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Probability Theory and Related Fields

Statistically Self-Similar Fractals

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Summary. Let X be a complete separable bounded metric space and μ a Borel probability measure on the space $\mathrm{Con}(X)^N$ of all N-tuples of contractions of X with the topology of pointwise convergence. Then there exists a unique μ-self-similar probability measure P_μ on the space $\mathscr{K}(X)$ of all non-empty compact subsets of X. Here a measure P on $\mathscr{K}(X)$ is called μ-self-similar if, for every Borel set $B \subset \mathscr{K}(X)$,

$$P(B) = \int P^N \left((K_0, \ldots, K_{N-1}) \,\middle|\, \bigcup_{i=0}^{N-1} S_i(K_i) \in B \right) d\mu(S_0, \ldots, S_{N-1}).$$

If, for μ-a.e. $(S_0, \ldots, S_{N-1})$, each S_i has an inverse which satisfies a Lipschitz condition then there is an $\alpha \geqq 0$ such that, for P_μ-a.e. $K \in \mathscr{K}(X)$, the Hausdorff dimension H-$\dim(K)$ is equal to α. If $X \subset \mathbb{R}^d$ is compact and has non-empty interior and if μ-a.e. $(S_0, \ldots, S_{N-1})$ consists of similarities which satisfy a certain disjointness condition w.r.t. X then α is determined by the equation

$$\int \sum_{i=0}^{N-1} \mathrm{Lip}(S_i)^\alpha \, d\mu(S_0, \ldots, S_{N-1}) = 1,$$

where $\mathrm{Lip}(S_i)$ denotes the (smallest) Lipschitz constant for S_i. Under fairly general assumptions the α-dimensional Hausdorff measure of P_μ-a.e. $K \in \mathscr{K}(X)$ equals 0.

If μ and X are chosen in a rather special way then P_μ-a.e. $K \in \mathscr{K}(X)$ is the graph of a homeomorphism of $[0, 1]$ (or a curve or the graph of a continuous function).

§ 1. Introduction

The term "fractal" was introduced by Mandelbrot for sets with a highly irregular structure including all sets of non-integer Hausdorff dimension. Man-

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delbrot and others have used such sets to model various physical phenomena (c.f. [8] and the references there). In that context those fractals seem to be of particular interest which have the additional property of being either strictly or statistically self-similar.

A theory of strictly self-similar compact sets has been developed by Moran [10] and Hutchinson [6]. A basic result of Hutchinson says that for every finite set of contractions $S_0, \ldots, S_{N-1}$ of a complete metric space there is a unique invariant non-empty compact set K, i.e. $K = \bigcup_{\rho=0}^{N-1} S_\rho(K)$. Lately, Falconer [4] and Mauldin-Williams [9] introduced and investigated general concepts of statistically self-similar fractals. In particular Falconer showed that Hutchinson's result has a probabilistic counterpart. (I came to know Falconer's results only after I had finished most of the following investigations.)

The starting point for the considerations in the present paper was a scheme used by Dubins-Freedman [2] to generate probability distribution functions at random (see also [5]) which is the prototype of the construction introduced here. Inspired by the work of Mauldin-Williams [9] a generalization of (parts of) Hutchinson's theory to the probabilistic setting is given which includes a slightly more general version of Falconer's result quoted above and makes it possible to answer a question of Mauldin-Williams concerning the Hausdorff measure of statistically self-similar fractals. As a byproduct the methods of proof used in this paper enable us to give new (and simpler) proofs for some of the results already contained in Falconer [4] and Mauldin-Williams [9].

Now we will give a more detailed preview of our results.

In § 2 we describe a method to construct a compact set from a given N-ary tree of contractions of a metric space, thereby imitating the construction of the classical Cantor set. We also obtain lower and upper estimates for the Hausdorff measures of the sets thus generated using the contraction (Lipschitz-) constants of the contractions involved.

In § 3 a general scheme for producing statistically self-similar fractals is introduced. To generate a fractal at random we start with a probability distribution μ on the set of all N-tuples of contractions of a given bounded separable complete metric space X. We define a probability measure P_μ on the space $\mathscr{K}(X)$ of all non-empty compact subsets of X in the following way: First we choose an N-tuple $(S_0, \ldots, S_{N-1})$ of contractions at random with respect to μ and set

$$A_1 = \bigcup_{\rho=0}^{N-1} S_\rho(X).$$

For every $\rho \in \{0, \ldots, N-1\}$ we independently choose an N-tuple $(S_{\rho 0}, \ldots, S_{\rho(N-1)})$ at random w.r.t. μ and set

$$A_2 = \bigcup_{\rho=0}^{N-1} S_\rho\left(\bigcup_{k=0}^{N-1} S_{\rho k}(X)\right).$$

We continue this process. Then $K = \bigcap_{n \in \mathbb{N}} \bar{A}_n$ is a typical P_μ-random fractal.

In §4 we show that the measure P_μ is characterized by the fact that

$$P_\mu(B) = \mu \otimes (P_\mu)^N \left(\left\{ ((S_0, \ldots, S_{N-1}), (K_0, \ldots, K_{N-1})) \,\middle|\, \bigcup_{\rho=0}^{N-1} S_\rho(K_\rho) \in B \right\} \right)$$

for every Borel set $B \subset \mathscr{K}(X)$. Under more restrictive assumptions this result is already contained in Falconer [4].

In §5 we show that a non-empty subset A of $\mathscr{K}(X)$ supports P_μ provided there is a complete metric d_A on A whose topology is weaker than that induced by the Hausdorff metric and provided μ-a.e. $(S_0, \ldots, S_{N-1})$ satisfies the following two conditions:

(i) $\forall K_0, \ldots, K_{N-1} \in A: \bigcup_{\rho=0}^{N-1} S_\rho(K_\rho) \in A$,

(ii) $\exists c \in (0, 1)\ \forall K_0, \ldots, K_{N-1} \in A\ \forall L_0, \ldots, L_{N-1} \in A$:

$$d_A\left(\bigcup_{\rho=0}^{N-1} S_\rho(K_\rho), \bigcup_{\rho=0}^{N-1} S_\rho(L_\rho) \right) \leqq c \max_{0 \leqq \rho \leqq N-1} d_A(K_\rho, L_\rho).$$

This result is used to show that the probability measures introduced by Dubins and Freedman [2] on the probability distribution functions are of the type P_μ for a suitably chosen μ. Moreover, conditions are stated under which P_μ is concentrated on graphs of continuous functions or curves.

In §6 we investigate certain martingales connected with N-ary trees of contractions. In particular we obtain an example of a martingale indexed by a countable set which converges in every $L^p(\infty > p \geqq 1)$ but whose pointwise lim sup is ∞ a.e. and whose pointwise lim inf is 0 a.e. The results in this section provide the basis for the determination of the Hausdorff measure and Hausdorff dimension of P_μ-random sets.

§7 contains the main results. First we show that, under rather weak assumptions, the Hausdorff dimensions of P_μ-random sets equal a constant α P_μ-a.e. Then we determine this constant α under the stronger assumption that $X \subset \mathbb{R}^d$ and all contractions involved are actually similarities which satisfy a certain disjointness condition with respect to X. Thereby we reprove results of Falconer [4] and Mauldin-Williams [9]. Moreover, we show that, for P_μ-a.e. compact set K, the α-dimensional Hausdorff measure of K is zero provided μ does *not* satisfy

$$\sum_{\rho=0}^{N-1} \operatorname{Lip}(S_\rho)^\alpha = 1 \qquad \mu\text{-a.e.},$$

where $\operatorname{Lip}(S)$ denotes the contraction (Lipschitz) constant of the contraction S. This last result answers a question of Mauldin-Williams ([9], 3.8) in the negative. Under the additional assumption that there is a $c > 0$ with $\operatorname{Lip}(S_\rho) \geqq c$ $(\rho = 0, \ldots, N-1)$ for μ-a.e. $(S_0, \ldots, S_{N-1})$ we prove that the following conditions are equivalent:

(i) $\sum_{\rho=0}^{N-1} \operatorname{Lip}(S_\rho)^\alpha = 1$ for μ-a.e. $(S_0, \ldots, S_{N-1})$,

(ii) $0 < \mathscr{H}^\alpha(K) < \infty$ for P_μ-a.e. K

(i.e., K is an α-set in the sense of Falconer [3] for P_μ-a.e. K).

§ 2. Fractals Constructed from Trees of Contractions

In this section we describe a Cantor-like construction of a compact subset of a bounded metric space X starting with an N-ary tree of contractions of X. Let us first fix the basic notation and definitions which will be used in the rest of the paper without further reference.

Let (X,d) be a complete separable metric space whose diameter $\operatorname{diam}(X)$ is finite.

For a map $S\colon X\to X$ let

$$\operatorname{Lip}(S)=\sup\left\{\frac{d(Sx,Sy)}{d(x,y)}\,\middle|\, x,y\in X,\ x\neq y\right\}$$

be the smallest Lipschitz constant for S which may be infinite. S is called a *contraction* if $\operatorname{Lip}(S)<1$.

By $\operatorname{Con}(X)$ we denote the set of all contractions of X. Let $\mathscr{K}(X)$ denote the space of all non-empty compact subsets of X with the Hausdorff metric η, i.e.

$$\eta(K,L)=\sup(\{d(x,L)\,|\,x\in K\}\cup\{d(K,y)\,|\,y\in L\}).$$

Then $(\mathscr{K}(X),\eta)$ is a complete separable metric space. Let $\mathbb{N}$ denote the positive integers and $\mathbb{N}_0=\mathbb{N}\cup\{0\}$. For $N\in\mathbb{N}$ let

$$D=D(N)=\bigcup_{m\in\mathbb{N}} D_m(N)$$

where

$$D_m=D_m(N)=\bigcup_{q=0}^{m}\{0,\dots,N-1\}^q$$

and $\{0,\dots,N-1\}^0:=\{\emptyset\}$, i.e. D is the set of finite sequences in $\{0,\dots,N-1\}$ including the empty sequence.

If $\sigma=(\sigma_0,\dots,\sigma_q)$ and $\tau=(\tau_0,\dots,\tau_r)$ are in D then $|\sigma|=q+1$ is the length of σ and $\sigma*\tau=(\sigma_0,\dots,\sigma_q,\tau_0,\dots,\tau_r)$ is the juxtaposition of σ and τ ($\emptyset*\sigma=\sigma$ and $\sigma*\emptyset=\sigma$).

Let $C=C(N)$ equal $\{0,\dots,N-1\}^{\mathbb{N}_0}$ with the product of the discrete topology on $\{0,\dots,N-1\}$.

For $q\in\mathbb{N}$ let $C_q=C_q(N)=\{0,\dots,N-1\}^q$.

For $\sigma\in D\cup C(N)$ and $n\in\mathbb{N}_0$ with $n\leqq|\sigma|$ if $\sigma\in D$ let

$$\sigma\,|\,n=(\sigma_0,\dots,\sigma_{n-1}).$$

We define a partial order on $D\cup C(N)$ by

$$\sigma\prec\tau \iff \tau\,|\,|\sigma|=\sigma.$$

We say that σ is preceding τ.

A subset $\Gamma\subset D$ is called a *covering*, if, for each $\tau\in C(N)$, there is an element $\sigma\in\Gamma$ preceding τ. If this σ is uniquely determined we call Γ *minimal*. Let Min denote the collection of all minimal coverings in D. It is easy to check that every element of Min is a finite set. We say that $\Gamma\in\mathrm{Min}$ is a *refinement* of $\Delta\in\mathrm{Min}$ and write $\Delta\prec\Gamma$ if, for every $\tau\in\Gamma$, there is a (unique) $\sigma\in\Delta$ with $\sigma\prec\tau$.

The elements of $\Omega=\Omega(X,N)=(\mathrm{Con}(X)^N)^D$ will be denoted by

$$\mathscr{S}=(\mathscr{S}_\sigma)_{\sigma\in D}$$

where $\mathscr{S}_\sigma=(S_{\sigma*0},\ldots,S_{\sigma*(N-1)})\in\mathrm{Con}(X)^N$.
For $\sigma=\emptyset$ we abbreviate

$$\mathscr{S}_\sigma=(S_0,\ldots,S_{N-1}).$$

Let $\Omega_0=\Omega_0(X,N)$ be the set of all $\mathscr{S}$ in Ω, such that, for every $\sigma=(\sigma_n)_{n\in\mathbb{N}_0}$,

$$\lim_{q\to\infty}\prod_{n=1}^{q}\mathrm{Lip}(S_{\sigma|n})=0.$$

2.1. Lemma. *Let $\mathscr{S}\in\Omega_0$ be given. Then, for every $\varepsilon>0$, there exists a $q_0\in\mathbb{N}$ such that, for all $q\geqq q_0$ and all $\sigma\in C_{q+1}$,*

$$\prod_{n=1}^{q}\mathrm{Lip}(S_{\sigma|n})<\varepsilon.$$

Proof. For $q\in\mathbb{N}$ the set

$$U_q=\left\{\sigma\in C(N)\,\middle|\,\prod_{n=1}^{q}\mathrm{Lip}(S_{\sigma|n})<\varepsilon\right\}$$

is open in $C(N)$. Since $\mathscr{S}\in\Omega_0$ the sets $(U_q)_{q\in\mathbb{N}}$ form an open covering of $C(N)$. By definition $U_q\subset U_{q+1}$ and hence the compactness of $C(N)$ implies that $C(N)=U_q$ for all sufficiently large q.

2.2. Theorem. *For every $\mathscr{S}\in\Omega_0$ the set*

$$K=K(\mathscr{S}):=\bigcap_{q\in\mathbb{N}}\bigcup_{\sigma\in C_{q+1}}\overline{S_{\sigma|1}\circ\ldots\circ S_{\sigma|q+1}(X)}$$

is compact. Moreover, for every family $(K_\sigma)_{\sigma\in D}$ in $\mathscr{K}(X)$,

$$K(\mathscr{S})=\lim_{q\to\infty}\bigcup_{\sigma\in C_{q+1}}S_{\sigma|1}\circ\ldots\circ S_{\sigma|q+1}(K_\sigma)$$

(where the limit is taken w.r.t. *the Hausdorff metric).*

Proof. First we will show that

$$\Big(\bigcup_{\sigma\in C_{q+1}}S_{\sigma|1}\circ\ldots\circ S_{\sigma|q+1}(K_\sigma)\Big)_{q\in\mathbb{N}_0}$$

is a Cauchy sequence in $(\mathscr{K}(X),\eta)$.

Let $\varepsilon>0$ be given. It is obvious that

$$\mathrm{diam}(S_{\sigma|1}\circ\ldots\circ S_{\sigma|q+1}(X))\leqq\left(\prod_{n=1}^{q+1}\mathrm{Lip}(S_{\sigma|n})\right)\mathrm{diam}(X).$$

By Lemma 2.1 there exists a $q_0\in\mathbb{N}$ such that the right hand side is less than ε for all $q\geqq q_0$ and all $\sigma\in C_{q+1}$. Now let $r>q\geqq q_0$ be arbitrary. For every

$\sigma \in C_{q+1}$ and every $x \in S_{\sigma|1} \circ \dots \circ S_{\sigma|q+1}(K_\sigma)$ we have

$$d(x, \bigcup_{\tau \in C_{r+1}} S_{\tau|1} \circ \dots \circ S_{\tau|r+1}(K_\tau)) \leqq d(x, S_{\sigma|1} \circ \dots \circ S_{\sigma|q+1}(L))$$

where $L = S_{\sigma*(\tau|1)} \circ \dots \circ S_{\sigma*(\tau|r-q)}(K_{\sigma*(\tau|r-q)})$ for some $\tau \in C_{r-q}$. Since $x \in S_{\sigma|1} \circ \dots \circ S_{\sigma|q+1}(X)$ we deduce

$$d(x, S_{\sigma|1} \circ \dots \circ S_{\sigma|q+1}(L)) \leqq \operatorname{diam}(S_{\sigma|1} \circ \dots \circ S_{\sigma|q+1}(X)) < \varepsilon.$$

Similarly we can show that

$$d(\bigcup_{\sigma \in C_{q+1}} S_{\sigma|1} \circ \dots \circ S_{\sigma|q+1}(K_\sigma), y) < \varepsilon$$

for every $y \in \bigcup_{\tau \in C_{r+1}} S_{\tau|1} \circ \dots \circ S_{\tau|r+1}(K_\tau)$.

