (A) and (B), then for sufficiently large m we have*

$$|\phi_m|_t^s\leqq K\Delta_{s-2^{-m},t}(w), \qquad 0\leqq s<t\leqq T, \tag{3.5}$$

where K is a constant depending only on the constants r_0, β, δ in Conditions (A), (B), T, $\|w\|_T$ and the modulus of uniform continuity of w on $[0,T]$.

Before proving the proposition, we prepare some lemmas under the same assumption as in Proposition 3.1. Define

$$T_{m,0}=\inf\{t\geqq 0:\ \xi_m(t)\in\partial D\},$$
$$t_{m,n}=\inf\{t>T_{m,n-1}:\ |\xi_m(t)-\xi_m(T_{m,n-1})|\geqq\delta/2\},$$
$$T_{m,n}=\inf\{t\geqq t_{m,n}:\ \xi_m(t)\in\partial D\}.$$

Notice that $T_{m,n-1}<t_{m,n}$ but it can happen that $t_{m,n}=T_{m,n}$.

Lemma 3.1. *For sufficiently large m, we have*

$$(3.6)\quad |\phi_m|_t^s\leqq\beta\{\Delta_{s,t}(\xi_m)+\Delta_{s,t}(w_m)\},\qquad s,t\in[T_{m,n-1},t_{m,n}]\cup(t_{m,n},T_{m,n}).$$

Proof. If $T_{m,n-1}\leqq s<t\leqq t_{m,n}$, then by Condition (B) we have

$$\begin{aligned}\langle \boldsymbol{l},\xi_m(t)-\xi_m(s)\rangle&=\langle \boldsymbol{l},w_m(t)-w_m(s)\rangle+\langle \boldsymbol{l},\phi_m(t)-\phi_m(s)\rangle\\&\geqq\langle \boldsymbol{l},w_m(t)-w_m(s)\rangle+\beta^{-1}|\phi_m|_t^s,\end{aligned}$$

where $\boldsymbol{l}=\boldsymbol{l}_{\xi_m(T_{m,n-1})}$ and m is taken to be sufficiently large so that $|\xi_m(t)-\xi_m(T_{m,n-1})|<\delta$ holds for $T_{m,n-1}\leqq t\leqq t_{m,n}$. Therefore

$$|\phi_m|_t^s\leqq\beta\{|\xi_m(t)-\xi_m(s)|+\Delta_{s,t}(w_m)\},\qquad T_{m,n-1}\leqq s<t\leqq t_{m,n}.$$

Since ϕ_m is constant in the interval $(t_{m,n},T_{m,n})$, we have

$$|\phi_m|_t^s\leqq\beta\{\Delta_{s,t}(\xi_m)+\Delta_{s,t}(w_m)\},\qquad s,t\in[T_{m,n-1},t_{m,n}]\cup(t_{m,n},T_{m,n}).$$

Lemma 3.2. *Let $T>0$ be any fixed time. Then for sufficiently large m we have*

$$(3.7)\quad \Delta_{s,t}(\xi_m)\leqq\sqrt{2}\{(1+\varepsilon^{-1})\Delta_{s,t}(w_m)+\varepsilon|\phi_m|_t^s\}\cdot\exp(\gamma|\phi_m|_t^s),\qquad 0\leqq s<t\leqq T,$$

where $\gamma=(r_0)^{-1}$ and ε is an arbitrary positive number.

Proof. By (2.7) we have

$$\begin{aligned}|\xi_m(t)-\xi_m(s)|^2&\leqq|w_m(t)-w_m(s)|^2+\gamma\int_{(s,t]}|\xi_m(u)-\xi_m(s)|^2\,d|\phi_m|_u\\&\quad+2\int_{(s,t]}\langle w_m(t)-w_m(u),d\phi_m(u)\rangle\\&\leqq\Delta_{s,t}^2(w_m)+2\Delta_{s,t}(w_m)|\phi_m|_t^s+\gamma\int_{(s,t]}|\xi_m(u)-\xi_m(s)|^2\,d|\phi_m|_u\\&=\Delta_{s,t}^2(w_m)+2\Delta_{s,t}(w_m)|\phi_m|_t^s+\gamma\int_{(s,t)}|\xi_m(u)-\xi_m(s)|^2\,d|\phi_m|_u\\&\quad+\gamma|\xi_m(t)-\xi_m(s)|^2\Delta|\phi_m|_t.\end{aligned}$$

Since $\Delta|\phi_m|_t\leqq|\Delta w_m(t)|<r_0/2=(2\gamma)^{-1}$, $0\leqq t\leqq T$, for sufficiently large m by Remark 3.1,

$$\begin{aligned}|\xi_m(t)-\xi_m(s)|^2&\leqq 2\Delta_{s,t}^2(w_m)+4\Delta_{s,t}(w_m)|\phi_m|_t^s\\&\quad+2\gamma\int_{(s,t)}|\xi_m(u)-\xi_m(s)|^2\,d|\phi_m|_u,\end{aligned}$$

and hence, using Lemma 2.2, we have

$$(3.8)\qquad |\xi_m(t)-\xi_m(s)|^2 \leqq 2\{\Delta^2_{s,t}(w_m)+2\Delta_{s,t}(w_m)\,|\phi_m|_t^s\}\cdot\exp(2\gamma\,|\phi_m|_t^s).$$

Therefore for any $\varepsilon>0$,

$$|\xi_m(t)-\xi_m(s)|^2 \leqq 2\{(1+\varepsilon^{-2})\,\Delta^2_{s,t}(w_m)+\varepsilon^2(|\phi_m|_t^s)^2\}\cdot\exp(2\gamma\,|\phi_m|_t^s),$$

that is,

$$|\xi_m(t)-\xi_m(s)| \leqq \sqrt{2}\{(1+\varepsilon^{-1})\,\Delta_{s,t}(w_m)+\varepsilon\,|\phi_m|_t^s\}\cdot\exp(\gamma\,|\phi_m|_t^s),\qquad 0\leqq s<t\leqq T.$$

The proof is finished.

Lemma 3.3. *Let $T>0$ be any fixed time. Then, for sufficiently large m we have*

$$(3.9)\qquad \Delta_{s,t}(\xi_m)\leqq K'\,\Delta_{s,t}(w_m),\qquad s,t\in[T_{m,n-1},t_{m,n}]\cup(t_{m,n},T_{m,n}),$$

provided that $T_{m,n}\leqq T$, where K' is a constant depending only on the constants r_0,β,δ in Conditions (A), (B), T and $\|w\|_T$.

Proof. (3.7) combined with (3.6) implies

$$\Delta_{s,t}(\xi_m)\leqq\sqrt{2}\{(1+\varepsilon^{-1}+\beta\varepsilon)\,\Delta_{s,t}(w_m)+\beta\varepsilon\,\Delta_{s,t}(\xi_m)\}$$
$$\cdot\exp[\beta\gamma\{\Delta_{s,t}(\xi_m)+\Delta_{s,t}(w_m)\}],\qquad s,t\in[T_{m,n-1},t_{m,n}].$$

If we put $\eta=\Delta_{s,t}(\xi_m)$, then

$$(3.10)\qquad \eta\leqq\sqrt{2}\{(1+\varepsilon^{-1}+\beta\varepsilon)\,\Delta_{s,t}(w_m)+\beta\varepsilon\eta\}\cdot\exp[\beta\gamma\{\eta+\Delta_{s,t}(w_m)\}],$$
$$s,t\in[T_{m,n-1},t_{m,n}].$$

Since $0\leqq\eta<2\delta$ for $s,t\in[T_{m,n-1},t_{m,n}]$, we have

$$\eta\leqq\sqrt{2}\{(1+\varepsilon^{-1}+\beta\varepsilon)\,\Delta_{s,t}(w_m)+\beta\varepsilon\eta\}\cdot\exp\{2\beta\gamma(\delta+\|w\|_T)\}.$$

If we take ε as $0<\varepsilon<(\sqrt{2}\beta)^{-1}\exp\{-2\beta\gamma(\delta+\|w\|_T)\}$,

$$\eta\leqq\frac{\sqrt{2}(1+\varepsilon^{-1}+\beta\varepsilon)\,\Delta_{s,t}(w_m)\cdot\exp\{2\beta\gamma(\delta+\|w\|_T)\}}{1-\sqrt{2}\beta\varepsilon\cdot\exp\{2\beta\gamma(\delta+\|w\|_T)\}},$$

and hence

$$(3.11)\qquad \Delta_{s,t}(\xi_m)\leqq K_\varepsilon\,\Delta_{s,t}(w_m),\qquad s,t\in[T_{m,n-1},t_{m,n}],$$

where

$$K_\varepsilon=\frac{\sqrt{2}(1+\varepsilon^{-1}+\beta\varepsilon)\cdot\exp\{2\beta\gamma(\delta+\|w\|_T)\}}{1-\sqrt{2}\beta\varepsilon\cdot\exp\{2\beta\gamma(\delta+\|w\|_T)\}}.$$

If $t_{m,n}<T_{m,n}$, by the definitions of $t_{m,n}$, $T_{m,n}$, we have

$$\Delta_{s,t}(\xi_m)=\Delta_{s,t}(w_m),\qquad s,t\in[t_{m,n},T_{m,n}).$$

Therefore,

$$\begin{aligned}\Delta_{s,t}(\xi_m) &\leqq \Delta_{s,t_{m,n}}(\xi_m)+\Delta_{t_{m,n},t}(\xi_m)\\ &\leqq K_\varepsilon \Delta_{s,t_{m,n}}(w_m)+\Delta_{t_{m,n},t}(w_m)\\ &\leqq (K_\varepsilon+1)\Delta_{s,t}(w_m), \qquad T_{m,n-1}\leqq s\leqq t_{m,n}<t<T_{m,n},\end{aligned}$$

and hence

$$\Delta_{s,t}(\xi_m)\leqq K'\Delta_{s,t}(w_m), \qquad s,t\in[T_{m,n-1},t_{m,n}]\cup(t_{m,n},T_{m,n}),$$

where

$$K'=\inf[K_\varepsilon+1:\ 0<\varepsilon<(\sqrt{2}\beta)^{-1}\exp\{-2\beta\gamma(\delta+\|w\|_T)\}].$$

The proof of Lemma 3.3 is finished.

Now we are going to prove Proposition 3.1. (3.9) combined with (3.6) implies

$$|\phi_m|_t^s\leqq\beta(K'+1)\Delta_{s,t}(w_m), \qquad s,t\in[T_{m,n-1},t_{m,n}]\cup(t_{m,n},T_{m,n}),$$

and hence

$$|\phi_m|_t^s\leqq K''\Delta_{s,t}(w_m), \qquad s,t\in[T_{m,n-1},T_{m,n}], \tag{3.12}$$

where $K''=\beta(K'+1)+1$. By (3.9) we have

$$\delta/2\leqq|\xi_m(t_{m,n})-\xi_m(T_{m,n-1})|\leqq K'\Delta_{T_{m,n-1},t_{m,n}}(w_m), \tag{3.13}$$

and hence

$$\Delta\leqq\Delta_{T_{m,n-1},t_{m,n}}(w_m), \tag{3.14}$$

where $\Delta=\delta/(2K')$. Since w is continuous, there exist an integer $m_0\geqq 1$ and $h>0$ such that for any $m\geqq m_0$,

$$\Delta_{0,T,h}(w_m)<\Delta.$$

Note that $T_{m,n}\leqq T$ implies $T_{m,n}-T_{m,n-1}\geqq h$ for $m\geqq m_0$. In fact, if $T_{m,n}-T_{m,n-1}<h$, we have

$$\Delta_{T_{m,n-1},t_{m,n}}(w_m)\leqq\Delta_{T_{m,n-1},T_{m,n}}(w_m)\leqq\Delta_{0,T,h}(w_m)<\Delta,$$

which contradicts (3.14). Therefore if $m\geqq m_0$, $T_{m,n}>T$ for $n>T/h$. By (3.12),

$$|\phi_m|_t^s\leqq(T/h+1)K''\Delta_{s,t}(w_m)\leqq K\Delta_{s-2^{-m},t}(w), \qquad 0\leqq s<t\leqq T,$$

where $K=(T/h+1)K''$. The proof of Proposition 3.1 is finished.