This shows

$$\eta(\bigcup_{\sigma \in C_{q+1}} S_{\sigma|1} \circ \dots \circ S_{\sigma|q+1}(K_\sigma), \bigcup_{\tau \in C_{r+1}} S_{\tau|1} \circ \dots \circ S_{\tau|r+1}(K_\tau)) \leqq \varepsilon.$$

Hence our claim is proved.

Since $(\mathscr{K}(X), \eta)$ is complete

$$K' = \lim_{q \to \infty} \bigcup_{\sigma \in C_{q+1}} S_{\sigma|1} \circ \dots \circ S_{\sigma|q+1}(K_\sigma)$$

exists.

For every $q \in \mathbb{N}$ and every $r \geqq q$ the set

$$\bigcup_{\tau \in C_{r+1}} S_{\tau|1} \circ \dots \circ S_{\tau|q+1}(K_\tau)$$

is contained in the closed set

$$\bigcup_{\sigma \in C_{q+1}} \overline{S_{\sigma|1} \circ \dots \circ S_{\sigma|q+1}(X)}.$$

Hence the same is true for the limit K'.

Since this holds for arbitrary q we obtain

$$K' \subset K = \bigcap_{q \in \mathbb{N}} \bigcup_{\sigma \in C_{q+1}} \overline{S_{\sigma|1} \circ \dots \circ S_{\sigma|q+1}(X)}.$$

To show the inverse inclusion assume $K' \subsetneqq K$.

Then there exists an $x \in K \setminus K'$. By the definition of K there is a $\sigma \in C(N)$ such that

$$x \in \overline{S_{\sigma|1} \circ \dots \circ S_{\sigma|q+1}(X)}$$

for all $q \in \mathbb{N}$.

For sufficiently large q we have

$$\tfrac{1}{2} d(x, K') > \operatorname{diam}(S_{\sigma|1} \circ \dots \circ S_{\sigma|q+1}(X)).$$

Thus $S_{\sigma|1} \circ \dots \circ S_{\sigma|q+1}(K_\sigma)$ is contained in the ball with radius $\frac{1}{2} d(x, K')$ and centre x, which implies

$$\eta(K', \bigcup_{\tau \in C_{q+1}} S_{\tau|1} \circ \ldots \circ S_{\tau|q+1}(K_\tau)) \geqq \tfrac{1}{2} d(x, K') > 0$$

for all large q, a contradiction.

Thus $K' = K = \bigcap_{q \in \mathbb{N}} \bigcup_{\sigma \in C_{q+1}} \overline{S_{\sigma|1} \circ \ldots \circ S_{\sigma|q+1}(X)}$ and the theorem is proved.

2.3. Remarks. a) It is easy to check that, for every $\mathscr{S} \in \Omega_0$,

$$\bigcap_{q \in \mathbb{N}} \bigcup_{\sigma \in C_{q+1}} \overline{S_{\sigma|1} \circ \ldots \circ S_{\sigma|q+1}(X)} = \bigcup_{\sigma \in C(N)} \bigcap_{q \in \mathbb{N}} \overline{S_{\sigma|1} \circ \ldots \circ S_{\sigma|q+1}(X)}.$$

Moreover, for every $\sigma \in C(N)$, the set

$$\bigcap_{q \in \mathbb{N}} \overline{S_{\sigma|1} \circ \ldots \circ S_{\sigma|q+1}(X)}$$

is a singleton and the map, which assigns to σ the single element of

$$\bigcap_{q \in \mathbb{N}} \overline{S_{\sigma|1} \circ \ldots \circ S_{\sigma|q+1}(X)}$$

is a continuous map from $C(N)$ onto

$$\bigcap_{q \in \mathbb{N}} \bigcup_{\sigma \in C_{q+1}} \overline{S_{\sigma|1} \circ \ldots \circ S_{\sigma|q+1}(X)}.$$

b) The above construction generalizes a construction of Hutchinson [6].

Next we will give a lower and upper estimate for the Hausdorff measure of the compact set constructed in the first part of this paragraph.

To this end we introduce some more notation.

Let $E \subset X$, $\delta \geqq 0$ and $\alpha \geqq 0$ be arbitrary. Define

$$\mathscr{H}_\delta^\alpha(E) = \inf\left\{ \sum_{n=1}^{\infty} \operatorname{diam}(G_n)^\alpha \,|\, E \subset \bigcup G_n,\ G_n \text{ open},\ \operatorname{diam}(G_n) \leqq \delta \right\},$$

$$\overline{\mathscr{H}}_\delta^\alpha(E) = \inf\left\{ \sum_{n=1}^{\infty} \operatorname{diam}(E_n)^\alpha \,|\, E \subset \bigcup E_n,\ \operatorname{diam}(E_n) \leqq \delta \right\},$$

and

$$\mathscr{H}^\alpha(E) = \sup_{\delta > 0} \mathscr{H}_\delta^\alpha(E) = \sup_{\delta > 0} \overline{\mathscr{H}}_\delta^\alpha(E).$$

Then $\mathscr{H}^\alpha$ is an outer measure on X such that all Borel sets are $\mathscr{H}^\alpha$-measurable. $\mathscr{H}^\alpha$ is called the *α-dimensional Hausdorff measure*. The *Hausdorff dimension* of E is defined by

$$H\text{-dim}(E) = \sup\{\alpha \geqq 0 \,|\, \mathscr{H}^\alpha(E) > 0\} = \inf\{\alpha \geqq 0 \,|\, \mathscr{H}^\alpha(E) < \infty\}.$$

For other basic properties of $\mathscr{H}_\delta^\alpha$, $\overline{\mathscr{H}}_\delta^\alpha$, $\mathscr{H}^\alpha$ and H-dim we refer to Rogers [11] or Falconer [3].

For reasons of completeness we include the following estimate of Hausdorff measure obtained by using the natural coverings.

2.4. Theorem. *Let $\mathscr{S} \in \Omega_0$ be given. Then, for every $\alpha > 0$,*

$$\mathscr{H}^\alpha(K(\mathscr{S})) \leqq \operatorname{diam}(X)^\alpha \sup_{\Gamma_0 \in \mathrm{Min}} \inf\left\{ \sum_{\sigma \in \Gamma} \prod_{\rho=1}^{|\sigma|} \operatorname{Lip}(S_{\sigma|\rho})^\alpha \,|\, \Gamma \in \mathrm{Min},\ \Gamma \succ \Gamma_0 \right\}.$$

Proof. Recall $K=K(\mathscr{S})=\bigcap_{q\in\mathbb{N}}\bigcup_{\sigma\in C_{q+1}}\overline{S_{\sigma|1}\circ\ldots\circ S_{\sigma|q+1}(X)}$.

Let $\delta>0$ be arbitrary. According to Lemma 2.1 there exists a $q\in\mathbb{N}$ such that, for every $\sigma\in D$ with $|\sigma|\geqq q$, we have

$$\operatorname{diam}(X)\prod_{\rho=1}^{|\sigma|}\operatorname{Lip}(S_{\sigma|\rho})\leqq\delta.$$

Let $\Gamma\in\operatorname{Min}$, $\Gamma\succ C_q$ be arbitrary. Then

$$K\subset\bigcup_{\sigma\in\Gamma}\overline{S_{\sigma|1}\circ\ldots\circ S_{\sigma||\sigma|}(X)}$$

and

$$\operatorname{diam}(\overline{S_{\sigma|1}\circ\ldots\circ S_{\sigma||\sigma|}(X)})\leqq\operatorname{diam}(X)\prod_{n=1}^{|\sigma|}\operatorname{Lip}(S_{\sigma|n})\leqq\delta.$$

Hence

$$\begin{aligned}\mathscr{H}_\delta^\alpha(K)&\leqq\sum_{\sigma\in\Gamma}\operatorname{diam}(\overline{S_{\sigma|1}\circ\ldots\circ S_{\sigma||\sigma|}(X)})^\alpha\\&\leqq\operatorname{diam}(X)^\alpha\sum_{\sigma\in\Gamma}\prod_{n=1}^{|\sigma|}\operatorname{Lip}(S_{\sigma|n})^\alpha.\end{aligned}$$

Since $\Gamma\succ C_q$ was arbitrary this implies

$$\begin{aligned}\mathscr{H}_\delta^\alpha(K)&\leqq\inf\left\{\operatorname{diam}(X)^\alpha\sum_{\sigma\in\Gamma}\prod_{n=1}^{|\sigma|}\operatorname{Lip}(S_{\sigma|n})^\alpha\,\middle|\,\Gamma\in\operatorname{Min},\ \Gamma\succ C_q\right\}\\&\leqq\operatorname{diam}(X)^\alpha\sup_{\Gamma_0\in\operatorname{Min}}\inf\left\{\sum_{\sigma\in\Gamma}\prod_{n=1}^{|\sigma|}\operatorname{Lip}(S_{\sigma|n})^\alpha\,\middle|\,\Gamma\in\operatorname{Min},\ \Gamma\succ\Gamma_0\right\}.\end{aligned}$$

The last inequality holds for every $\delta>0$ and, therefore, yields the statement of the theorem.

For the statement of the next theorem let us recall that a *similarity* S: $X\to X$ is a map such that there exists a constant $c>0$ with $d(Sx,Sy)=c\,d(x,y)$ for all $x,y\in X$. Obviously we have $c=\operatorname{Lip}(S)$.

The following result is closely related to Theorem 7.3 in Falconer [4].

2.5. Theorem. *Let $X\subset\mathbb{R}^d$ be a compact set with non-empty interior $\mathring{X}$. Let $\mathscr{S}\in\Omega_0$ be such that for every $\sigma\in D$ and all $\rho,\rho'\in\{0,\ldots,N-1\}$ the map $S_{\sigma*\rho}$ is a similarity with*

$$S_{\sigma*\rho}(\mathring{X})\cap S_{\sigma*\rho}(\mathring{X})=\emptyset\quad\text{if }\rho\neq\rho'.\qquad(*)$$

Then there exists a constant $c>0$ depending only on X and the dimension d such that, for every $\alpha\geqq0$,

$$\begin{aligned}&c\operatorname{diam}(X)^\alpha\sup_{\Gamma_0\in\operatorname{Min}}\inf\left\{\sum_{\sigma\in\Gamma}\operatorname{Lip}(S_\sigma)^d\prod_{n=1}^{|\sigma|}\operatorname{Lip}(S_{\sigma|n})^\alpha\,\middle|\,\Gamma\in\operatorname{Min},\ \Gamma\succ\Gamma_0\right\}\\&\leqq\mathscr{H}^\alpha(K(\mathscr{S})).\end{aligned}$$

Proof. Since X is compact we have $K=K(\mathscr{S})=\bigcap_{q\in\mathbb{N}}\bigcup_{\sigma\in C_{q+1}}S_{\sigma|1}\circ\ldots\circ S_{\sigma||\sigma|}(X)$.

Let $\delta>0$ be arbitrary and let $(U_\nu)_{1\leqq\nu\leqq m}$ be an open covering of K with $\operatorname{diam}(U_\nu)\leqq\delta$ and $U_\nu\cap K\neq\emptyset$ for $\nu=1,\dots,m$. Define

$$\Gamma_\nu=\Big\{\sigma\in D\,|\,S_{\sigma|1}\circ\dots\circ S_{\sigma||\sigma|}(X)\cap K\cap U_\nu\neq\emptyset,\quad \operatorname{diam}(X)\prod_{n=1}^{|\sigma|-1}\operatorname{Lip}(S_{\sigma|n})\geqq\operatorname{diam}(U_\nu),\quad \operatorname{diam}(X)\prod_{n=1}^{|\sigma|}\operatorname{Lip}(S_{\sigma|n})<\operatorname{diam}(U_\nu)\Big\}.$$

According to the last two conditions in the definition of Γ_ν we have neither $\sigma\prec\tau$ nor $\tau\prec\sigma$ if $\sigma,\tau\in\Gamma_\nu$ are different. Due to condition (*) of the theorem this implies

$$S_{\sigma|1}\circ\dots\circ S_{\sigma||\sigma|}(\mathring{X})\cap S_{\tau|1}\circ\dots\circ S_{\tau||\tau|}(\mathring{X})=\emptyset \tag{1}$$

for all $\sigma,\tau\in\Gamma_\nu$ with $\sigma\neq\tau$.

Since all the maps belonging to $\mathscr{S}$ are similarities we deduce, for $\sigma\in\Gamma_\nu$,

$$\operatorname{diam}(S_{\sigma|1}\circ\dots\circ S_{\sigma||\sigma|}(X))=\operatorname{diam}(X)\prod_{n=1}^{|\sigma|}\operatorname{Lip}(S_{\sigma|n})<\operatorname{diam}(U_\nu),$$

hence

$$\begin{aligned}S_{\sigma|1}\circ\dots\circ S_{\sigma||\sigma|}(X)&\subset\{x\in\mathbb{R}^n\,|\,d(x,U_\nu\cap K)<\operatorname{diam}(U_\nu)\}\\&\subset B_{2\operatorname{diam}(U_\nu)}(x_\nu)\qquad\text{for any } x_\nu\in U_\nu\cap K,\end{aligned}\tag{2}$$

where $B_r(x)$ denotes the open ball of radius r and center x. Now let λ^d denote the d-dimensional Lebesgue measure. Since all maps in $\mathscr{S}$ are similarities we have

$$\lambda^d(S_{\sigma|1}\circ\dots\circ S_{\sigma||\sigma|}(\mathring{X}))=\lambda^d(\mathring{X})\prod_{n=1}^{|\sigma|}\operatorname{Lip}(S_{\sigma|n})^d. \tag{3}$$

Combining (1), (2), (3) and the definition of Γ_ν yields

$$\begin{aligned}(2\operatorname{diam}(U_\nu))^d\lambda^d(B_1(0))&=\lambda^d(B_{2\operatorname{diam}(U_\nu)}(x_\nu))\\&\underset{(1),(2)}{\geqq}\sum_{\sigma\in\Gamma_\nu}\lambda^d(S_{\sigma|1}\circ\dots\circ S_{\sigma||\sigma|}(\mathring{X}))\\&\underset{(3)}{\geqq}\sum_{\sigma\in\Gamma_\nu}\lambda^d(\mathring{X})\prod_{n=1}^{|\sigma|}\operatorname{Lip}(S_{\sigma|n})^d\\&\geqq\sum_{\sigma\in\Gamma_\nu}\lambda^d(\mathring{X})\operatorname{diam}(U_\nu)^d\operatorname{Lip}(S_\sigma)^d.\end{aligned}$$

Hence

$$\sum_{\sigma\in\Gamma_\nu}\operatorname{Lip}(S_\sigma)^d\leqq\frac{2^d\lambda^d(B_1(0))}{\lambda^d(\mathring{X})}=:\frac{1}{c}. \tag{4}$$

Using the definition of the Γ_ν's we obtain

$$\sum_{\nu=1}^{m} \operatorname{diam}(U_\nu)^\alpha \geqq \sum_{\nu=1}^{m} \max_{\sigma\in\Gamma_\nu} \operatorname{diam}(X)^\alpha \prod_{n=1}^{|\sigma|} \operatorname{Lip}(S_{\sigma|n})^\alpha$$
$$\geqq \sum_{\nu=1}^{m} (1/\sum_{\sigma\in\Gamma_\nu} \operatorname{Lip}(S_\sigma)^d) \sum_{\sigma\in\Gamma_\nu} \operatorname{Lip}(S_\sigma)^d \operatorname{diam}(X)^\alpha \prod_{n=1}^{|\sigma|} \operatorname{Lip}(S_{\sigma|n})^\alpha$$
$$\geqq c \operatorname{diam}(X)^\alpha \sum_{\nu=1}^{m} \sum_{\sigma\in\Gamma_\nu} \operatorname{Lip}(S_\sigma)^d \prod_{n=1}^{|\sigma|} \operatorname{Lip}(S_{\sigma|n})^\alpha$$
$$\geqq c \operatorname{diam}(X)^\alpha \sum_{\sigma\in\cup\Gamma_\nu} \operatorname{Lip}(S_\sigma)^d \prod_{n=1}^{|\sigma|} \operatorname{Lip}(S_{\sigma|n})^\alpha. \tag{5}$$

By Remark 2.3a it is easy to check that $\bigcup_{\nu=1}^{m} \Gamma_\nu$ is a covering.