§4. Solvability of the Deterministic Problem

In this section we prove the following theorem.

Theorem 4.1. *Suppose that a domain D in $\mathbf{R}^d$ satisfies Conditions (A) and (B). Then for any $w\in\mathbf{W}(\mathbf{R}^d)$ with $w(0)\in\bar{D}$ there exists a unique solution $\xi(t,w)$ of the equation* (1.1), *and $\xi(t,w)$ is continuous in (t,w).*

Proof. The uniqueness was proved in Theorem 1.1 of [4], under Condition (A) only. In fact, the uniqueness follows immediately from (2.6).

We prove the existence. Let $T>0$ be any constant. In what follows m is assumed to be so large that (3.5) holds. First we notice that there exists a constant C, depending only on the constants r_0, β, δ in Conditions (A), (B), T, $\|w\|_T$ and the modulus of uniform continuity of w on $[0,T]$, such that

$$|\phi_m|_t \leqq C, \quad 0 \leqq t \leqq T;$$

in fact, it is enough to take $C = K\Delta_{0,T}(w)$ where K is the constant in (3.5). For $m<n$ and $0 \leqq t \leqq T$ we see that

$$|w_m(t)-w_n(t)| \leqq \Delta_{0,t,2^{-m}}(w),$$

$$\int_{(0,t]} \langle w_m(t)-w_m(s)-w_n(t)+w_n(s), d\phi_m(s)-d\phi_n(s)\rangle \leqq 4\,C\Delta_{0,t,2^{-m}}(w).$$

Then by (2.6), setting $\gamma = (r_0)^{-1}$, we have

$$\begin{aligned}|\xi_m(t)-\xi_n(t)|^2 &\leqq |w_m(t)-w_n(t)|^2 + \gamma \int_{(0,t]} |\xi_m(s)-\xi_n(s)|^2\, d(|\phi_m|_s+|\phi_n|_s)\\ &\quad +2\int_{(0,t]} \langle w_m(t)-w_m(s)-w_n(t)+w_n(s), d\phi_m(s)-d\phi_n(s)\rangle\\ &\leqq \Delta^2_{0,t,2^{-m}}(w) + \gamma \int_{(0,t)} |\xi_m(s)-\xi_n(s)|^2\, d(|\phi_m|_s+|\phi_n|_s)\\ &\quad +\gamma|\xi_m(t)-\xi_n(t)|^2 \Delta(|\phi_m|_t+|\phi_n|_t) + 8\,C\Delta_{0,t,2^{-m}}(w).\end{aligned}$$

Since Remark 3.1 implies

$$\Delta(|\phi_m|_t+|\phi_n|_t) \leqq r_0/2 = (2\gamma)^{-1}, \quad 0 \leqq t \leqq T$$

for sufficiently large m and n, we have

$$\begin{aligned}|\xi_m(t)-\xi_n(t)|^2 &\leqq 2\Delta^2_{0,t,2^{-m}}(w) + 16\,C\Delta_{0,t,2^{-m}}(w)\\ &\quad +2\gamma \int_{(0,t)} |\xi_m(s)-\xi_n(s)|^2\, d(|\phi_m|_s+|\phi_n|_s).\end{aligned}$$

Hence by Lemma 2.2, we have

$$\begin{aligned}|\xi_m(t)-\xi_n(t)|^2 &\leqq 2\{\Delta_{0,t,2^{-m}}(w)+8\,C\}\,\Delta_{0,t,2^{-m}}(w)\cdot \exp\{2\gamma(|\phi_m|_t+|\phi_n|_t)\}\\ &\leqq 2\{\Delta_{0,t,2^{-m}}(w)+8\,C\}\,\Delta_{0,t,2^{-m}}(w)\cdot \exp(4\,C\gamma),\end{aligned}$$

which tends to 0 as $m\to\infty$. Therefore ξ_m and ϕ_m converge uniformly on $[0,T]$ as $m\to\infty$. We denote the limits by ξ and ϕ respectively. Then it follows immediately from Proposition 3.1 that ξ and ϕ are continuous in t. Therefore all we have to show is the following:

(4.1) $$|\phi|_t = \int_{(0,t]} \mathbb{1}_{\partial D}(\xi(s))\, d|\phi|_s,$$

(4.2) $$\phi(t) = \int_{(0,t]} \mathbf{n}(s)\, d|\phi|_s, \quad \mathbf{n}(s)\in\mathcal{N}_{\xi(s)}, \quad \xi(s)\in\partial D.$$

By Remark 1.1 and (1.3),

$$\int_{(0,t]} \langle \eta(s)-\xi_n(s), d\phi_n(s)\rangle + \frac{1}{2r_0}\int_{(0,t]} |\eta(s)-\xi_n(s)|^2\, d|\phi_n|_s \geqq 0, \tag{4.3}$$

for any $\eta \in \mathbf{W}(\bar{D})$, $0 \leqq t \leqq T$. Put

$$I_1 = \int_{(0,t]} \langle \eta(s)-\xi_n(s), d\phi_n(s)\rangle,$$
$$I_2 = \int_{(0,t]} |\eta(s)-\xi_n(s)|^2\, d|\phi_n|_s.$$

We now prove the following 1° and 2°.

1°. $\lim_{n\to\infty} I_1 = \int_{(0,t]} \langle \eta(s)-\xi(s), d\phi(s)\rangle$.