Define
$$\Gamma = \{\sigma\in\bigcup\Gamma_\nu \mid \forall\tau\in\bigcup\Gamma_\nu\colon\ \tau\prec\sigma \Rightarrow \tau=\sigma\}.$$

Then Γ is a minimal covering.

Now define $\Gamma_\delta = \left\{\sigma\in D \,\middle|\, \prod_{n=1}^{|\sigma|-1} \operatorname{Lip}(S_{\sigma|n}) \geqq \delta,\ \prod_{n=1}^{|\sigma|} \operatorname{Lip}(S_{\sigma|n}) < \delta\right\}$.

Then Γ_δ is also minimal with $\Gamma_\delta \prec \Gamma$.

Thus (5) implies

$$\sum_{\nu=1}^{m} \operatorname{diam}(U_\nu)^\alpha \geqq c \operatorname{diam}(X)^\alpha \sum_{\sigma\in\Gamma} \operatorname{Lip}(S_\sigma)^d \prod_{n=1}^{|\sigma|} \operatorname{Lip}(S_{\sigma|n})^\alpha$$
$$\geqq c \operatorname{diam}(X)^\alpha \inf\left\{\sum_{\sigma\in\Gamma'} \operatorname{Lip}(S_\sigma)^d \prod_{n=1}^{|\sigma|} \operatorname{Lip}(S_{\sigma|n})^\alpha \,\middle|\, \Gamma'\in\operatorname{Min}, \Gamma'\succ\Gamma_\delta\right\}.$$

Since K is compact and (U_ν) is an arbitrary finite open covering of K with $\operatorname{diam}(U_\nu)\leqq\delta$ we deduce

$$\mathscr{H}^\alpha(K) \geqq \mathscr{H}^\alpha_\delta(K)$$
$$\geqq c \operatorname{diam}(X)^\alpha \inf\left\{\sum_{\sigma\in\Gamma} \operatorname{Lip}(S_\sigma)^d \prod_{n=1}^{|\sigma|} \operatorname{Lip}(S_{\sigma|n})^\alpha \,\middle|\, \Gamma'\in\operatorname{Min}, \Gamma'\succ\Gamma_\delta\right\}. \tag{6}$$

Let $\Gamma_0\in\operatorname{Min}$ be arbitrary. Since $\operatorname{Lip}(S_\tau)>0$ for all $\tau\in D$ there exists a $\delta>0$ with

$$\delta < \min\left\{\prod_{n=1}^{|\sigma|} \operatorname{Lip}(S_{\sigma|n}) \,\middle|\, \sigma\in\Gamma_0\right\}.$$

Obviously this implies $\Gamma_0 \prec \Gamma_\delta$.

Hence (6) yields the assertion of the theorem.

2.6. Remark. If $(S_{\sigma*0}, \ldots, S_{\sigma*(N-1)})$ satisfies condition $(*)$ in Theorem 2.5 then $(S_{\sigma*0}, \ldots, S_{\sigma*(N-1)})$ satisfies Hutchinson's open set condition ([6], pp. 735/736).

An analysis of the proof of Theorem 2.5 shows that the theorem remains true under the following weaker assumption:

$X \subset \mathbb{R}^d$ is compact and $\mathscr{S} \in \Omega_0$ is such that there exists a Borel subset W of X of positive Lebesbue measure satisfying $S_{\sigma*\rho}(W) \subset W$ and $S_{\sigma*\rho}(W) \cap S_{\sigma*\rho'}(W) = \emptyset$ if $\rho \neq \rho'$ for all $\sigma \in D$ and $\rho, \rho' \in \{0, \ldots, N-1\}$.

This last condition is implied by the open set condition of Hutchinson.

§ 3. A Probability P_μ on $\mathscr{K}(X)$ Induced by a Probability μ on N-tuples of Contractions of X

As everywhere in the paper (X, d) denotes a complete separable metric space of finite diameter. The space $\mathrm{Con}(X)$ of contractions of X will be equipped with the topology of pointwise convergence. It is easy to check that $\mathrm{Con}(X)$ is a separable metrizable space which is the countable union of completely metrizable subsets and hence a Suslin space. The function Lip: $\mathrm{Con}(X) \to [0, 1]$, $S \to \mathrm{Lip}(S)$ is lower-semicontinuous since it is the supremum of the continuous functions $S \mapsto \dfrac{d(Sx, Sy)}{d(x, y)}$ $(x \neq y,\ x, y \in X)$.

For $N \in \mathbb{N}$ the space

$$\Omega = (\mathrm{Con}(X)^N)^D$$

will be equipped with the product topology. Since D is countable the space Ω is a metrizable Suslin space and the product of the Borel field of $\mathrm{Con}(X)$ is equal to the Borel field of Ω. In the following μ is a Borel probability measure on $\mathrm{Con}(X)^N$. Let μ^D denote the corresponding product measure on $\Omega = (\mathrm{Con}(X)^N)^D$. By $(\mu^D)^N$ we denote the product of the μ^D's on Ω^N.

3.1. Proposition. *Define* φ: $\mathrm{Con}(X)^N \times \Omega^N \to \Omega$ *by*

$$\varphi((S_0, \ldots, S_{N-1}), (\mathscr{S}^{(0)}, \ldots, \mathscr{S}^{(N-1)})) := \mathscr{S},$$

where

$$\mathscr{S}_\emptyset = (S_0, \ldots, S_{N-1})$$

and

$$\mathscr{S}_{n*\sigma} = \mathscr{S}_\sigma^{(n)} \quad \text{for } \sigma \in D,\ n \in \{0, \ldots, N-1\}.$$

Then φ *is Borel measurable such that, for every Borel set* $B \subset \Omega$,

$$\mu \otimes (\mu^D)^N (\varphi^{-1}(B)) = \mu^D(B),$$

i.e. the image

$$\mu \otimes (\mu^D)^N \circ \varphi^{-1} \text{ of } \mu \otimes (\mu^D)^N \text{ w.r.t. } \varphi \text{ is equal to } \mu^D.$$

Proof. Obviously φ is Borel-measurable and the remaining assertion follows from the elementary properties of the product measure μ^D (Fubini's theorem).

3.2. Lemma. *Let* g: $\mathrm{Con}(X)^N \to [0, 1)$ *be Borel measurable. Then*

$$\Omega_g := \left\{ \mathscr{S} \in \Omega \mid \forall \sigma \in C(N): \prod_{n=1}^{\infty} g(S_{(\sigma|n-1)*0}, \ldots, S_{(\sigma|n-1)*(N-1)}) = 0 \right\}$$

is a Borel set with $\mu^D(\Omega_g) = 1$.

Proof. We have

$$\Omega_g=\left\{\mathscr{S}\,|\,\forall m\in\mathbb{N}\ \forall\sigma\in C(N)\ \exists q\in\mathbb{N}:\right.$$
$$\left.\prod_{n=1}^{q} g(S_{(\sigma|n-1)*0},\dots,S_{(\sigma|n-1)*(N-1)})<\frac{1}{m}\right\}.$$

Using the compactness of $C(N)$ in the same way as in the proof of Lemma 2.1 we obtain

$$\Omega_g=\left\{\mathscr{S}\,|\,\forall m\in\mathbb{N}\ \exists q\in\mathbb{N}\ \forall\sigma\in C_q\colon \prod_{n=1}^{q} g(S_{(\sigma|n-1)*0},\dots,S_{(\sigma|n-1)*(N-1)})<\frac{1}{m}\right\}$$
$$=\bigcap_{m\in\mathbb{N}}\bigcup_{q\in\mathbb{N}}\bigcap_{\sigma\in C_q}\left\{\mathscr{S}\,\middle|\,\prod_{n=1}^{q} g(S_{(\sigma|n-1)*0},\dots,S_{(\sigma|n-1)*(N-1)})<\frac{1}{m}\right\}$$

which is a Borel set.

For $a>0$ set

$$B_a=\left\{\mathscr{S}\in\Omega\,|\,\exists\sigma\in C(N)\colon \prod_{n=1}^{\infty} g(S_{(\sigma|n-1)*0},\dots,S_{(\sigma|n-1)*(N-1)})\geqq a\right\}.$$

As above one can see that B_a is Borel.

Define $p\colon(0,1)\to[0,1]$ by $p(a)=\mu^D(B_a)$.

Then p is a non-increasing function. We will show that p vanishes identically. Once this has been established the proof is complete because $\Omega_g=\Omega\setminus\bigcup_{a>0}B_a$.

It follows from Proposition 3.1 that, for every $a\in(0,1)$, we have

$$p(a)=\mu\otimes(\mu^D)^N\left(\left\{((S_0,\dots,S_{N-1}),(\mathscr{S}^{(0)},\dots,\mathscr{S}^{(N-1)}))\,|\,\exists\sigma\in C(N)\ \exists\rho\in\{0,\dots,N-1\}\colon\right.\right.$$
$$\left.\left. g(S_0,\dots,S_{N-1})\cdot\prod_{n=1}^{\infty} g(S^{(\rho)}_{(\sigma|n-1)*0},\dots,S^{(\rho)}_{(\sigma|n-1)*(N-1)})\geqq a\right\}\right)$$
$$\leqq\sum_{\rho=0}^{N-1}\mu\otimes(\mu^D)^N\left(\left\{((S_0,\dots,S_{N-1}),(\mathscr{S}^{(0)},\dots,\mathscr{S}^{(N-1)}))\,|\,\exists\sigma\in C(N)\colon g(S_0,\dots,S_{N-1})\right.\right.$$
$$\left.\left.\cdot\prod_{n=1}^{\infty} g(S^{(\rho)}_{(\sigma|n-1)*0},\dots,S^{(\rho)}_{(\sigma|n-1)*(N-1)})\geqq a\right\}\right)$$
$$\leqq N\mu(\{(S_0,\dots,S_{N-1})\,|\,g(S_0,\dots,S_{N-1})\geqq a\})\,p(a). \tag{1}$$

Since $g<1$ there exists a $b\in(0,1)$ with

$$\mu(\{(S_0,\dots,S_{N-1})\,|\,g(S_0,\dots,S_{N-1})\geqq b\})<\frac{1}{N}.$$

It follows from (1) with $a=b$ that

$$p(b)=0.$$

Define $\eta=\inf\{a\in(0,1)\,|\,p(a)=0\}$.

Assume $\eta>0$. Then there is an $a>\eta$ with $ab<\eta$. As before we deduce

$$p(ab)\leqq\sum_{\rho=0}^{N-1}\mu\otimes(\mu^D)^N\left(\left\{((S_0,\ldots,S_{N-1}),(\mathscr{S}^{(0)},\ldots,\mathscr{S}^{(N-1)}))\,|\,\exists\sigma\in C(N):\right.\right.$$
$$\left.\left. g(S_0,\ldots,S_{N-1})\prod_{n=1}^{\infty}g(S^{(\rho)}_{(\sigma|n-1)*0},\ldots,S^{(\rho)}_{(\sigma|n-1)*(N-1)})\geqq ab\right\}\right).$$

Since $a>\eta$ we have $p(a)=0$, hence

$$\prod_{n=1}^{\infty}g(S^{(\rho)}_{(\sigma|n-1)*0},\ldots,S^{(\rho)}_{(\sigma|n-1)*(N-1)})\leqq a$$

for μ^D-a.e. $\mathscr{S}^{(\rho)}$. This leads to

$$p(ab)\leqq N\mu(\{(S_0,\ldots,S_{N-1})\,|\,g(S_0,\ldots,S_{N-1})\geqq b\})\,p(ab),$$

which implies $p(ab)=0$, a contradiction.

Thus $\eta=0$ and p vanishes identically.

3.3. Theorem. *The set* $\Omega_0=\left\{\mathscr{S}\in(\mathrm{Con}(X)^N)^D\,|\,\forall\sigma\in C(N):\ \prod_{n=1}^{\infty}\mathrm{Lip}(S_{\sigma|n})=0\right\}$ *is a Borel set with* $\mu^D(\Omega_0)=1$.

Proof. Define g: $\mathrm{Con}(X)^N\to[0,1)$ by $g(S_0,\ldots,S_{N-1})=\max_{0\leqq\rho\leqq N-1}\mathrm{Lip}(S_\rho)$ and use Lemma 3.2.

3.4. Lemma. *For every* $m\in\mathbb{N}$ *the map* $\mathrm{Con}(X)^m\to\mathrm{Con}(X)$, $(S_0,\ldots,S_{m-1})\mapsto S_0\circ\ldots\circ S_{m-1}$ *is continuous.*

Proof. It suffices to prove the lemma for $m=2$.

Let $\varepsilon>0$, $(S_0^{(0)},S_1^{(0)})\in\mathrm{Con}(X)^2$ and $x\in X$ be given.

For every $(S_0,S_1)\in\mathrm{Con}(X)^2$ with $d(S_1x,S_1^{(0)}x)<\frac{\varepsilon}{2}$ and $d(S_0\circ S_1^{(0)}x, S_0^{(0)}\circ S_1^{(0)}x)<\frac{\varepsilon}{2}$ we get

$$d(S_0\circ S_1x,S_0^{(0)}\circ S_1^{(0)}x)\leqq d(S_0\circ S_1x,S_0\circ S_1^{(0)}x)+d(S_0\circ S_1^{(0)}x,S_0^{(0)}\circ S_1^{(0)}x)$$
$$\leqq\mathrm{Lip}(S_0)\,d(S_1x,S_1^{(0)}x)+\frac{\varepsilon}{2}<\varepsilon.$$

Thus the map $\mathrm{Con}(X)^2\to X$, $(S_0,S_1)\to S_0\circ S_1x$ is continuous. Since $\mathrm{Con}(X)$ carries the topology of pointwise convergence this implies the lemma.

3.5. Lemma. *The map* $\mathrm{Con}(X)\times\mathscr{K}(X)\to\mathscr{K}(X)$, $(S,K)\mapsto S(K)$ *is continuous.*

Proof. For $S,T\in\mathrm{Con}(X)$ and $K,L\in\mathscr{K}(X)$ we have

$$\eta(S(K),T(L))\leqq\eta(S(K),S(L))+\eta(S(L),T(L))$$
$$\leqq\sup(\{d(Sx,S(L))\,|\,x\in K\}\cup\{d(S(K),Sy)\,|\,y\in L\})$$
$$+\sup(\{d(Sy,T(L))\,|\,y\in L\}\cup\{d(S(L),Ty)\,|\,y\in L\})$$
$$\leqq\mathrm{Lip}(S)\,\eta(K,L)+\sup(\{d(Sx,Tx)\,|\,x\in L\}).$$

Let $\varepsilon>0$ be given and let $T\in\mathrm{Con}(X)$ and $L\in\mathscr{K}(X)$ be fixed. Then there are $x_1,\ldots,x_k\in L$ such that

$$L\subset B_{\varepsilon/3}(x_1)\cup\ldots\cup B_{\varepsilon/3}(x_k).$$

For every $S\in\mathrm{Con}(X)$ with $d(Sx_\rho,Tx_\rho)<\frac{\varepsilon}{3}$ for $\rho=1,\ldots,k$ and every $x\in L\cap B_{\varepsilon/3}(x_\rho)$ we deduce

$$\begin{aligned} d(Sx,Tx)&\leqq d(Sx,Sx_\rho)+d(Sx_\rho,Tx_\rho)+d(Tx_\rho,Tx)\\ &\leqq \mathrm{Lip}(S)\,d(x,x_\rho)+\varepsilon/3+\mathrm{Lip}(T)\,d(x_\rho,x)\\ &<\varepsilon,\end{aligned}$$

hence

$$\sup\{d(Sx,Tx)\mid x\in L\}\leqq\varepsilon.$$

Thus for every $S\in\mathrm{Con}(X)$ with $d(Sx_\rho,Tx_\rho)<\frac{\varepsilon}{3}$ $(\rho=1,\ldots,k)$ and for every $K\in\mathscr{K}(X)$ with $\eta(K,L)<\varepsilon$, we have

$$\eta(S(K),T(L))<2\varepsilon.$$

Hence the lemma is proved.

The following result is well-known (cf. Kuratowski [7], Vol. I, p. 166).

3.6. Lemma. *The map $\mathscr{K}(X)\times\mathscr{K}(X)\to\mathscr{K}(X)$, $(K,L)\mapsto K\cup L$ is continuous.*

3.7. Theorem. *Let $\tilde K\in\mathscr{K}(X)$ be arbitrary and define*

ψ: $\Omega\to\mathscr{K}(X)$ by

$$\psi(\mathscr{S})=\begin{cases}\bigcap_{q\in\mathbb{N}}\bigcup_{\sigma\in C_{q+1}}\overline{S_{\sigma|1}\circ\ldots\circ S_{\sigma||\sigma|}(X)}, & \mathscr{S}\in\Omega_0\\ \tilde K, & \mathscr{S}\notin\Omega_0.\end{cases}$$

Then ψ is a Borel measurable map.

Proof. According to Theorem 2.2 ψ is a well-defined map. It follows from Lemma 3.4 through 3.6 that for every $q\in\mathbb{N}$ and every family $(K_\sigma)_{\sigma\in D}$ in $\mathscr{K}(X)$, the map

$$\Omega\to\mathscr{K}(X),\qquad \mathscr{S}\mapsto\bigcup_{\sigma\in C_{q+1}}S_{\sigma|1}\circ\ldots\circ S_{\sigma||\sigma|}(K_\sigma)$$

is continuous. Thus Theorem 2.2 implies that, on Ω_0, the map ψ is the pointwise limit of a sequence of Borel measurable maps. Since Ω_0 is a Borel set by Theorem 3.3 this implies the assertion of the theorem.

3.8. Definition. For a Borel probability measure μ on $\mathrm{Con}(X)^N$ let P_μ be the image measure of μ^D w.r.t. ψ, i.e. for every Borel set $B\subset\mathscr{K}(X)$, $P_\mu(B)=\mu^D(\psi^{-1}(B))$.

3.9. Remark. a) It follows from Theorem 2.2 that a P_μ-random set can be constructed as follows:

Take an arbitrary set $K\in\mathscr{K}(X)$. Choose an N-tuple $(S_0,\ldots,S_{N-1})$ at random w.r.t. the measure μ. Form the set $S_0(K)\cup\ldots\cup S_{N-1}(K)$. Then, for $\rho=0,\ldots,N-1$, choose independently an N-tuple $(S_{\rho,0},\ldots,S_{\rho,N-1})$ at random w.r.t. μ.

Form the set

$$S_0(S_{0,1}(K)\cup\ldots\cup S_{0,N-1}(K))\cup\ldots\cup S_{N-1}(S_{N-1,0}(K)\cup\ldots\cup S_{N-1,N-1}(K)).$$

Continue this process. The limit w.r.t. the Hausdorff metric is a typical P_μ-random object.

b) The result described in a) is a stochastic version of a result of Hutchinson ([6], p. 725).

§ 4. Characterization of P_μ as the Unique μ-Self-Similar Measure on $\mathscr{K}(X)$

4.1. Definition. Let μ be a Borel probability measure on $\mathrm{Con}(X)^N$. A probability measure P on $\mathscr{K}(X)$ is called μ-statistically-self-similar (or μ-self-similar) if, for every Borel set $B\subset\mathscr{K}(X)$,

$$P(B)=\mu\otimes P^N\left(\left\{((S_0,\ldots,S_{N-1}),(K_0,\ldots,K_{N-1}))\in\mathrm{Con}(X)^N\times\mathscr{K}(X)^N\,\middle|\,\bigcup_{\rho=0}^{N-1}S_\rho(K_\rho)\in B\right\}\right).$$

The following lemma is an immediate consequence of this definition.

4.2. Lemma. *Let φ: $\mathscr{K}(X)\to\bar{\mathbb{R}}_+$ be Borel measurable. Then, for any μ-self-similar measure P on $\mathscr{K}(X)$,*

$$\int\varphi\,dP=\iint\varphi\left(\bigcup_{\rho=0}^{N-1}S_\rho(K_\rho)\right)dP^N(K_0,\ldots,K_{N-1})\,d\mu(S_0,\ldots,S_{N-1}).$$

4.3. Definition. For a Borel probability measure μ on $\mathrm{Con}(X)^N$ define

$$T_\mu\colon\ \mathscr{P}(\mathscr{K}(X))\to\mathscr{P}(\mathscr{K}(X))$$

by

$$[T_\mu(Q)](B)=\mu\otimes Q^N\left(\left\{((S_0,\ldots,S_{N-1}),(K_0,\ldots,K_{N-1}))\,\middle|\,\bigcup_{\rho=0}^{N-1}S_\rho(K_\rho)\in B\right\}\right),$$

where $\mathscr{P}(\mathscr{K}(X))$ denotes the Borel probability measures on $\mathscr{K}(X)$.

4.4. Remark. A probability $P\in\mathscr{P}(\mathscr{K}(X))$ is μ-self-similar if and only if $T_\mu(P)=P$, i.e. if P is a fixed point of T_μ.

4.5. Theorem. *Let μ be a Borel probability on* $\mathrm{Con}(X)^N$. *Then P_μ is the unique μ-self-similar probability measure on $\mathscr{K}(X)$. Moreover, for every $Q\in\mathscr{P}(\mathscr{K}(X))$, the sequence $(T_\mu^n(Q))_{n\in\mathbb{N}}$ converges to P_μ in the weak topology.*

Proof. First we will show that $T_\mu(P_\mu)=P_\mu$, *i.e. that* P_μ is μ-self-similar.

Define $\tilde\varphi$: $\mathrm{Con}(X)^N\times\mathscr{K}(X)^N\to\mathscr{K}(X)$ by

$$\tilde\varphi((S_0,\ldots,S_{N-1}),(K_0,\ldots,K_{N-1}))=\bigcup_{\rho=0}^{N-1}S_\rho(K_\rho).$$

By Lemma 3.5 and 3.6 the map $\tilde\varphi$ is continuous.

Moreover $T_\mu(Q)$ is the image measure of $\mu\otimes Q^N$ w.r.t. $\tilde{\varphi}$ for any $Q\in\mathscr{P}(\mathscr{K}(X))$.

Now consider the map φ: $\mathrm{Con}(X)^N\times\Omega^N\to\Omega$ as defined in Proposition 3.1 and the map $\hat{\psi}$: $\mathrm{Con}(X)^N\times\Omega^N\to\mathrm{Con}(X)^N\times\mathscr{K}(X)^N$ defined by

$$\hat{\psi}((S_0,\dots,S_{N-1}),(\mathscr{S}^{(0)},\dots,\mathscr{S}^{(N-1)}))$$
$$=((S_0,\dots,S_{N-1}),(\psi(\mathscr{S}^{(0)}),\dots,\psi(\mathscr{S}^{(N-1)}))),$$

where ψ is defined in Theorem 3.7.

Then it is easy to check that $\tilde{\varphi}\circ\hat{\psi}=\psi\circ\varphi$.

Next we note that the image of $\mu\otimes(\mu^D)^N$ w.r.t. $\hat{\psi}$ is $\mu\otimes P_\mu^N$.

According to Proposition 3.1 the image of $\mu\otimes(\mu^D)^N$ w.r.t. φ is μ^D. Since $P_\mu=\mu^D\circ\psi^{-1}$ we deduce, by combining these results, that $T_\mu(P_\mu)=P_\mu$.

Next we will show that $\lim_{n\to\infty}T_\mu^n(Q)=P_\mu$ for any $Q\in\mathscr{P}(\mathscr{K}(X))$.

Let $A\subset\mathscr{K}(X)$ be a closed set. Using induction on n it is easy to prove

$$[T_\mu^n(Q)](A)=\mu^D\otimes Q^D(\{(\mathscr{S},(K_\sigma)_{\sigma\in D})\in\Omega\times\mathscr{K}(X)^D\mid \bigcup_{\sigma\in C_n}S_{\sigma|1}\circ\dots\circ S_{\sigma|n}(K_\sigma)\in A\}).$$

Hence we obtain

$$\begin{aligned}&\limsup_{n\to\infty}[T_\mu^n(Q)](A)\\&=\inf_m\sup_{n\geq m}\mu^D\otimes Q^D(\{(\mathscr{S},(K_\sigma)_{\sigma\in D})\mid \bigcup_{\sigma\in C_n}S_{\sigma|1}\circ\dots\circ S_{\sigma|n}(K_\sigma)\in A\})\\&\leqq\mu^D\otimes Q^D(\bigcap_m\bigcup_{n\geq m}\{(\mathscr{S},(K_\sigma)_{\sigma\in D})\mid \bigcup_{\sigma\in C_n}S_{\sigma|1}\circ\dots\circ S_{\sigma|n}(K_\sigma)\in A\})\\&\leqq\mu^D\otimes Q^D(\{(\mathscr{S},(K_\sigma)_{\sigma\in D})\in\Omega_0\times\mathscr{K}(X)^D\mid \lim_{n\to\infty}\bigcup_{\sigma\in C_n}S_{\sigma|1}\circ\dots\circ S_{\sigma|n}(K_\sigma)\in A\}).\end{aligned}$$

By Theorem 2.2 and the definition of ψ this last expression equals

$$\mu^D\otimes Q^D(\{(\mathscr{S},(K_\sigma)_{\sigma\in D})\in\Omega_0\times\mathscr{K}(X)^D\mid\psi(\mathscr{S})\in A\})$$

which, in turn, is equal to $\mu^D(\psi^{-1}(A))$.

Thus, by the definition of P_μ, we have shown

$$\limsup_{n\to\infty}[T_\mu^n(Q)](A)\leqq P_\mu(A).$$

Since this is true for an arbitrary closed subset A of $\mathscr{K}(X)$ we deduce that $(T_\mu^n(Q))_{n\in\mathbb{N}}$ converges to P_μ in the weak topology. This last fact also implies that, except for P_μ, there is no fixed point for T_μ.

Hence the theorem is proved.

4.6. Remark. The preceding theorem was mainly inspired by the techniques used in [5]. It has independently been proved by Falconer [4] in a slightly more restrictive case: Falconer defined a metric on $\mathscr{P}(X)$ such that the corresponding topology is stronger than the weak topology and such that T_μ is a contraction w.r.t. this metric provided there is a $c<1$ such that, for μ-a.e.

$(S_0, \ldots, S_{N-1}) \in \mathrm{Con}(X)^N$, $\mathrm{Lip}(S_\rho) \leqq c$ $(\rho = 0, \ldots, N-1)$. Moreover, Falconer showed that under this condition P_μ is the unique fixed point of T_μ.

4.7. Problem. Is there a natural property which characterizes those probabilities on $\mathscr{K}(X)$ that are μ-self-similar w.r.t. some μ?

4.8. Remark. If μ is a point mass $\varepsilon_{(T_0, \ldots, T_{N-1})}$ then μ^D is also a point mass. Hence it follows from the definition of P_μ as image of μ^D that P_μ is a point mass, i.e. $P_\mu = \varepsilon_K$ for a compact set K. It follows from Theorem 4.5 that $K = T_0(K) \cup \ldots \cup T_{N-1}(K)$. Thus Theorem 4.5 contains Hutchinson's result ([6], p. 724) as a special case.

§ 5. Sets Supporting the Measure P_μ

In this section we give a sufficient condition for a subset of $\mathscr{K}(X)$ to support the measure P_μ. We use this condition to show how one can use the construction described in § 3 to generate random curves and random homeomorphisms.

In the following μ is always a Borel probability on $\mathrm{Con}(X)^N$.

5.1. Theorem. *Let $A \subset \mathscr{K}(X)$ be a non-empty set and d_A a bounded metric on A such that (A, d_A) is a complete metric space whose topology is not stronger than the topology induced by the Hausdorff metric. Suppose that for μ-a.e. $(S_0, \ldots, S_{N-1}) \in \mathrm{Con}(X)^N$ the following two conditions are satisfied:*

(i) $\forall K_0, \ldots, K_{N-1} \in A$: $\bigcup_{\rho=0}^{N-1} S_\rho(K_\rho) \in A$.

(ii) $\exists c \in (0,1)$ $\forall K_0, \ldots, K_{N-1} \in A$ $\forall L_0, \ldots, L_{N-1} \in A$:

$$d_A\left(\bigcup_{\rho=0}^{N-1} S_\rho(K_\rho), \bigcup_{\rho=0}^{N-1} S_\rho(L_\rho)\right) \leqq c \max_{0 \leqq \rho \leqq N-1} d_A(K_\rho, L_\rho).$$

Then P_μ is supported by A, i.e. A is P_μ-measurable and $P_\mu(A) = 1$.

Proof. Let $W \subset \mathrm{Con}(X)^N$ be a Borel set with $\mu(W) = 1$ such that for all $(S_0, \ldots, S_{N-1}) \in W$ conditions (i) and (ii) are satisfied. Define g: $\mathrm{Con}(X)^N \to [0,1)$ by

$$g(S_0, \ldots, S_{N-1}) = \begin{cases} \sup\left\{ \dfrac{d_A\left(\bigcup_{\rho=0}^{N-1} S_\rho(K_\rho), \bigcup_{\rho=0}^{N-1} S_\rho(L_\rho)\right)}{\max_{0 \leqq \rho \leqq N-1} d_A(K_\rho, L_\rho)} \,\middle|\, (K_\rho), (L_\rho) \in A^N,\ (K_\rho) \neq (L_\rho) \right\}, & \text{for } (S_0, \ldots, S_{N-1}) \in W, \\ 0, & \text{elsewhere.} \end{cases}$$

Since d_A: $A \times A \to \mathbb{R}$ is continuous with respect to the product of the topology induced by the Hausdorff metric it follows that the restriction of g to W is lower-semi-continuous. Since W is a Borel set this implies that g is Borel measurable.

Let V_g be the set of all $\mathscr{S}\in W^D$ such that, for all $\sigma\in C(N)$,

$$\prod_{n=1}^{\infty} g(S_{(\sigma|n-1)*0},\ldots,S_{\sigma|n-1)*(N-1)})=0.$$

By the choice of W as a set of full μ-measure it follows from Lemma 3.2 that $\mu^D(V_g)=1$.

Let $K\in A$ be arbitrary. We will show that, for $\mathscr{S}\in V_g$, the sequence

$$(\bigcup_{\sigma\in C_{q+1}} S_{\sigma|1}\circ\ldots\circ S_{\sigma|q+1}(K))_{q\in\mathbb{N}}$$

is Cauchy in (A,d_A).

Let $\mathscr{S}\in V_g$ and $\varepsilon>0$ be given. As in the proof of Lemma 2.1 it can be seen that there exists a $q_0\in\mathbb{N}$ with

$$\forall\sigma\in C_{q_0}\colon \prod_{n=1}^{q_0} g(S_{(\sigma|n-1)*0},\ldots,S_{(\sigma|n-1)*(N-1)})<\varepsilon.$$

For $q>m\geqq q_0$ we deduce

$$\begin{aligned}
&d_A(\bigcup_{\sigma\in C_{m+1}} S_{\sigma|1}\circ\ldots\circ S_{\sigma|m+1}(K),\bigcup_{\sigma\in C_{q+1}} S_{\sigma|1}\circ\ldots\circ S_{\sigma|q+1}(K))\\
&=d_A\left(\bigcup_{n=0}^{N-1} S_n(\bigcup_{\sigma\in C_m} S_{n*(\sigma|1)}\circ\ldots\circ S_{n*(\sigma|m)}(K)),\bigcup_{n=0}^{N-1} S_n(\bigcup_{\sigma\in C_q} S_{n*(\sigma|1)}\circ\ldots\circ S_{n*(\sigma|q)}(K)\right)\\
&\underset{\text{(ii)}}{\leqq} g(S_0,\ldots,S_{N-1})\max_{0\leqq n\leqq N-1} d_A(\bigcup_{\sigma\in C_m} S_{n*(\sigma|1)}\ldots S_{n*(\sigma|m)}(K),\bigcup_{\sigma\in C_q} S_{n*(\sigma|1)}\ldots S_{n*(\sigma|q)}(K)).
\end{aligned}$$

By induction it follows that there exists a $\tau\in C_{m+1}$ with

$$\begin{aligned}
&d_A(\bigcup_{\sigma\in C_{m+1}} S_{\sigma|1}\ldots S_{\sigma|m+1}(K),\bigcup_{\sigma\in C_{q+1}} S_{\sigma|1}\circ\ldots\circ S_{\sigma|q+1}(K))\\
&\quad\leqq\prod_{n=1}^{m+1} g(S_{(\tau|n)*0},\ldots,S_{(\tau|n)*(N-1)})\,d_A(K,\bigcup_{\kappa\in C_{q-m}} S_{(\tau|q+1)*\kappa}(K))\leqq\varepsilon\operatorname{diam}_A(A)
\end{aligned}$$

where $\operatorname{diam}_A(A)$ denotes the diameter of the metric space (A,d_A). Hence our claim is proved.