Proof. We write

$$\begin{aligned} I_1 &= \int_{(0,t]} \langle \zeta(s), d\phi_n(s)\rangle + \int_{(0,t]} \langle \xi(s)-\xi_n(s), d\phi_n(s)\rangle \\ &= I_1' + I_1'', \end{aligned}$$

where $\zeta(s)=\eta(s)-\xi(s)$. Then I_1'' is dominated in modulus by $C\|\xi-\xi_n\|_t$, which tends to 0 as $n\to\infty$. To handle I_1' let $0=t_0<t_1<\cdots<t_m=t$ be an equipartition $(t_k-t_{k-1}=t/m)$ of $[0,t]$ and set $\zeta^m(s)=\zeta(t_k)$ for $t_k<s\leqq t_{k+1}$, $k=0,1,\cdots,m-1$. For any fixed $\varepsilon>0$ we take m so that $\|\zeta-\zeta^m\|_t\leqq\varepsilon$ holds. Then

$$\Big|\int_{(0,t]} \langle \zeta(s)-\zeta^m(s), d\phi_n(s)\rangle\Big| \leqq C\|\zeta-\zeta^m\|_t \leqq C\varepsilon,$$

and similarly

$$\Big|\int_{(0,t]} \langle \zeta(s)-\zeta^m(s), d\phi(s)\rangle\Big| \leqq C\varepsilon$$

because $|\phi|_t^s \leqq \varliminf_{n\to\infty} |\phi_n|_t^s$. Therefore we have

$$\begin{aligned} &\Big|I_1' - \int_{(0,t]} \langle \zeta(s), d\phi(s)\rangle\Big| \\ &\leqq \Big|\int_{(0,t]} \langle \zeta(s)-\zeta^m(s), d\phi_n(s)\rangle\Big| + \Big|\int_{(0,t]} \langle \zeta(s)-\zeta^m(s), d\phi(s)\rangle\Big| \\ &\quad + \left|\sum_{k=0}^{m-1} \zeta(t_k)\{\phi_n(t_{k+1})-\phi_n(t_k)\} - \sum_{k=0}^{m-1} \zeta(t_k)\{\phi(t_{k+1})-\phi(t_k)\}\right| \\ &\leqq 2C\varepsilon + o(1), \qquad n\to\infty, \end{aligned}$$

proving that $\lim_{n\to\infty} I_1 = \lim_{n\to\infty} I_1' = \int_{(0,t]} \langle \zeta(s), d\phi(s)\rangle$.

Let da_u be any weak limit on $[0,T]$ of $d|\phi_n|_u$ as $n\to\infty$ via some subsequence $n_1<n_2<\cdots$.

2°. $\lim I_2 = \int_{(0,t]} |\eta(s)-\xi(s)|^2\, da_s$ if t is a point of continuity for a_u, where the limit is taken as $n\to\infty$ via $n_1<n_2<\cdots$.

Proof. We have

$$
\begin{aligned}
&\Big|\int_{(0,t]} |\eta(s)-\xi_n(s)|^2\, d\,|\phi_n|_s - \int_{(0,t]} |\eta(s)-\xi(s)|^2\, da_s\Big| \\
&\quad \le \Big|\int_{(0,t]} |\eta(s)-\xi_n(s)|^2\, d\,|\phi_n|_s - \int_{(0,t]} |\eta(s)-\xi(s)|^2\, d\,|\phi_n|_s\Big| \\
&\qquad + \Big|\int_{(0,t]} |\eta(s)-\xi(s)|^2\, d\,|\phi_n|_s - \int_{(0,t]} |\eta(s)-\xi(s)|^2\, da_s\Big|.
\end{aligned}
$$

If t is a point of continuity for a_u, the second term tends to 0 as $n\to\infty$ via $n_1<n_2<\cdots$. If we put $\zeta_n(s)=\eta(s)-\xi_n(s)$, then the first term is dominated by

$$\sup_{0\le s\le t} |\{|\zeta_n(s)|^2-|\zeta(s)|^2\}|\cdot C,$$

which tends to 0 as $n\to\infty$. Thus the proof of 2° is finished.

Now we are going to complete the proof of the theorem by making use of a method similar to that used in the proof of Theorem 1.1 of Lions and Sznitman [4]. From $|\phi|_t^s \le \varliminf_{n\to\infty} |\phi_n|_t^s$ and the definition of the measure da_s, we have

$$d\,|\phi|_s \le da_s, \tag{4.4}$$

and hence there is an $\mathbf{R}^d$-valued bounded measurable function h_s such that

$$d\phi(s)=h_s\, da_s. \tag{4.5}$$

Making n tend to ∞ in (4.3) and then using 1°, 2° and (4.5), we have

$$\int_{(0,t]} \langle \eta(s)-\xi(s), h_s\rangle\, da_s + \frac{1}{2r_0}\int_{(0,t]} |\eta(s)-\xi(s)|^2\, da_s \ge 0, \qquad 0\le t\le T, \tag{4.6}$$

or equivalently

$$\langle \eta(s)-\xi(s), h_s\rangle + \frac{1}{2r_0}|\eta(s)-\xi(s)|^2 \ge 0, \quad da_s\text{-a.e.} \tag{4.7}$$

Define a continuous function χ $(0\le\chi\le 1)$ on $\mathbf{R}^d$ by

$$\chi=\begin{cases} 1 & \text{on a compact set included in } D,\\ 0 & \text{on } \mathbf{R}^d\setminus D.\end{cases}$$

Since

$$
\begin{aligned}
&\Big|\int_{(0,t]} \chi(\xi(s))\, d\,|\phi_n|_s - \int_{(0,t]} \chi(\xi_n(s))\, d\,|\phi_n|_s\Big| \\
&\quad \le \sup_{0\le s\le t} |\chi(\xi(s))-\chi(\xi_n(s))|\cdot C \to 0, \qquad n\to\infty,
\end{aligned}
$$

we have

$$
\begin{aligned}
0&=\lim \int_{(0,t]} \chi(\xi_n(s))\, d\,|\phi_n|_s = \lim \int_{(0,t]} \chi(\xi(s))\, d\,|\phi_n|_s \\
&= \int_{(0,t]} \chi(\xi(s))\, da_s \ge 0.
\end{aligned}
$$

Hence $\int_{(0,t]} \chi(\xi(s))\,da_s=0$. Letting χ increase to the indicator function $\mathbb{1}_D$, we have

$$\int_{(0,t]} \mathbb{1}_D(\xi(s))\,da_s=0, \tag{4.8a}$$

or equivalently

$$\xi(s)\in\partial D, \qquad da_s\text{-a.e.} \tag{4.8b}$$

Thus $\xi(s)\in\partial D$ in (4.7). Therefore by Condition (A) and Remark 1.1 there exist $\theta(s)\geqq 0$ and $\mathbf{n}(s)\in\mathcal{N}_{\xi(s)}$ (if $\xi(s)\in\partial D$) such that $h_s=\theta(s)\mathbf{n}(s)$, da_s-a.e. Then (4.5) implies that $d|\phi|_s=|h_s|\,da_s=\theta(s)\,da_s$ and hence

$$d\phi(s)=h_s\,da_s=\theta(s)\mathbf{n}(s)\,da_s=\mathbf{n}(s)\,d|\phi|_s,$$

which is nothing but (4.2). Now (4.1) is also clear. (4.1) and (4.2) mean that ξ is a solution of (1.1). The continuity of ξ in (t,w) follows from Remark 4.1 below. The proof of Theorem 4.1 is finished.