Since (A,d_A) is complete the sequence

$$(\bigcup_{\sigma\in C_{q+1}} S_{\sigma|1}\ldots S_{\sigma|q+1}(K))_{q\in\mathbb{N}}$$

converges in (A,d_A).

If, in addition, $\mathscr{S}\in\Omega_0$ then the above sequence converges w.r.t. the Hausdorff metric η to $\psi(\mathscr{S})$. Since the topology induced by d_A is not stronger than the topology induced by η it follows that the two limits agree, hence that $\psi(\mathscr{S})\in A$ for μ^D-a.e. $\mathscr{S}$. Since $\Omega=(\mathrm{Con}(X)^N)^D$ and $\mathscr{K}(X)$ are Suslin spaces and since ψ is Borel measurable this implies that A is measurable w.r.t. $\mu^D\circ\psi^{-1}=P_\mu$ and satisfies $P_\mu(A)=1$.

5.2. Corollary. *The support of the measure P_μ is equal to the intersection of all non-empty closed subsets A of $\mathscr{K}(X)$ such that, for μ-a.e. $(S_0, \ldots, S_{N-1}) \in \mathrm{Con}(X)^N$ and for all $(K_0, \ldots, K_{N-1}) \in A^N$,*

$$\bigcup_{\rho=0}^{N-1} S_\rho(K_\rho) \in A.$$

Proof. By Theorem 5.1 each closed set A with the properties stated in the corollary supports P_μ. Thus the assertion of the corollary follows.

5.3. Examples. a) Random continuous functions.

Let E be a compact subset of $\mathbb{R}^d$ and $[a,b]$ a compact non-trivial interval in $\mathbb{R}$. Let $X=[a,b]\times E$ carry the Euclidean metric d. Consider the space $\mathscr{C}([a,b],E)$ of all graphs of continuous functions from $[a,b]$ to E with the supremum metric. Let (A, d_A) be a non-empty closed subspace of $\mathscr{C}([a,b],E)$.

Suppose that μ is a probability on $\mathrm{Con}(X)^N$ such that μ-a.e. $(S_0, \ldots, S_{N-1}) \in \mathrm{Con}(X)^N$ satisfies the following conditions:

(i) $S_\rho = S'_\rho \times S''_\rho$, where $S'_\rho(]a,b[) \cap S'_{\rho'}(]a,b[) = \emptyset$ for $\rho \neq \rho'$,

(ii) $\forall K_0, \ldots, K_{N-1} \in A\colon \bigcup_{\rho=0}^{N-1} S_\rho(K_\rho) \in A$.

Then P_μ is supported by A.

Proof. The metric d_A generates the same topology on A as the Hausdorff metric η (Kuratowski [7], Vol. I, p. 223). Moreover, (A, d_A) is obviously a bounded complete metric space. To apply Theorem 5.1 it suffices, therefore, to show that condition (ii) in that theorem is satisfied. Let

$$(K_\rho)_{0 \leqq \rho \leqq N-1} \quad \text{and} \quad (L_\rho)_{0 \leqq \rho \leqq N-1} \quad \text{in } A \text{ be given.}$$

We will prove that

$$d_A\left(\bigcup_{\rho=0}^{N-1} S_\rho(K_\rho), \bigcup_{\rho=0}^{N-1} S_\rho(L_\rho)\right) \leqq \max_{0 \leqq \rho \leqq N-1} \mathrm{Lip}(S_\rho) \max_{0 \leqq \rho \leqq N-1} d_A(K_\rho, L_\rho)$$

provided $(S_0, \ldots, S_{N-1})$ satisfies (i) and (ii). Since $\bigcup_{\rho=0}^{N-1} S_\rho(K_\rho)$ and $\bigcup_{\rho=0}^{N-1} S_\rho(L_\rho)$ are graphs of continuous functions on $[a,b]$ by (ii) and since $\bigcup_{\rho=0}^{N-1} S'_\rho(]a,b[)$ is dense in $[a,b]$ it is enough to show that, for every $x \in \bigcup_{\rho=0}^{N-1} S'_\rho(]a,b[)$, every y with $(x,y) \in \bigcup_{\rho=0}^{N-1} S_\rho(K_\rho)$, and every z with $(x,z) \in \bigcup_{\rho=0}^{N-1} S_\rho(L_\rho)$ we have $\|y-z\| \leqq \max_\rho \mathrm{Lip}(S_\rho) \max_\rho d_A(K_\rho, L_\rho)$ (where $\|\ \|$ denotes the Euclidean norm). Suppose $x \in S'_\rho(]a,b[)$. Then (i) implies $(x,y) \in S_\rho(K_\rho)$ and $(x,z) \in S_\rho(L_\rho)$. There exists a $(u,v) \in K_\rho$ with $S_\rho(u,v) = (x,y)$. Since L_ρ is a graph there exists a unique $w \in E$ with $(u,w) \in L_\rho$. Then $S_\rho(u,w) = (S'_\rho(u), S''_\rho(w)) = (x, S''_\rho(w))$. Since $\bigcup_{i=0}^{N-1} S_i(L_i)$ is a graph this implies $S''_\rho(w)=z$. Then we deduce

$$\|y-z\|=\|S''_\rho(v)-S''_\rho(w)\|=\|S_\rho(u,v)-S_\rho(u,w)\|$$
$$\leqq \operatorname{Lip}(S_\rho)\,\|(u,v)-(u,w)\|\leqq \operatorname{Lip}(S_\rho)\,d_A(K_\rho,L_\rho)$$

which proves our claim.

By Theorem 5.1 we conclude that P_μ is supported by A.

To give a brief specific example let $[a,b]=[0,1]=E$ and let $A=\{f\in\mathscr{C}([0,1],[0,1])\,|\,f(0)=0,\ f(1)=1\}$. Let ν be the normalized Lebesgue measure on $\Delta=\{(x_1,x_2)\in[0,1]^2\,|\,x_1+x_2\leqq 1\}$. For $(x_1,x_2),(y_1,y_2)\in\Delta$ define contractions $S_0^{x_1,x_2,y_1,y_2},\ldots$ by

$$S_0^{x_1,x_2,y_1,y_2}(u,v)=(x_1u,(1-y_2)v),$$
$$S_1^{x_1,x_2,y_1,y_2}(u,v)=(x_1+(1-u)(1-x_1-x_2),1-y_2+(1-v)(y_1+y_2-1)),$$
$$S_2^{x_1,x_2,y_1,y_2}(u,v)=(1-x_2+ux_2,y_1+v(1-y_1)).$$

Let μ be the image of $\nu\otimes\nu$ with respect to the map

$$((x_1,x_2),(y_1,y_2))\to(S_0^{x_1,x_2,y_1,y_2},S_1^{x_1,x_2,y_1,y_2},S_2^{x_1,x_2,y_1,y_2}).$$

Then μ obviously satisfies (i) and (ii). Hence the corresponding measure P_μ is concentrated on A.

b) Random curves joining two points.

Let $E\subset\mathbb{R}^d$ be compact with the Euclidean metric. Let μ be a probability measure on $\operatorname{Con}(E)^N$ such that there are $a,b\in E$ and, for μ-a.e. $(T_0,\ldots,T_{N-1})\in\operatorname{Con}(E)^N$, $T_0(a)=a$, $T_{N-1}(b)=b$, and $T_\rho(b)=T_{\rho+1}(a)$ for $\rho=0,\ldots,N-2$. Then P_μ-a.e. $K\in\mathscr{K}(E)$ is a curve (i.e. continuous image of $[0,1]$) joining a and b.

Proof. Let $X=[0,1]\times E$ be equipped with the euclidean metric. Define φ: $\operatorname{Con}(E)^N\to\operatorname{Con}(X)^N$ by

$$\varphi(T_0,\ldots,T_{N-1})=(S_0,\ldots,S_{N-1})$$

with

$$S_\rho(x,y)=\left(\frac{1}{N}x+\frac{\rho}{N},T_\rho y\right).$$

Then φ is continuous. Define $\bar\mu=\mu\circ\varphi^{-1}$. If

$$A=\{f\in\mathscr{C}([0,1],E)\,|\,f(0)=a,\ f(1)=b\}$$

then A is easily seen to satisfy condition (i) and (ii) in the preceding example for $\bar\mu$-a.e. $(S_0,\ldots,S_{N-1})\in\operatorname{Con}(X)$. Hence $P_{\bar\mu}$ is supported by A. Now define ξ: $\mathscr{K}(X)\to\mathscr{K}(E)$ by

$$\xi(K)=\{y\in E\,|\,\exists x\in[0,1]:\ (x,y)\in K\}.$$

Then ξ is Borel measurable and – using Theorem 4.5 – it can be checked that $P_\mu=P_{\bar\mu}\circ\xi^{-1}$.

Hence P_μ is supported by $\{\xi(K)|K\in A\}$ and each $\xi(K)$ is a curve joining a and b.

Remark. The present example is a stochastic version of Hutchinson's construction of parametrized curves ([6], pp. 730–731).

c) Random homeomorphisms.

Let $X=[0,1]^2$ be equipped with the Euclidean metric. Let $H\subset\mathcal{K}(X)$ be the set of all (graphs of) increasing homeomorphisms from $[0,1]$ onto itself. Define

$$d_H(Gr(h), Gr(h'))=\|h-h'\|_\infty+\|h^{-1}-h'^{-1}\|_\infty,$$

where $\|\ \|_\infty$ denotes the supremum norm and $Gr(h)$ the graph of the homeomorphism h. Then the topology of (H, d_H) coincides with the topology induced by the Hausdorff metric. Moreover, (H, d_H) is a complete metric space.

Define $\varphi\colon (0,1)\times(0,1)\to\mathrm{Con}([0,1]^2)^2$ by

$$\varphi(x,y)=(S_{x,y}, T_{x,y})$$

with

$$S_{x,y}(u,v)=(xu, yv)$$

and

$$T_{x,y}(u,v)=(x,y)+((1-x)u,(1-y)v).$$

Then φ is continuous.

For all $(x,y)\in(0,1)\times(0,1)$ and all $h, h'\in H$ we have $S_{x,y}(Gr(h))\cup T_{x,y}(Gr(h'))\in H$. Moreover

$$\begin{aligned}&d_H(S_{x,y}(Gr(h))\cup T_{x,y}(Gr(h')), S_{x,y}(Gr(g))\cup T_{x,y}(Gr(g')))\\ &\qquad\leqq\max\{x,y,1-x,1-y\}\max\{d_H(Gr(h),Gr(g)), d_H(Gr(h'),Gr(g'))\}.\end{aligned}$$

Let ν be any probability measure on $(0,1)\times(0,1)$ and $\mu=\nu\circ\varphi^{-1}$. Then Theorem 5.1 implies that P_μ is supported by H.

Remark. The measures P_μ of the present example have been introduced by Dubins-Freedman [2]. A detailed investigation of some of these measures can be found in [5].

§ 6. Probabilistic Tools for the Investigation of P_μ-Random Fractals

It is the purpose of this section to develop the tools for determining the Hausdorff dimension and Hausdorff measure of P_μ-random sets. To a large extend our results and techniques are inspired by the work of Mauldin-Williams [9].

In this section $g\colon \mathrm{Con}(X)\to[0,1)$ is a Borel measurable function. Recall that $\Omega=(\mathrm{Con}(X)^N)^D$. For $\Gamma\subset D$ and $\beta>0$ define $f_{\Gamma,\beta}\colon\Omega\to\mathbb{R}_+$ by

$$f_{\Gamma,\beta}(\mathcal{S})=\sum_{\sigma\in\Gamma}\prod_{n=1}^{|\sigma|}g(S_{\sigma|n})^\beta,\qquad f_{\{\emptyset\},\beta}\equiv 1,$$

and abbreviate $f_{C_q,\beta}$ by $f_{q,\beta}$.

As always, μ is a probability measure on $\mathrm{Con}(X)^N$.

6.1. Theorem. *The function* $\mathbb{R}_+\to\mathbb{R}_+$, $\beta\mapsto\int\sum_{\rho=0}^{N-1}g(S_\rho)^\beta\,d\mu(S_0,\dots,S_{N-1})$ (*where* $0^0:=0$) *is decreasing.*

If $\int \sum_{\rho=0}^{N-1} g(S_\rho)^0 \, d\mu(S_0, \ldots, S_{N-1}) > 1$ *then there exists a unique* $\alpha > 0$ *with* $\int \sum_{\rho=0}^{N-1} g(S_\rho)^\alpha \, d\mu(S_0, \ldots, S_{N-1}) = 1$.

Proof. The first part of the theorem is obviously true. The second part follows from the fact that the map

$$\beta \to \int \sum_{\rho=0}^{N-1} g(S_\rho)^\beta \, d\mu(S_0, \ldots, S_{N-1})$$

is continuous and strictly decreasing with

$$\lim_{\beta \to \infty} \int \sum_{\rho=0}^{N-1} g(S_\rho)^\beta \, d\mu(S_0, \ldots, S_{N-1}) = 0.$$

6.2. Definition. Let $\alpha = \alpha(g)$ denote the α in the conclusion of the preceding theorem, i.e. $\int \sum_{\rho=0}^{N-1} g(S_\rho)^\alpha \, d\mu(S_0, \ldots, S_{N-1}) = 1$.

The following theorem was proved by Mauldin-Williams [9] in a more general setting. Due to the special situation we are considering we are able to give a simple proof here.

6.3. Theorem (Mauldin-Williams [9]). *Suppose*

$$\int \sum_{\rho=0}^{N-1} g(S_\rho)^0 \, d\mu(S_0, \ldots, S_{N-1}) > 1.$$

Let $\alpha = \alpha(g)$. *For* $q \in \mathbb{N}$ *let* $\mathfrak{A}_q$ *be the* σ*-field of all Borel subsets in* $\Omega = (\mathrm{Con}(X)^N)^D$ *depending only on coordinates from* $D_q = \bigcup_{\rho=0}^{q} \{0, \ldots, N-1\}^\rho$.

Then, for every $p \in \mathbb{N}$, $(f_{q,\alpha})_{q \in \mathbb{N}}$ *is an* L^p*-bounded martingale w.r.t. to* $(\mathfrak{A}_q)_{q \in \mathbb{N}}$ *which converges* μ^D*-a.e. and in* $L^p(\Omega)$ *to a function* $f = f^{(g)}$.

If, for μ*-a.e.* $(S_0, \ldots, S_{N-1})$, $g(S_\rho) > 0$ *for* $\rho = 0, \ldots, N-1$ *then* $f > 0$ μ^D*-a.e.*

Proof. Clearly $(f_{q,\alpha})_{q \in \mathbb{N}}$ is a martingale w.r.t. $(\mathfrak{A}_q)_{q \in \mathbb{N}}$. By induction on $p \in \mathbb{N}$ we will prove that $(f_{q,\alpha})_{q \in \mathbb{N}}$ is L^p-bounded. Since $f_{q,\alpha} \geq 0$ and $(f_{q,\alpha})_{q \in \mathbb{N}}$ is a martingale it is obviously L^1-bounded.