Theorem 4.2. *If (ξ,ϕ) is the solution of (1.1) for $w\in\mathbf{W}(\mathbf{R}^d)$ with $w(0)\in\bar{D}$, for any finite $T>0$, we have*

$$|\phi|_t^s\leqq K\Delta_{s,t}(w), \qquad 0\leqq s<t\leqq T,$$

where K is a constant depending only on the constants r_0, β, δ in Conditions (A), (B), T, $\|w\|_T$ and the modulus of uniform continuity of w on $[0,T]$.

The proof is immediate from Proposition 3.1.

Remark 4.1. Let (ξ,ϕ) and (ξ',ϕ') be the solutions of the Skorohod equations $\xi=w+\phi$ and $\xi'=w'+\phi'$, respectively.

(i) By (2.6) and Gronwall's inequality, we immediately have

$$|\xi(t)-\xi'(t)|^2\leqq\{|w(t)-w'(t)|^2+4(|\phi|_t+|\phi'|_t)\,\|w-w'\|_t\}$$
$$\cdot\exp\{(|\phi|_t+|\phi'|_t)/r_0\}, \qquad 0\leqq t\leqq T.$$

(ii) Similarly, (2.7) and an application of Gronwall's inequality yield

$$|\xi(t)-\xi(s)|^2\leqq\{|w(t)-w(s)|^2+2|\phi|_t^s\Delta_{s,t}(w)\}\cdot\exp(|\phi|_t^s/r_0), \qquad 0\leqq s<t\leqq T.$$

(See also Lemma 1.1 of [4] for similar inequalities.)

Finally, we assume that the domain D satisfies the following Condition (D) and study the Lipschitz continuity of the solution ξ of (1.1) in w with respect to the total variation in the w-space.

Condition (D). Condition (A) is satisfied and there exist constants $C_1\geqq 0$, $C_2\in(0,r_0)$ such that

$$|\bar{x}-\bar{y}|\leqq(1+C_1\varepsilon)|x-y|$$

holds for any $x,y\in R^d$ with $|x-\bar{x}|\leqq C_2$, $|y-\bar{y}|\leqq C_2$, where $\varepsilon=\max\{|x-\bar{x}|,|y-\bar{y}|\}$.

Let $w,w'\in\mathbf{W}(\mathbf{R}^d)$ with $w(0),w'(0)\in\bar{D}$ and (ξ,ϕ), (ξ',ϕ') be the unique solutions of the Skorohod equations $\xi=w+\phi$ and $\xi'=w'+\phi'$, respectively. Then we have the following proposition.

Proposition 4.1. *Under Conditions (B) and (D), we have for each $T>0$*

(4.9) $$|\xi(t)-\xi'(t)|\leqq K\{\|w-w'\|_t+|w(0)-w'(0)|\}, \qquad 0\leqq t\leqq T,$$

where $K>0$ is a constant depending only on the constants C_1, C_2, r_0, β, δ in Conditions (B) and (D), $T, \{\Delta_{0,T,h}(w), 0<h\leqq T\}$ and $\{\Delta_{0,T,h}(w'), 0<h\leqq T\}$.

Remark 4.2. (i) We can take $K=\exp(2CC_1)$ where C and C_1 are the constants in Proposition 3.1 and Condition (D), respectively.

(ii) Any convex domain satisfies Condition (D) with $C_1=0$ and hence (4.9) holds with $K=1$.

(iii) In most applications w is supposed to be a Brownian path in which case $\|w\|_t=\infty$, but (4.9) will be of some use if we consider the Skorohod equation $\xi(t)=w(t)+\int_0^t b(\xi(s))\,ds+\phi(t)$ with Lipschitz continuous b.

Proof of Proposition 4.1. Define w_m, ξ_m and ϕ_m by (3.1), (3.3) and (3.4) respectively and define also w'_m, ξ'_m and ϕ'_m similarly. Then by Condition (D),

$$\begin{aligned}
&|\xi_m(k2^{-m})-\xi'_m(k2^{-m})|\\
&=|\overline{\{\xi_m((k-1)2^{-m})+w_m(k2^{-m})-w_m((k-1)2^{-m})\}}\\
&\qquad -\overline{\{\xi'_m((k-1)2^{-m})+w'_m(k2^{-m})-w'_m((k-1)2^{-m})\}}|\\
&\leqq [1+C_1\{\|\phi_m\|_{k2^{-m}}^{(k-1)2^{-m}}+\|\phi'_m\|_{k2^{-m}}^{(k-1)2^{-m}}\}]\\
&\quad\cdot\{|\xi_m((k-1)2^{-m})-\xi'_m((k-1)2^{-m})|\\
&\quad+|w_m(k2^{-m})-w_m((k-1)2^{-m})-w'_m(k2^{-m})+w'_m((k-1)2^{-m})|\}.
\end{aligned}$$