Now assume $p > 1$ and that, for $m < p$, $(f_{q,\alpha})_{q \in \mathbb{N}}$ is L^m-bounded. Define $M = \sup\{\|f_{q,\alpha}\|_m \mid q \in \mathbb{N}, m < p\}$ where $\| \ \|_m$ denotes the L^m-norm. Then $M < \infty$. We claim that $(f_{q,\alpha})_{q \in \mathbb{N}}$ is L^p-bounded. By the definition of $f_{q+1,\alpha}$ we have

$$\begin{aligned}
\|f_{q+1,\alpha}\|_p^p &= \int f_{q+1,\alpha}^p \, d\mu^D \\
&= \iint \left(\sum_{\rho=0}^{N-1} g(S_\rho)^\alpha f_{q,\alpha}(\mathscr{S}^{(\rho)}) \right)^p d(\mu^D)^N (\mathscr{S}^{(0)}, \ldots, \mathscr{S}^{(N-1)}) \, d\mu(S_0, \ldots, S_{N-1}) \\
&= \sum_{\nu_0 + \ldots + \nu_{N-1} = p} \frac{p!}{\nu_0! \ldots \nu_{N-1}!} \int g(S_0)^{\nu_0 \alpha} \cdot \ldots \cdot g(S_{N-1})^{\nu_{N-1} \alpha} \\
&\quad \cdot \|f_{q,\alpha}\|_{\nu_0}^{\nu_0} \cdot \ldots \cdot \|f_{q,\alpha}\|_{\nu_{N-1}}^{\nu_{N-1}} \, d\mu(S_0, \ldots, S_{N-1})
\end{aligned}$$

$$= \int (g(S_0)^{p\alpha} + \ldots + g(S_{N-1})^{p\alpha})\, d\mu(S_0, \ldots, S_{N-1})\, \|f_{q,\alpha}\|_p^p$$
$$+ \sum_{\substack{\nu_0 + \ldots + \nu_{N-1} = p \\ \nu_0, \ldots, \nu_{N-1} < p}} \left[\frac{p!}{\nu_0! \ldots \nu_{N-1}!} \int g(S_0)^{\nu_0 \alpha} \cdot \ldots \cdot g(S_{N-1})^{\nu_{N-1}\alpha}\, d\mu(S_0, \ldots, S_{N-1}) \right]$$
$$\cdot \|f_{q,\alpha}\|_{\nu_0}^{\nu_0} \cdot \ldots \cdot \|f_{q,\alpha}\|_{\nu_{N-1}}^{\nu_{N-1}}.$$

Using backward induction on q this leads to

$$\|f_{q+1,\alpha}\|_p^p = [\int g(S_0)^{p\alpha} + \ldots + g(S_{N-1})^{p\alpha}\, d\mu(S_0, \ldots, S_{N-1})]^q\, \|f_{1,\alpha}\|_p^p$$
$$+ \sum_{\substack{\nu_0 + \ldots + \nu_{N-1} = p \\ \nu_0, \ldots, \nu_{N-1} < p}} \left[\frac{p!}{\nu_0! \ldots \nu_{N-1}!} \int g(S_0)^{\nu_0 \alpha} \cdot \ldots \cdot g(S_{N-1})^{\nu_{N-1}\alpha}\, d\mu(S_0, \ldots, S_{N-1}) \right.$$
$$\cdot \left[\sum_{\rho=0}^{q-1} (\int (g(S_0)^{p\alpha} + \ldots + g(S_{N-1})^{p\alpha})\, d\mu(S_0, \ldots, S_{N-1}))^\rho \right.$$
$$\left. \cdot \|f_{q-\rho,\alpha}\|_{\nu_0}^{\nu_0} \cdot \ldots \cdot \|f_{q-\rho,\alpha}\|_{\nu_{N-1}}^{\nu_{N-1}} \right]$$
$$\leqq [\int g(S_0)^{\alpha p} + \ldots + g(S_{N-1})^{p\alpha}\, d\mu(S_0, \ldots, S_{N-1})]^q\, \|f_{1,\alpha}\|_p^p$$
$$+ M^p \int (g(S_0)^\alpha + \ldots + g(S_{N-1})^\alpha)^p\, d\mu(S_0, \ldots, S_{N-1})$$
$$\cdot \sum_{\rho=0}^{q-1} (\int (g(S_0)^{p\alpha} + \ldots + g(S_{N-1})^{p\alpha})\, d\mu(S_0, \ldots, S_{N-1}))^\rho.$$

Since $0 \leqq f_{1,\alpha} \leqq N$, and by the choice of α,

$$\int g(S_0)^{p\alpha} + \ldots + g(S_{N-1})^{p\alpha}\, d\mu(S_0, \ldots, S_{N-1}) < 1$$

we deduce that $(\|f_{q,\alpha}\|_p^p)_{q \in \mathbb{N}}$ is bounded. This proves our claim.

By the martingale convergence theorem there is an f with $f \in L^p(\Omega)$ for all $p \in \mathbb{N}$, such that $(f_{q,\alpha})_{q \in \mathbb{N}}$ converges to f μ^D-a.e. and in $L^p(\Omega)$.

It remains to show that $f > 0$ μ^D-a.e. provided $g(S_\rho) > 0$ for μ-a.e. $(S_0, \ldots, S_{N-1})$.

We will postpone the proof until we have proved the following lemma.

6.4. Lemma. *Let the assumptions of the preceding theorem be satisfied and let f be as in that theorem.*

For $\sigma \in D$ let $\mathscr{S}^\sigma$ be defined by

$$\mathscr{S}_\tau^\sigma = \mathscr{S}_{\sigma * \tau} \qquad (\tau \in D).$$

Then, for μ^D-a.e. $\mathscr{S}$, the following conditions are satisfied:

(i) *For every $\sigma \in D$, $\lim_{q \to \infty} f_{q,\alpha}(\mathscr{S}^\sigma) = f(\mathscr{S}^\sigma)$.*

(ii) *For every minimal covering $\Gamma \subset D$,*

$$f(\mathscr{S}) = \sum_{\sigma \in \Gamma} \prod_{n=1}^{|\sigma|} g(S_{\sigma|n})^\alpha f(\mathscr{S}^\sigma).$$

Proof. Since $(f_{q,\alpha})_q$ converges to f μ^D-a.e. the first statement follows immediately from the fact that D is countable.

Let Γ be minimal and let $\mathscr{S}\in\Omega$ satisfy (i).

For every $q\in\mathbb{N}$ with $\Gamma\prec C_q$ we obtain

$$\begin{aligned} f_{q,\alpha}(\mathscr{S}) &= \sum_{\sigma\in C_q}\prod_{n=1}^{|\sigma|} g(S_{\sigma|n})^\alpha \\ &= \sum_{\sigma\in\Gamma}\sum_{\tau\in C_{q-|\sigma|}}\prod_{n=1}^{|\sigma|} g(S_{\sigma|n})^\alpha \prod_{\rho=1}^{q-|\sigma|} g(S_{\sigma*(\tau|\rho)})^\alpha \\ &= \sum_{\sigma\in\Gamma}\prod_{n=1}^{|\sigma|} g(S_{\sigma|n})^\alpha \sum_{\tau\in C_{q-|\sigma|}}\prod_{\rho=1}^{q-|\sigma|} g(S_{\sigma*(\tau|\rho)})^\alpha \\ &= \sum_{\sigma\in\Gamma}\prod_{n=1}^{|\sigma|} g(S_{\sigma|n})^\alpha f_{q-|\sigma|,\alpha}(\mathscr{S}^\sigma). \end{aligned}$$

Taking the limit over q yields the assertion of the lemma.

Continuation of the Proof of Theorem 6.3. We will show that $f>0$ μ^D-a.e. provided $g(S_\rho)>0$ for μ-a.e. $(S_0,\dots,S_{N-1})$.

Using Proposition 3.1 and Lemma 6.4 we deduce

$$\begin{aligned} \mu^D(\{\mathscr{S}\,|\,f(\mathscr{S})=0\}) &= \mu\otimes(\mu^D)^N\left(\left\{((S_0,\dots,S_{N-1}),(\mathscr{S}^{(0)},\dots,\mathscr{S}^{(N-1)}))\,\middle|\,\sum_{\rho=0}^{N-1} g(S_\rho)^\alpha f(\mathscr{S}^{(\rho)})=0\right\}\right) \\ &= \mu\otimes(\mu^D)^N(\{(S_0,\dots,S_{N-1}),(\mathscr{S}^{(0)},\dots,\mathscr{S}^{(N-1)})\,|\; \forall\rho=0,\dots,N-1:\ g(S_\rho)^\alpha f(\mathscr{S}^{(\rho)})=0\}). \end{aligned}$$

Since $g(S_\rho)>0$ for μ-a.e. $(S_0,\dots,S_{N-1})$ we deduce

$$\mu^D(\{\mathscr{S}\,|\,f(\mathscr{S})=0\})=(\mu^D(\{\mathscr{S}\,|\,f(\mathscr{S})=0\}))^N.$$

This implies $\mu^D(\{\mathscr{S}\,|\,f(\mathscr{S})=0\})=0$ or $=1$.

Since $\int f\,d\mu^D=\int f_{q,\alpha}\,d\mu^D=1$ we get

$$\mu^D(\{\mathscr{S}\,|\,f(\mathscr{S})=0\})=0.$$

Thus the proof of Theorem 6.3 is completed.

6.5. Corollary. *Suppose* $\int\sum_{\rho=0}^{N-1} g(S_\rho)^0\,d\mu(S_0,\dots,S_{N-1})>1$. *Let* $\alpha=\alpha(g)$. *Then*

$$\sup_{\Gamma_0\in\mathrm{Min}}\inf\{f_{\Gamma,\alpha}(\mathscr{S})\,|\,\Gamma\in\mathrm{Min},\,\Gamma\succ\Gamma_0\}<\infty$$

for μ^D*-a.e.* $\mathscr{S}\in\Omega$.

Proof. For μ^D-a.e. $\mathscr{S}$ we have

$$\sup_{\Gamma_0\in\mathrm{Min}}\inf\{f_{\Gamma,\alpha}(\mathscr{S})\,|\,\Gamma\in\mathrm{Min},\,\Gamma\succ\Gamma_0\}\le\sup_{q_0\in\mathbb{N}}\inf_{q\ge q_0} f_{q,\alpha}(\mathscr{S})=f(\mathscr{S}).$$

Since $\int f\,d\mu^D<\infty$ by Theorem 6.3 the corollary is proved.

Our next aim is to obtain a lower bound for

$$\liminf_{\Gamma \in \mathrm{Min}} f_{\Gamma,\beta} \quad \text{provided } \beta < \alpha(g).$$

6.6. Lemma. *Suppose* $\int \sum_{\rho=0}^{N-1} g(S_\rho)^0 \, d\mu(S_0, \ldots, S_{N-1}) > 1$. *For* $\beta < \alpha = \alpha(g)$ *and* μ^D*-a.e.* $\mathscr{S} \in \Omega$ *there exists an* $m \in \mathbb{N}$ *such that, for every* $\sigma \in D$ *with* $|\sigma| \geqq m$,

$$\prod_{n=1}^{|\sigma|} g(S_{\sigma|n})^\alpha f(\mathscr{S}^\sigma) \leqq \prod_{n=1}^{|\sigma|} g(S_{\sigma|n})^\beta,$$

(*where* $f = f^{(g)}$ *is defined in Theorem* 6.3).

Proof. Let $\sigma \in D$ and $p \in \mathbb{N}$ be arbitrary. Then, by Čebyshev's inequality,

$$\mu^D\left(\left\{\mathscr{S} \,\middle|\, \prod_{n=1}^{|\sigma|} g(S_{\sigma|n})^{\alpha-\beta} f(\mathscr{S}^\sigma) > 1\right\}\right) \leqq \int \prod_{n=1}^{|\sigma|} g(S_{\sigma|n})^{p(\alpha-\beta)} \, d\mu^D(\mathscr{S}) \int f(\mathscr{S})^p \, d\mu^D(\mathscr{S}).$$

Taking the union of the sets on the left-hand side when σ runs through C_q yields

$$\mu^D\left(\left\{\mathscr{S} \,|\, \exists \sigma \in C_q\colon \prod_{n=1}^{q} g(S_{\sigma|n})^{\alpha-\beta} f(\mathscr{S}^\sigma) > 1\right\}\right) \leqq \int \sum_{\sigma \in C_q} \prod_{n=1}^{|\sigma|} g(S_{\sigma|n})^{p(\alpha-\beta)} \, d\mu^D(\mathscr{S}) \int f(\mathscr{S})^p \, d\mu^D(\mathscr{S}).$$

For $p \in \mathbb{N}$ with $p(\alpha-\beta) > \alpha$ we have

$$\int \sum_{\rho=0}^{N-1} g(S_\rho)^{p(\alpha-\beta)} \, d\mu(S_0, \ldots, S_{N-1}) < 1.$$

Since

$$\int \sum_{\sigma \in C_q} \prod_{n=1}^{|\sigma|} g(S_{\sigma|n})^{p(\alpha-\beta)} \, d\mu^D(\mathscr{S}) = \left[\int \sum_{\rho=0}^{N-1} g(S_\rho)^{p(\alpha-\beta)} \, d\mu(S_0, \ldots, S_{N-1})\right]^q$$

and,

$\int f^p \, d\mu^D < \infty$ by Theorem 6.3, we deduce

$$\sum_{q=1}^{\infty} \mu^D\left(\left\{\mathscr{S} \,|\, \exists \sigma \in C_q\colon \prod_{n=1}^{q} g(S_{\sigma|n})^{\alpha-\beta} f(\mathscr{S}^\sigma) > 1\right\}\right) < \infty.$$

By the Borel-Cantelli lemma this yields

$$\mu^D\left(\bigcap_{m \in \mathbb{N}} \bigcup_{q \geqq m} \left\{\mathscr{S} \,|\, \exists \sigma \in C_q\colon \prod_{n=1}^{q} g(S_{\sigma|n})^{\alpha-\beta} f(\mathscr{S}^\sigma) > 1\right\}\right) = 0.$$

Taking the complement of the set on the left-hand side leads to the conclusion of the lemma.

6.7. Theorem. *Suppose* $\int \sum_{\rho=0}^{N-1} g(S_\rho)^0 \, d\mu(S_0, \ldots, S_{N-1}) > 1$. *Let* $\alpha = \alpha(g)$. *For* $\beta < \alpha$ *and* μ^D*-a.e.* $\mathscr{S} \in \Omega$,

$$\sup_{\Gamma_0 \in \mathrm{Min}} \inf\{f_{\Gamma,\beta}(\mathscr{S}) \mid \Gamma \in \mathrm{Min}, \Gamma \succ \Gamma_0\} \geqq f(\mathscr{S}).$$

Proof. Let $\mathscr{S} \in \Omega$ satisfy the following conditions

$$\exists m \in \mathbb{N} \;\; \forall \sigma \in D\colon \; |\sigma| \geqq m \Rightarrow \prod_{n=1}^{|\sigma|} g(S_{\sigma|n})^\alpha f(\mathscr{S}^\sigma) \leqq \prod_{n=1}^{|\sigma|} g(S_{\sigma|n})^\beta \tag{$*$}$$

and

$$\forall \Gamma \in \mathrm{Min}\colon \; f(\mathscr{S}) = \sum_{\sigma \in \Gamma} \prod_{n=1}^{|\sigma|} g(S_{\sigma|n}) \, f(\mathscr{S}^\sigma). \tag{$\substack{*\\*}$}$$

Then we obtain, for every $\Gamma \in \mathrm{Min}$ with $\Gamma \succ C_m$,

$$f(\mathscr{S}) = \sum_{\sigma \in \Gamma} \prod_{n=1}^{|\sigma|} g(S_{\sigma|n})^\alpha f(\mathscr{S}^\sigma) \leqq \sum_{\sigma \in \Gamma} \prod_{n=1}^{|\sigma|} g(S_{\sigma|n})^\beta = f_{\Gamma,\beta}(\mathscr{S}),$$

i.e.

$$f(\mathscr{S}) \leqq \sup_{\Gamma_0 \in \mathrm{Min}} \inf\{f_{\Gamma,\beta}(\mathscr{S}) \mid \Gamma \in \mathrm{Min}, \Gamma \succ \Gamma_0\}.$$

By Lemma 6.6 and Lemma 6.4 μ^D-a.e. $\mathscr{S} \in \Omega$ satisfies conditions $(*)$ and $(\substack{*\\*})$. Thus the theorem is proved.