If we put

$$\begin{aligned}
x_{m,k}&=|\xi_m(k2^{-m})-\xi'_m(k2^{-m})|,\\
a_{m,k}&=\|\phi_m\|_{k2^{-m}}^{(k-1)2^{-m}}+\|\phi'_m\|_{k2^{-m}}^{(k-1)2^{-m}},\\
b_{m,k}&=|w_m(k2^{-m})-w_m((k-1)2^{-m})-w'_m(k2^{-m})+w'_m((k-1)2^{-m})|,
\end{aligned}$$

then

$$\begin{aligned}
x_{m,k}&\leqq(1+C_1a_{m,k})(x_{m,k-1}+b_{m,k})\\
&\leqq\exp(C_1a_{m,k})(x_{m,k-1}+b_{m,k})\\
&\leqq\exp(C_1a_{m,k})\{\exp(C_1a_{m,k-1})(x_{m,k-2}+b_{m,k-1})+b_{m,k}\}\\
&\leqq\exp\{C_1(a_{m,k}+a_{m,k-1})\}(x_{m,k-2}+b_{m,k}+b_{m,k-1})\\
&\leqq\cdots\\
&\leqq\exp\left(C_1\sum_{i=1}^k a_{m,i}\right)\left(x_{m,0}+\sum_{i=1}^k b_{m,i}\right).
\end{aligned}$$

Since ξ_m, ϕ_m and w_m are constant in $[k2^{-m},(k+1)2^{-m})$, we have

$$|\xi_m(t)-\xi'_m(t)|\leqq\exp\{C_1(\|\phi_m\|_t+\|\phi'_m\|_t)\}\cdot\{\|w_m-w'_m\|_t+|w(0)-w'(0)|\}.$$

By Proposition 3.1, we have for any $T>0$,

$$|\phi_m|_t+|\phi'_m|_t \leqq 2C, \qquad 0\leqq t\leqq T,$$

and hence, for $0\leqq t\leqq T$,

$$|\xi_m(t)-\xi'_m(t)|\leqq K\{|w_m-w'_m|_t+|w(0)-w'(0)|\},$$

where $K=\exp(2CC_1)$. Letting $m\to\infty$, we have (4.9). The proof is finished.

Before closing this section we state an interesting example, due to Saisho and Tanaka [8], of a domain satisfying Conditions (A) and (B).

Example. We write $\mathbf{x}=(x_1,x_2,\cdots,x_n)$ for a point of $\mathbf{R}^{nd}$, where each x_k denotes a point of $\mathbf{R}^d$. Let D be a domain in $\mathbf{R}^{nd}$ defined by

$$D=\{(x_1,x_2,\cdots,x_n):\ |x_i-x_j|>\rho \text{ for } 1\leqq\forall i<\forall j\leqq n\},$$

where ρ is a given positive constant. Saisho and Tanaka [8] proved that D satisfies Conditions (A) and (B). Suppose we are given $w_1,w_2,\cdots,w_n\in\mathbf{W}(\mathbf{R}^d)$ and write $\mathbf{w}=(w_1,w_2,\cdots,w_n)$. We assume that $\mathbf{w}(0)\in\bar{D}$, that is, $|w_i(0)-w_j(0)|\geqq\rho$ for $1\leqq\forall i<\forall j\leqq n$. Then by Theorem 4.1 there exists a unique solution of the Skorohod equation $\boldsymbol{\xi}(t)=\mathbf{w}(t)+\boldsymbol{\phi}(t)$ for the domain D. When $w_1(t),w_2(t),\cdots,w_n(t)$ are independent d-dimensional Brownian motions, $\boldsymbol{\xi}(t)=(\xi_1(t),\xi_2(t),\cdots,\xi_n(t))$ descrives the motion of n mutually reflecting Brownian balls of diameter ρ in the space $\mathbf{R}^d$. For details see Saisho and Tanaka [8].

§ 5. Solvability of the Skorohod SDE

Let D be a domain in $\mathbf{R}^d$ satisfying Conditions (A) and (B) and suppose that we are given coefficients

$$\sigma:\ \bar{D}\to\mathbf{R}^d\otimes\mathbf{R}^d, \qquad b:\ \bar{D}\to\mathbf{R}^d$$

satisfying

$$|\sigma(x)-\sigma(y)|\leqq L|x-y|, \qquad |b(x)-b(y)|\leqq L|x-y|,$$
$$|\sigma(x)|\leqq L, \qquad |b(x)|\leqq L$$

for any $x,y\in\bar{D}$ with some constant $L\geqq 0$. Let $(\Omega,\mathscr{F},P)$ be a complete probability space with a filtration $\{\mathscr{F}_t\}_{t\geqq 0}$ such that each $\mathscr{F}_t$ contains all P-null sets and $\mathscr{F}_t=\bigcap_{\varepsilon>0}\mathscr{F}_{t+\varepsilon}$. In this section we study a Skorohod SDE

$$dX(t)=\sigma(X(t))\,dB(t)+b(X(t))\,dt+d\Phi(t), \qquad X(0)\in\bar{D}, \tag{5.1}$$

where $B(t)$ is a d-dimensional $\mathscr{F}_t$-Brownian motion and $X(0)$ is $\mathscr{F}_0$-measurable. The main result is the following theorem.

Theorem 5.1. *There exists a unique strong solution of* (5.1).

Remark 5.1. From the proof below we see that the existence of a (not necessarily strong) solution of (5.1) holds only under the assumption that $\sigma(x)$ and $b(x)$ are bounded continuous.

The meaning of a strong solution is the same as in Definition IV-1.6 of [3].

To prove the theorem we first consider a Skorohod SDE

$$dX_n(t)=\sigma(X_n(h_n(t)))\,dB(t)+b(X_n(h_n(t)))\,dt+d\Phi_n(t), \tag{5.2}$$

with initial condition $X_n(0)=X(0)$, where

$$h_n(0)=0,$$
$$h_n(t)=(k-1)2^{-n},\quad (k-1)2^{-n}<t\leqq k2^{-n},\quad k=1,2,\cdots,\quad n\geqq 1.$$

By Theorem 4.1, we have a unique solution of (5.2); in fact, once $X_n(t)$ is obtained for $0\leqq t\leqq k2^{-n}$, $X_n(t)$ for $k2^{-n}<t\leqq(k+1)2^{-n}$ is uniquely determined as the solution of the Skorohod equation:

$$X_n(t)=X_n(k2^{-n})+\sigma(X_n(k2^{-n}))\{B(t)-B(k2^{-n})\}$$
$$+b(X_n(k2^{-n}))(t-k2^{-n})+\Phi_n(t).$$

Put

$$Y_n(t)=X(0)+\int_0^t\sigma(X_n(h_n(s)))\,dB(s)+\int_0^t b(X_n(h_n(s)))\,ds, \tag{5.3}$$

and denote by P_n the probability measure on $\mathbf{C}([0,T]\to\mathbf{R}^d\times\mathbf{R}^d)$ introduced by the process $\{(B(t),Y_n(t)),\ 0\leqq t\leqq T\}$, where T is an arbitrarily fixed time.