6.8. Theorem. *Suppose that, for* μ*-a.e.* $(S_0, \ldots, S_{N-1}) \in \mathrm{Con}(X)^N$ *and* $\rho = 0, \ldots, N-1$, $g(S_\rho) > 0$. *Let* $\beta < \alpha = \alpha(g)$ *and* $d \in \mathbb{N}$ *be arbitrary. Then, for* μ^D*-a.e.* $\mathscr{S}$,

$$\sup_{\Gamma_0} \inf\left\{ \sum_{\sigma \in \Gamma} g(S_\sigma)^d \prod_{n=1}^{|\sigma|} g(S_{\sigma|n})^\beta \,\middle|\, \Gamma \in \mathrm{Min}, \Gamma \succ \Gamma_0 \right\} > 0.$$

Proof. Since $\int \sum_{\rho=0}^{N-1} g(S_\rho)^\beta \, d\mu(S_0, \ldots, S_{N-1}) > 1$ there exists an $\eta > 0$ such that, for $A = \{(S_0, \ldots, S_{N-1}) \mid g(S_\rho) \geqq \eta$ for $\rho = 0, \ldots, N-1\}$,

$$\int_A \sum_{\rho=0}^{N-1} g(S_\rho)^\beta \, d\mu(S_0, \ldots, S_{N-1}) > 1. \tag{$*$}$$

Define $g_\eta\colon \mathrm{Con}(X) \to [0, 1)$ by

$$g_\eta(S) = \begin{cases} 0, & g(S) < \eta \\ g(S), & g(S) \geqq \eta. \end{cases}$$

Then $(*)$ implies $\alpha(g_\eta) > \beta$. Let $f^{(\eta)}$ stand for $f^{(g_\eta)}$. For all $\mathscr{S} \in \Omega$ we have

$$\sup_{\Gamma_0} \inf_{\Gamma \succ \Gamma_0} \sum_{\sigma \in \Gamma} g(S_\sigma)^d \prod_{n=1}^{|\sigma|} g(S_{\sigma|n})^\beta \geqq \sup_{\Gamma_0} \inf_{\Gamma \succ \Gamma_0} \sum_{\sigma \in \Gamma} g_\eta(S_\sigma)^d \prod_{n=1}^{|\sigma|} g_\eta(S_{\sigma|n})^\beta$$

$$\geqq \eta^d \sup_{\Gamma_0} \inf_{\Gamma \succ \Gamma_0} \sum_{\sigma \in \Gamma} \prod_{n=1}^{|\sigma|} g_\eta(S_{\sigma|n})^\beta.$$

By Theorem 6.7 the last expression in the above inequality is greater or equal to $f^{(n)}(\mathscr{S})$ for μ^D-a.e. $\mathscr{S}$. Since, by Theorem 6.3, $\int f^{(n)} d\mu^D > 0$ we deduce that

$$\sup_{\Gamma_0} \inf_{\Gamma \succ \Gamma_0} \sum_{\sigma\in\Gamma} g(S_\sigma)^d \prod_{n=1}^{|\sigma|} g(S_{\sigma|n})^\beta > 0 \tag{$\ast\atop\ast$}$$

with positive probability.

To complete the proof we will show that the left-hand side in $({\ast\atop\ast})$ is either 0 with probability 1 or >0 with probability 1.

By Proposition 3.1 we have

$$\begin{aligned}
&\mu^D\left(\left\{\mathscr{S}\,\middle|\,\sup_{\Gamma_0}\inf_{\Gamma\succ\Gamma_0}\sum_{\sigma\in\Gamma} g(S_\sigma)^d \prod_{n=1}^{|\sigma|} g(S_{\sigma|n})^\beta = 0\right\}\right)\\
&= \mu\otimes(\mu^D)^N\left(\left\{((S_0,\dots,S_{N-1}),(\mathscr{S}^{(0)},\dots,\mathscr{S}^{(N-1)}))\,\middle|\right.\right.\\
&\qquad\left.\left.\sup_{\Gamma_0}\inf_{\Gamma\succ\Gamma_0}\sum_{\rho=0}^{N-1} g(S_\rho)^\beta \sum_{\substack{\sigma\\ \rho*\sigma\in\Gamma}} g(S_\sigma^{(\rho)})^d \prod_{n=1}^{|\sigma|} g(S_{\sigma|n}^{(\rho)})^\beta = 0\right\}\right)\\
&= \mu\otimes(\mu^D)^N\left(\left\{((S_0,\dots,S_{N-1}),(\mathscr{S}^{(0)},\dots,\mathscr{S}^{(N-1)}))\,\middle|\right.\right.\\
&\qquad\left.\left.\sum_{\rho=0}^{N-1} g(S_\rho)^\beta \sup_{\Gamma'}\inf_{\Gamma\succ\Gamma'}\sum_{\sigma\in\Gamma} g(S_\sigma^{(\rho)})^d \prod_{n=1}^{|\sigma|} g(S_{\sigma|n}^{(\rho)})^\beta = 0\right\}\right)\\
&= \left[\mu^D\left(\left\{\mathscr{S}\,\middle|\,\sup_{\Gamma'}\inf_{\Gamma\succ\Gamma'}\sum_{\sigma\in\Gamma} g(S_\sigma)^d \prod_{n=1}^{|\sigma|} g(S_{\sigma|n}) = 0\right\}\right)\right]^N;
\end{aligned}$$

since $g(S_\rho) > 0$ for μ-a.e. $(S_0, \dots, S_{N-1})$.

This implies our claim and the theorem is proved.

6.9. Definition. For $\Gamma \subset D$ let

$$|\Gamma| = \max\{|\sigma|:\ \sigma\in\Gamma\}.$$

6.10. Lemma. *Suppose* $\int \sum_{\rho=0}^{N-1} g(S_\rho)^0\, d\mu(S_0, \dots, S_{N-1}) > 1$. *Let* $\alpha = \alpha(g)$. *For* $n \in \mathbb{N}$ *define* $h_n\colon \Omega \to \mathbb{R}_+$ *by*

$$h_n(\mathscr{S}) = \inf\{f_{\Gamma,\alpha}(\mathscr{S}) \mid \Gamma\in\mathrm{Min},\ \Gamma \neq \{\emptyset\},\ |\Gamma| \leqq n\}.$$

Then $(h_n)_{n\in\mathbb{N}}$ *is a non-increasing sequence of Borel measurable functions with the following properties*

(i) $\forall n\in\mathbb{N}\ \forall \mathscr{S}\in\Omega\colon\ h_{n+1}(\mathscr{S}) = \sum_{\rho=0}^{N-1} g(S_\rho)^\alpha \min(1, h_n(\mathscr{S}^\rho))$.

(ii) $h := \inf_{n\in\mathbb{N}} h_n = \inf_{\Gamma\in\mathrm{Min}\setminus\{\{\emptyset\}\}} f_{\Gamma,\alpha} \geqq 0$.

(iii) *The following properties are equivalent:*

a) $h > 0$ *on a set of positive* μ^D*-measure.*

b) $h > 0$ μ^D*-a.e.*

c) $\sum_{\rho=0}^{N-1} g(S_\rho)^\alpha = 1$ μ*-a.e.*

Proof. By definition $(h_n)_{n\in\mathbb{N}}$ is a decreasing sequence of Borel measurable functions.

(i) For $\Gamma\in\mathrm{Min}$ and $\rho\in\{0,\ldots,N-1\}$ let

$$\Gamma(\rho)=\{\sigma\in D\,|\,\rho*\sigma\in\Gamma\}.$$

It can be shown that $\Gamma(\rho)\in\mathrm{Min}$ for $\rho=0,\ldots,N-1$.

Obviously, for every $\mathscr{S}$,

$$f_{\Gamma,\alpha}(\mathscr{S})=\sum_{\rho=0}^{N-1} g(S_\rho)^\alpha f_{\Gamma(\rho),\alpha}(\mathscr{S}^\rho).$$

Using this last identity we obtain

$$\begin{aligned}h_{n+1}(\mathscr{S})&=\inf\{f_{\Gamma,\alpha}(\mathscr{S})\,|\,\Gamma\in\mathrm{Min},\,\Gamma\neq\{\emptyset\},\,|\Gamma|\leqq n+1\}\\&=\sum_{\rho=0}^{N-1} g(S_\rho)^\alpha\inf\{f_{\Gamma(\rho),\alpha}(\mathscr{S}^\rho)\,|\,\Gamma\in\mathrm{Min},\,\Gamma\neq\{\emptyset\},\,|\Gamma|\leqq n+1\}\\&=\sum_{\rho=0}^{N-1} g(S_\rho)^\alpha\min(1,\inf\{f_{\Lambda,\alpha}(\mathscr{S}^\rho)\,|\,\Lambda\in\mathrm{Min},\,\Lambda\neq\{\emptyset\},\,|\Lambda|\leqq n\}\\&=\sum_{\rho=0}^{N-1} g(S_\rho)^\alpha\min(1,h_n(\mathscr{S}^\rho)).\end{aligned}$$

(ii) is obviously true.

(iii) a)$\Rightarrow$c). It follows from (i) and (ii) that

$$h(\mathscr{S})=\sum_{\rho=0}^{N-1} g(S_\rho)^\alpha\min(1,h(\mathscr{S}^\rho)) \tag{1}$$

for all $\mathscr{S}\in\Omega$.

Since $g\leqq 1$ Eq. (1) implies that h is bounded by N, hence μ^D-integrable.

Using Proposition 3.1 and (1) yields

$$\begin{aligned}\int h\,d\mu^D&=\iint\sum_{\rho=0}^{N-1} g(S_\rho)^\alpha\min(1,h(\mathscr{S}^{(\rho)}))\,d(\mu^D)^N(\mathscr{S}^{(0)},\ldots,\mathscr{S}^{(N-1)})\,d\mu(S_0,\ldots,S_{N-1})\\&=\sum_{\rho=0}^{N-1}\int g(S_\rho)^\alpha\,d\mu(S_0,\ldots,S_{N-1})\int\min(1,h(\mathscr{S}))\,d\mu^D(\mathscr{S}).\end{aligned}$$

Since, by the definition of α,

$$\sum_{\rho=0}^{N-1}\int g(S_\rho)^\alpha\,d\mu(S_0,\ldots,S_{N-1})=1.$$

we deduce

$$\int h\,d\mu^D=\int\min(1,h)\,d\mu^D$$

and, hence,

$$h\leqq 1\quad \mu^D\text{-a.e.} \tag{2}$$

Let η be the essential supremum of h. Using (1) and Proposition 3.1 another time we obtain

$$\sum_{\rho=0}^{N-1} g(S_\rho)^\alpha h(\mathscr{S}^{(\rho)}) \leqq \eta \tag{3}$$

for μ-a.e. $(S_0, \ldots, S_{N-1}) \in \mathrm{Con}(X)^N$ and $(\mu^D)^N$-a.e. $(\mathscr{S}^{(0)}, \ldots, \mathscr{S}^{(N-1)}) \in \Omega^N$.

By our assumption $\eta > 0$. Thus dividing the last equation by η and observing that $h \leqq \eta$ μ^D-a.e. yields

$$\sum_{\rho=0}^{N-1} g(S_\rho)^\alpha \leqq 1 \tag{4}$$

for μ-a.e. $(S_0, \ldots, S_{N-1})$.

Since $\int \sum_{\rho=0}^{N-1} g(S_\rho)^\alpha \, d\mu(S_0, \ldots, S_{N-1}) = 1$ this leads to

$$\sum_{\rho=0}^{N-1} g(S_\rho)^\alpha = 1$$

for μ-a.e. $(S_0, \ldots, S_{N-1})$.

c)$\Rightarrow$b) If $\sum_{\rho=0}^{N-1} \mathrm{Lip}(S_\rho)^\alpha = 1$ for μ-a.e. $(S_0, \ldots, S_{N-1})$ then

$$h_1 = 1 \quad \mu^D\text{-a.e.}$$

and, therefore, by (i)

$$h_n = 1 \quad \mu^D\text{-a.e.}$$

hence

$$h = 1 \quad \mu^D\text{-a.e.}$$

b)$\Rightarrow$a) is trivial.

6.11. Theorem. *Suppose* $\int \sum_{\rho=0}^{N-1} g(S_\rho)^0 \, d\mu(S_0, \ldots, S_{N-1}) > 1$. *Let* $\alpha = \alpha(g)$. *Then the following conditions are equivalent*

(i) $\sum_{\rho=0}^{N-1} g(S_\rho)^\alpha = 1$ *for* μ*-a.e.* $(S_0, \ldots, S_{N-1})$.

(ii) $\sup_{\Gamma_0 \in \mathrm{Min}} \inf \{ f_{\Gamma,\alpha}(\mathscr{S}) \mid \Gamma \in \mathrm{Min}, \Gamma \succ \Gamma_0 \} > 0$.

for μ^D*-a.e.* $\mathscr{S}$

(iii) $\mu^D(\{\mathscr{S} \mid \sup_{\Gamma_0} \inf_{\Gamma \succ \Gamma_0} f_{\Gamma,\alpha}(\mathscr{S}) > 0\}) > 0$.

Proof. (i)$\Rightarrow$(ii): Under the hypothesis that $\sum_{\rho=0}^{N-1} g(S_\rho)^\alpha = 1$ μ-a.e. it is easy to check that, for every $\Gamma \in \mathrm{Min}$,

$$f_{\Gamma,\alpha} = 1 \quad \mu^D\text{-a.e.}$$

This implies

$$\sup_{\Gamma_0} \inf_{\Gamma \succ \Gamma_0} f_{\Gamma,\alpha} = 1 \quad \mu^D\text{-a.e.}$$

(ii)⇒(iii) is obviously true.

(iii)⇒(i): Let $\Gamma_0 \in \mathrm{Min}$ be given. For $\Gamma \succ \Gamma_0$ and $\sigma \in \Gamma_0$ let $\Gamma_\sigma = \{\tau \in D \mid \sigma * \tau \in \Gamma\}$. Then $\Gamma_\sigma \in \mathrm{Min}$, and for every $\mathscr{S} \in \Omega$, we obtain

$$\begin{aligned}
\inf_{\Gamma \succ \Gamma_0} f_{\Gamma,\alpha}(\mathscr{S}) &= \inf_{\Gamma \succ \Gamma_0} \sum_{\sigma \in \Gamma_0} \left[\left(\prod_{n=1}^{|\sigma|} g(S_{\sigma|n})^\alpha\right) \sum_{\tau \in \Gamma_\sigma} \prod_{\rho=1}^{|\tau|} g(S_{\sigma*(\tau|\rho)})^\alpha\right] \\
&= \sum_{\sigma \in \Gamma_0} \left[\left(\prod_{n=1}^{|\sigma|} g(S_{\sigma|n})^\alpha\right) \inf_{\Gamma \succ \Gamma_0} \sum_{\tau \in \Gamma_\sigma} \prod_{\rho=1}^{|\tau|} g(S_{\sigma*(\tau|\rho)})^\alpha\right] \\
&= \sum_{\sigma \in \Gamma_0} \left[\left(\prod_{N-1}^{|\sigma|} g(S_{\sigma|n})^\alpha\right) \min(1, \inf_{\Gamma \in \mathrm{Min}\setminus\{\{\emptyset\}\}} f_{\Gamma,\alpha}(\mathscr{S}^\sigma))\right] \\
&= \sum_{\sigma \in \Gamma_0} \left(\prod_{n=1}^{|\sigma|} g(S_{\sigma|n})^\alpha\right) \min(1, h(\mathscr{S}^\sigma)),
\end{aligned}$$

where h is defined in Lemma 6.10.

According to (iii) there is a Borel set $B \subset \Omega$ with $\mu^D(B) > 0$ such that, for every $\mathscr{S} \in B$, there exists a Γ_0 with $\inf_{\Gamma \succ \Gamma_0} f_{\Gamma,\alpha}(\mathscr{S}) > 0$. By the preceding considerations this implies that, for every $\mathscr{S} \in B$, there is a $\Gamma_0 \in \mathrm{Min}$ and a $\sigma \in \Gamma_0$ with $h(\mathscr{S}^\sigma) > 0$. For $\sigma \in D$ let $\Omega(\sigma) = \{\mathscr{S} \mid h(\mathscr{S}^\sigma) > 0\}$.

Since $B \subset \bigcup_{\sigma \in D} \Omega(\sigma)$ we get $\mu^D(\bigcup_{\sigma \in D} \Omega(\sigma)) > 0$. Hence there exists a $\sigma \in D$ with $\mu^D(\Omega(\sigma)) > 0$. Since $\mu^D(\Omega(\sigma)) = \mu^D(\{\mathscr{S} \mid h(\mathscr{S}) > 0\})$ Lemma 6.10 yields condition (i) in the theorem.

6.12. Remark. Using the techniques developed in this section it can be shown that, if $\sum_{\rho=0}^{N-1} g(S_\rho)^\alpha = 1$ does not hold μ-a.e., then for every $p \in \mathbb{N}$, $(f_{\Gamma,\alpha})_{\Gamma \in \mathrm{Min}}$ is an L^p-bounded martingale with a countable index set satisfying

$$\limsup_{\Gamma \in \mathrm{Min}} f_{\Gamma,\alpha} = \infty \quad \mu^D\text{-a.e.}$$

and

$$\liminf_{\Gamma \in \mathrm{Min}} f_{\Gamma,\alpha} = 0 \quad \mu^D\text{-a.e.}$$

Another example of a martingale with a countable index set which is not a.e.-convergent has been given by Dieudonné [1].