Lemma 5.1. *The family* $\{P_n, n\geqq 1\}$ *is tight.*

Proof. It is easy to see that for $0<\varepsilon<1/2$

$$\lim_{\lambda\to\infty}\sup_{n\geqq 1}P\left\{\sup_{0\leqq s<t\leqq T}\frac{|Y_n(t)-Y_n(s)|}{|t-s|^\varepsilon}>\lambda\right\}=0. \tag{5.4}$$

In fact, denoting by $M_n^i(t)$ the i-th component of

$$M_n(t)=\int_0^t\sigma(X_n(h_n(s)))\,dB(s),$$

we have

$$\sup_{0\leqq s<t\leqq T}\frac{|M_n(t)-M_n(s)|}{|t-s|^\varepsilon}\leqq\sum_{i=1}^d\sup_{0\leqq s<t\leqq T}\frac{|M_n^i(t)-M_n^i(s)|}{|t-s|^\varepsilon}.$$

Since M_n^i is a continuous martingale and its quadratic variation process $\langle M_n^i, M_n^i\rangle_t$ is dominated by const. t, we have

$$\sup_{0\leqq s<t\leqq T}\frac{|M_n^i(t)-M_n^i(s)|}{|t-s|^\varepsilon}\leqq_d \text{const.}\sup_{0\leqq s<t\leqq T}\frac{|W(t)-W(s)|}{|t-s|^\varepsilon},$$

where $W(t)$ is a 1-dimensional Brownian motion and "$\leqq_d$" means the stochastic domination, that is, $X\leqq_d Y$ if and only if $P(X\geqq x)\leqq P(Y\geqq x)$ for any $x\in\mathbf{R}$. Therefore the martingale part $M_n(t)$ of $Y_n(t)$ can be handled by using a well-known result of Lévy on the modulus of uniform continuity of Brownian paths. The bounded variation part can be treated directly. The tightness of $\{P_n\}$ follows from (5.4).

By Lemma 5.1 there exists a subsequence $n_1 < n_2 < \cdots$ such that P_{n_k} converges weakly as $k \to \infty$. To simplify the notation we assume that P_n itself converges weakly as $n \to \infty$. Then by Skorohod's realization theorem of almost sure convergence we can find, on a suitable probability space $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{P})$, a sequence of processes $(\tilde{B}_n(t), \tilde{Y}_n(t))$, $n \geqq 1$, satisfying the following two conditions.

(5.7) For each n the process $(\tilde{B}_n(t), \tilde{Y}_n(t))$, $0 \leqq t \leqq T$, is equivalent in law to the process $(B(t), Y_n(t))$, $0 \leqq t \leqq T$.

(5.8) $\tilde{B}_n(t)$ and $\tilde{Y}_n(t)$ converge uniformly in $t \in [0, T]$ (a.s.) as $n \to \infty$ to some processes $\tilde{B}(t)$ and $\tilde{Y}(t)$ respectively.

Let $(\tilde{X}_n(t), \tilde{\Phi}_n(t))$ and $(\tilde{X}(t), \tilde{\Phi}(t))$ be the solutions of the Skorohod equations

$$\tilde{X}_n(t) = \tilde{Y}_n(t) + \tilde{\Phi}_n(t), \tag{5.9a}$$

$$\tilde{X}(t) = \tilde{Y}(t) + \tilde{\Phi}(t), \tag{5.9b}$$

respectively. Then the continuity result in Theorem 4.1 implies that $(\tilde{X}_n(t), \tilde{\Phi}_n(t))$ converges to $(\tilde{X}(t), \tilde{\Phi}(t))$ uniformly in $t \in [0, T]$ (a.s.) as $n \to \infty$. From (5.3) and (5.7) it also follows that

$$\tilde{Y}_n(t) = \tilde{X}(0) + \int_0^t \sigma(\tilde{X}_n(h_n(s)))\, d\tilde{B}_n(s) + \int_0^t b(\tilde{X}_n(h_n(s)))\, ds. \tag{5.10}$$

Lemma 5.2. *$(\tilde{X}(t), \tilde{\Phi}(t))$ is a solution of the Skorohod SDE*

$$\tilde{X}(t) = \tilde{X}(0) + \int_0^t \sigma(\tilde{X}(s))\, d\tilde{B}(s) + \int_0^t b(\tilde{X}(s))\, ds + \tilde{\Phi}(t). \tag{5.11}$$

Proof. It is enough to prove

$$\tilde{Y}(t) = \tilde{X}(0) + \int_0^t \sigma(\tilde{X}(s))\, d\tilde{B}(s) + \int_0^t b(\tilde{X}(s))\, ds.$$

But this follows from

$$\int_0^t \sigma(\tilde{X}_n(h_n(s)))\, d\tilde{B}_n(s) \to \int_0^t \sigma(\tilde{X}(s))\, d\tilde{B}(s) \quad \text{in probability, } n \to \infty,$$

which can be easily proved by approximating stochastic integrals by Riemann-Stieltjes sums for each fixed t.

In the rest of the paper r_0, β and δ are the constants appearing in Conditions (A) and (B).

Lemma 5.3 (cf. [4], Remark 3.1). *D satisfies Condition (C') of §1 with $\gamma = 2r_0\beta^{-1}$.*

Proof. Let $\boldsymbol{l} = \boldsymbol{l}_{x_0}$ be the unit vector appearing in Condition (B). Then the assertion of the lemma holds with $f(x) = \langle \boldsymbol{l}, x - x_0 \rangle$.

Let X_t and X'_t be solutions of (5.1) with the same initial value. Suppose that the supports of the coefficients σ and b are included in $B(x_0, \delta)$ for some

$x_0 \in \partial D$. Then by Lemma 5.3 we have

$$\begin{aligned}&\langle X_s - X'_s, d\Phi_s - d\Phi'_s\rangle - \frac{1}{\gamma}|X_s - X'_s|^2 \langle l, d\Phi_s + d\Phi'_s\rangle \\ &= -\left\{\langle X'_s - X_s, d\Phi_s\rangle + \frac{1}{\gamma}|X_s - X'_s|^2 \langle l, d\Phi_s\rangle\right\} \\ &\quad -\left\{\langle X_s - X'_s, d\Phi'_s\rangle + \frac{1}{\gamma}|X_s - X'_s|^2 \langle l, d\Phi'_s\rangle\right\} \leqq 0.\end{aligned}$$

Therefore, employing a method similar to [4], pp. 524–525, we can prove

$$E\{\|X - X'\|_t^4\} \leqq \text{const.} \int_0^t E\{\|X - X'\|_s^4\}\, ds,$$

which implies $X_t = X'_t$, $t \geqq 0$. Thus we have the pathwise uniqueness of solutions of (5.1) under the restricted condition on σ and b. Combining this with Lemma 5.2, we have the following lemma by an argument similar to [3], Theorem IV-1.1.

Lemma 5.4. *Suppose that the supports of σ and b are included in $B(x_0, \delta)$ for some $x_0 \in \partial D$. Then there is a unique strong solution of* (5.1).

Lemma 5.5. *For any $x_0 \in \partial D$, the pathwise uniqueness of solutions of* (5.1) *with initial value x_0 holds for $0 \leqq t \leqq \tau$, where τ is the exit time of $B(x_0, r)$, $0 < r < \delta$. More precisely, if $X_i(t)$, $i = 1, 2$ are $\mathscr{F}_t$-adapted solutions of* (5.1) *and if $\tau_i = \inf\{t > 0\colon X_i(t) \notin B(x_0, r)\}$, $i = 1, 2$, then $\tau_1 = \tau_2$* a.s. *and $X_1(t) = X_2(t)$ for $0 \leqq t \leqq \tau_1$,* a.s.

Proof. For $0 < r < \delta$ we define Lipschitz continuous functions $\hat{\sigma}\colon \bar{D} \to \mathbf{R}^d \otimes \mathbf{R}^d$ and $\hat{b}\colon \bar{D} \to \mathbf{R}^d$ such that

$$\hat{\sigma}(x) = \begin{cases}\sigma(x) & \text{if } x \in B(x_0, r) \cap \bar{D}, \\ 0 & \text{if } x \in B(x_0, \delta)^c \cap \bar{D},\end{cases}$$

$$\hat{b}(x) = \begin{cases}b(x) & \text{if } x \in B(x_0, r) \cap \bar{D}, \\ 0 & \text{if } x \in B(x_0, \delta)^c \cap \bar{D},\end{cases}$$

and then consider the following Skorohod SDE:

$$d\hat{Y}(t) = \hat{\sigma}(\hat{Y}(t))\, dB(t) + \hat{b}(\hat{Y}(t))\, dt + d\hat{\Psi}(t). \tag{5.12}$$

Lemma 5.4 implies that (5.12) has a unique strong solution. Let $X(t)$ be a solution of (5.1) starting at x_0 and put $\tau = \inf\{t > 0\colon X(t) \notin B(x_0, r)\}$. We may assume $\tau < \infty$ a.s.; otherwise, it is enough to consider $\tau \wedge n$. Next, define $\hat{B}(u) = B(\tau + u) - B(\tau)$ and $\hat{\mathscr{F}}_u = \mathscr{F}_{\tau+u} = \{\Lambda \in \mathscr{F}\colon \Lambda \cap \{\tau + u \leqq t\} \in \mathscr{F}_t,\ \forall t \geqq 0\}$, $u \geqq 0$. Then it is easy to see that $\hat{B}$ is an $\hat{\mathscr{F}}_u$-Brownian motion and that $\hat{t} = t - \tau \wedge t$ is an $\hat{\mathscr{F}}_u$-stopping time for each fixed $t \geqq 0$. Since (5.12) has a unique strong solution,

$$\hat{Y}(t) = X(\tau) + \int_0^t \hat{\sigma}(\hat{Y}(s))\, d\hat{B}(s) + \int_0^t \hat{b}(\hat{Y}(s))\, ds + \hat{\Psi}(t) \tag{5.12'}$$

has a unique $\hat{\mathscr{F}}_u$-adapted solution. We put

$$Y(t)=\begin{cases} X(t), & t\leqq\tau,\\ \hat{Y}(t-\tau), & t>\tau.\end{cases}$$

Then, the uniqueness assertion in the lemma follows from the pathwise uniqueness for (5.12) once we prove that Y is a solution of (5.12). Set $\hat{M}(t)=\int_0^t \hat{\sigma}(\hat{Y}(s))\,d\hat{B}(s)$. Then, approximating the stochastic integral by a Riemann-Stieltjes sum and noting that $\hat{\tau}$ is an $\hat{\mathscr{F}}_u$-stopping time, we have

$$\mathbb{1}_{\{\tau<t\}}\hat{M}(\hat{\tau})=\mathbb{1}_{\{\tau<t\}}\int_0^\infty \mathbb{1}_{\{\tau<s\leqq t\}}\hat{\sigma}(Y(s))\,dB(s),$$

and hence

$$\begin{aligned} Y(t)&=\mathbb{1}_{\{t\leqq\tau\}}Y(t)+\mathbb{1}_{\{t>\tau\}}Y(t)\\ &=x_0+\int_0^t \hat{\sigma}(Y(s))\,dB(s)+\int_0^t \hat{b}(Y(s))\,ds+\Psi(t),\end{aligned}$$

where

$$\Psi(t)=\begin{cases}\Phi(t), & t\leqq\tau,\\ \hat{\Psi}(t-\tau)+\Phi(\tau), & t>\tau.\end{cases}$$

Thus Y solves (5.12). The proof of Lemma 5.5 is finished.

From Lemma 5.5 we can easily prove the following lemma.

Lemma 5.6. *The pathwise uniqueness of solutions of* (5.1) *holds.*

Using Lemma 5.2 and Lemma 5.6 we can finally prove Theorem 5.1 by a method similar to [3], Theorem IV-1.1.

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