§ 7. Hausdorff-Dimension and Hausdorff-Measure of P_μ-Random Fractals

In this section we prove that – under rather weak conditions – there is a number $\alpha \in \bar{\mathbb{R}}_+$ such that P_μ-a.e. compact set has Hausdorff dimension α. Under more restrictive assumptions we calculate the number α, thereby reproving more general results of Mauldin-Williams [9] and Falconer [4] in our special situation. Moreover we show that in most cases the α-dimensional Hausdorff measure of P_μ-a.e. compact set is 0. This result answers a question of Mauldin-Williams [9] in the negative.

7.1. Lemma. *Let $\beta \in \mathbb{R}_+$ and $\delta > 0$ be given. Then the following maps from $\mathscr{K}(X)$ to $\bar{\mathbb{R}}_+$ are Borel measurable.*

(i) $K \to \mathscr{H}_\delta^\beta(K)$.

(ii) $K \mapsto \mathscr{H}^\beta(K)$.

(iii) $K \to H\text{-}\dim(K)$.

Proof. (i) Let $K_0 \in \mathscr{K}(X)$ and $\varepsilon > 0$ be arbitrary. Then there exists a finite open covering $(G_n)_n$ of K_0 with $\operatorname{diam}(G_n) \leqq \delta$ for all n and

$$\sum_n \operatorname{diam}(G_n)^\beta < \mathscr{H}_\delta^\beta(K_0) + \varepsilon < \infty.$$

Let $G = \bigcup_n G_n$ and $\delta' = d(K, X \setminus G) > 0$. For every $K \in \mathscr{K}(X)$ with $\eta(K_0, K) < \delta'$ we have $K \subset G$ and, therefore, $(G_n)_n$ is an open covering of K, hence

$$\mathscr{H}_\delta^\beta(K) < \sum_n \operatorname{diam}(G_n)^\beta < \mathscr{H}_\delta^\beta(K_0) + \varepsilon.$$

Thus, the function $K \to \mathscr{H}_\delta^\beta(K)$ is upper-semi-continuous, hence Borel measurable.

(ii) Since $\mathscr{H}^\alpha(K) = \sup_{n \in \mathbb{N}} \mathscr{H}_{\frac{1}{n}}^\alpha(K)$ the second assertion follows immediately from the first one.

(iii) For $\beta \in \mathbb{R}_+$ the set

$$\{K \in \mathscr{K}(X) \mid H\text{-}\dim(K) > \beta\}$$

equals

$$\bigcup_{n \in \mathbb{N}} \{K \in \mathscr{K}(X) \mid \mathscr{H}^{\beta + \frac{1}{n}}(K) > 0\}.$$

Hence (iii) follows from (ii).

The following $0-1$-law does not seem to be an immediate consequence of one of the classical $0-1$ laws.

7.2. Theorem. *Suppose that, for μ-a.e. $(S_0, \ldots, S_{N-1}) \in \operatorname{Con}(X)^N$ and every $\rho \in \{0, \ldots, N-1\}$, there exists a $c > 0$ with $d(S_\rho x, S_\rho y) \geqq c\, d(x, y)$ for all $x, y \in X$. Let $\beta \geqq 0$ be given. Then P_μ has the following properties*

(i) $P_\mu(\{K \in \mathscr{K}(X) \mid \mathscr{H}^\beta(K) = 0\}) = 0$ *or* $= 1$.

(ii) $P_\mu(\{K \in \mathscr{K}(X) \mid \mathscr{H}^\beta(K) = \infty\}) = 0$ *or* $= 1$.

Proof. (i) According to Theorem 4.5 we have

$$\begin{aligned} P_\mu(\{K \mid \mathscr{H}^\beta(K) = 0\}) &= \mu \otimes (P_\mu)^N \left(\left\{ ((S_0, \ldots, S_{N-1}), (K_0, \ldots, K_{N-1})) \right| \right. \\ &\qquad \left. \left. \mathscr{H}^\beta \left(\bigcup_{\rho=0}^{N-1} S_\rho(K_\rho) \right) = 0 \right\} \right) \\ &= \mu \otimes P_\mu^N(\{((S_0, \ldots, S_{N-1}), (K_0, \ldots, K_{N-1})) \mid \\ &\qquad \mathscr{H}^\beta(S_\rho(K_\rho)) = 0 \quad \text{for } \rho = 0, \ldots, N-1\}). \end{aligned}$$

According to our assumption we have, for μ-a.e. $(S_0, \ldots, S_{N-1})$ and every ρ, that $\mathscr{H}^\beta(S_\rho(K_\rho))=0$ if and only if $\mathscr{H}^\beta(K_\rho)=0$. This yields

$$P_\mu(\{K \mid \mathscr{H}^\beta(K)=0\})=(P_\mu(\{K \mid \mathscr{H}^\beta(K)=0\}))^N$$

which proves (i).

(ii) is proved in the same way.

7.3. Corollary. *Let the assumptions of Theorem 7.2 be satisfied. Then there exists an* $\alpha \geqq 0$ *such that*

$$H\text{-}\dim(K)=\alpha$$

for P_μ*-a.e.* $K \in \mathscr{K}(X)$.

Proof. Define $\alpha=\inf\{\beta \geqq 0 \mid \mathscr{H}^\beta(K)=0$ for P_μ-a.e. $K \in \mathscr{K}(X)\}$. If $\alpha=\infty$ we obviously have H-$\dim(K) \leqq \alpha$ for P_μ-a.e. Suppose $\alpha<\infty$. Then there exists a sequence $\beta_n \downarrow \alpha$ such that $\mathscr{H}^{\beta_n}(K)=0$ for P_μ-a.e. K and all $n \in \mathbb{N}$. This obviously yields H-$\dim(K) \leqq \alpha$ for P_μ-a.e. $K \in \mathscr{K}(X)$.

If $\alpha=0$ then the corollary is proved. Suppose $\alpha>0$. Let $(\beta_n)_n$ be sequence in $(0, \alpha)$ with $\beta_n \uparrow \alpha$. By Theorem 7.2(i) and the definition of α we have $\mathscr{H}^{\beta_n}(K)>0$ for P_μ-a.e. K and every $n \in \mathbb{N}$. This implies H-$\dim(K) \geqq \alpha$ for P_μ-a.e. K.

The following theorem gives an upper bound for the Hausdorff-dimension of P_μ-random fractals. It has been proved by Mauldin-Williams [9] in a more general context.

7.4. Theorem. *Suppose* $\int \sum_{\rho=0}^{N-1} (\operatorname{Lip}(S_\rho))^0 \, d\mu(S_0, \ldots, S_{N-1})>1$ *and let* $\alpha=\alpha(\operatorname{Lip})$, *i.e.*

$$\int \sum_{\rho=0}^{N-1} \operatorname{Lip}(S_\rho)^\alpha \, d\mu(S_0, \ldots, S_{N-1})=1.$$

Then $\mathscr{H}^\alpha(K)<\infty$ *and, in particular,* H-$\dim(K) \leqq \alpha$ *for* P_μ*-a.e.* $K \in \mathscr{K}(X)$.

Proof. Let $\psi: \Omega \rightarrow \mathscr{K}(X)$ be as defined in Theorem 3.7. By definition $P_\mu=\mu^D \circ \psi^{-1}$. To prove the theorem it is, therefore, enough to show $\mathscr{H}^\alpha(\psi(\mathscr{S}))<\infty$ for μ^D-a.e. $\mathscr{S}$. According to Theorem 3.3 and Theorem 2.4 we have

$$\mathscr{H}^\alpha(\psi(\mathscr{S})) \leqq \operatorname{diam}(X)^\alpha \sup_{\Gamma_0} \inf_{\Gamma \succ \Gamma_0} f_{\Gamma, \alpha}(\mathscr{S})$$

for μ^D-a.e. $\mathscr{S}$. By Corollary 6.5 this last expression is finite μ^D-a.e.

As the following example shows it may happen that H-$\dim(K)=\beta$ for P_μ-a.e. K and $\beta<\alpha=\alpha(\operatorname{Lip})$. (This holds for the graphs of most of the random homeomorphisms constructed in Example 5.3c.)

7.5. Example. Consider $X=[0,1]^2$ with the Euclidean metric.

Define $T_0: X \rightarrow X$ by $T_0(x, y)=\left(\frac{1}{2} x, \frac{\sqrt{3}}{2} y\right)$

and $T_1: X \rightarrow X$ by $T_1(x, y)=\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)+\left(\frac{1}{2}, \left(1-\frac{\sqrt{3}}{2}\right) y\right)$.

Let $\mu=\varepsilon_{(T_0,T_1)}$ be the Dirac measure on $\mathrm{Con}(X)^2$ concentrated in the point (T_0,T_1). Then we have $\mathrm{Lip}(T_0)=\frac{\sqrt{3}}{2}$ and $\mathrm{Lip}(T_1)=\frac{1}{2}$.

Hence $\alpha=\alpha(\mathrm{Lip})$ is determined by the following equation:

$$\begin{aligned}1&=\int(\mathrm{Lip}(S_0)^\alpha+\mathrm{Lip}(S_1)^\alpha)\,d\mu(S_0,S_1)\\&=\mathrm{Lip}(T_0)^\alpha+\mathrm{Lip}(T_1)^\alpha=\left(\frac{\sqrt{3}}{2}\right)^\alpha+\left(\frac{1}{2}\right)^\alpha\end{aligned}$$

which yields $\alpha=2$.

But according to Example 5.3c) we know that P_μ-a.e. $K\in\mathscr{K}(X)$ is the graph of an increasing homeomorphism of $[0,1]$ onto itself and, therefore, has Hausdorff dimension 1.

The following theorem states conditions which ensure that $\alpha=\alpha(\mathrm{Lip})$ is equal to the Hausdorff dimension of P_μ-a.e. compact set. It is a special case of a result of Mauldin-Williams [9]. The result of Mauldin-Williams has also independently been proved by Falconer [4] under the additional assumption that the Lipschitz constants of the maps involved are all uniformly bounded away from zero. In the deterministic case $\mu=\varepsilon_{(S_0,\ldots,S_{N-1})}$ the following result was first proved by Moran [10]. Another dimension formula for deterministic self-similar sets can be found in Ruelle [12].

7.6. Theorem (Mauldin-Williams [9], Falconer [4]). *Let $X\subset\mathbb{R}^d$ be compact with $\mathring{X}\neq\emptyset$. Let d be the Euclidean metric on X. Suppose that, for μ-a.e. $(S_0,\ldots,S_{N-1})\in\mathrm{Con}(X)^N$, the following conditions are satisfied:*

a) $\forall\rho\in\{0,\ldots,N-1\}$: *$S_\rho$ is a similarity*

$$(i.e.\ \exists r>0\ \forall x,y\in X:\ d(S_\rho x,S_\rho y)=r\,d(x,y)).$$

b) $\forall\rho,\rho'\in\{0,\ldots,N-1\}$: $\rho\neq\rho'\Rightarrow S_\rho(\mathring{X})\cap S_{\rho'}(\mathring{X})=\emptyset$.

Let $\alpha\geqq 0$ be such that

$$\int\sum_{\rho=0}^{N-1}\mathrm{Lip}(S_\rho)^\alpha\,d\mu(S_0,\ldots,S_{N-1})=1.$$

Then H-$\dim(K)=\alpha$ for P_μ-a.e. $K\in\mathscr{K}(X)$.

Proof. By Theorem 7.4 we have H-$\dim(K)\leqq\alpha$ for P_μ-a.e. $K\in\mathscr{K}(X)$. To prove the converse inequality it suffices to show that, for all $\beta<\alpha$,

$$\mathscr{H}^\beta(K)>0\qquad\text{for } P_\mu\text{-a.e. } K\in\mathscr{K}(X). \tag{1}$$

By the definition of P_μ this is equivalent to

$$\mathscr{H}^\beta(\psi(\mathscr{S}))>0\qquad\text{for } \mu^D\text{-a.e. } \mathscr{S}, \tag{2}$$

where ψ is defined in Theorem 3.7.

According to conditions a) and b) we know that the assumptions of Theorem 2.5 are satisfied for μ^D-a.e. $\mathscr{S}$. Hence Theorem 2.5 yields that there exists a $c>0$ such that

$$c\,\mathrm{diam}(X)^\beta\sup_{\Gamma_0}\inf_{\Gamma>\Gamma_0}\sum_{\sigma\in\Gamma}\mathrm{Lip}(S_\sigma)^d\prod_{n=1}^{|\sigma|}\mathrm{Lip}(S_{\sigma|n})^\beta\leqq\mathscr{H}^\beta(\psi(\mathscr{S})) \tag{3}$$

for μ^D-a.e. $\mathscr{S}$. By assumption a) we have $\mathrm{Lip}(S_\rho)>0$ ($\rho=0,\ldots,N-1$) for μ-a.e. $(S_0,\ldots,S_{N-1})\in\mathrm{Con}(X)^N$. Thus Theorem 6.8 is applicable and yields that the first expression in (3) is positive for μ^D-a.e. $\mathscr{S}$. Hence (2) is proved.

In the following theorem the α-dimensional Hausdorff measure of P_μ-a.e. $K\in\mathscr{K}(X)$ is determined, where $\alpha=\alpha(\mathrm{Lip})$. The theorem answers a question of Mauldin-Williams ([9], Question 3.8) in the negative.

7.7. Theorem. *Let the assumptions of Theorem 7.6 be satisfied. Suppose*

$$\mu\left(\left\{(S_0,\ldots,S_{N-1})\,\middle|\,\sum_{\rho=0}^{N-1}\mathrm{Lip}(S_\rho)^\alpha\neq 1\right\}\right)>0.$$

Then, $\mathscr{H}^\alpha(K)=0$ for P_μ-a.e. $K\in\mathscr{K}(X)$.

Proof. By the definition of P_μ we have to prove

$$\mathscr{H}^\alpha(\psi(\mathscr{S}))=0 \quad \text{for } \mu^D\text{-a.e. } \mathscr{S}. \tag{$*$}$$

By Theorem 2.4 and Theorem 3.3 we deduce

$$\mathscr{H}^\alpha(\psi(\mathscr{S}))\leqq \mathrm{diam}(X)^\alpha \sup_{\Gamma_0}\inf_{\Gamma>\Gamma_0}\sum_{\sigma\in\Gamma}\prod_{n=1}^{|\sigma|}\mathrm{Lip}(S_{\sigma|n})^\alpha$$

for μ^D-a.e. $\mathscr{S}$. Hence Theorem 6.11 implies ($*$).

7.8. Theorem. *Let the assumptions of Theorem 7.6 be satisfied. Suppose that there exists a $\delta>0$ such that $\mathrm{Lip}(S_\rho)\geqq\delta$ for $\rho=0,\ldots,N-1$ and μ-a.e. $(S_0,\ldots,S_{N-1})\in\mathrm{Con}(X)^N$. Then the following statements are equivalent:*

(i) $\sum_{\rho=0}^{N-1}\mathrm{Lip}(S_\rho)^\alpha=1$ *for μ-a.e. $(S_0,\ldots,S_{N-1})\in\mathrm{Con}(X)^N$.*

(ii) $\mathscr{H}^\alpha(K)>0$ *for P_μ-a.e. $K\in\mathscr{K}(X)$.*

(iii) $P_\mu(\{K\in\mathscr{K}(X)\,|\,\mathscr{H}^\alpha(K)>0\})>0$.

Proof. (i)$\Rightarrow$(ii). By the definition of P_μ we have to prove

$$\mathscr{H}^\alpha(\psi(\mathscr{S}))>0 \quad \text{for } \mu^D\text{-a.e. } \mathscr{S}. \tag{1}$$

By Theorem 2.5 and Theorem 3.3 we know that there is a $c>0$ with

$$c\,\mathrm{diam}(X)^\alpha \sup_{\Gamma_0}\inf_{\Gamma>\Gamma_0}\sum_{\sigma\in\Gamma}\mathrm{Lip}(S_\sigma)^d\prod_{n=1}^{|\sigma|}\mathrm{Lip}(S_{\sigma|n})^\alpha\leqq\mathscr{H}^\alpha(\psi(\mathscr{S})), \tag{2}$$

for μ^D-a.e. $\mathscr{S}$. According to our assumptions this yields

$$c\,\delta^q\,\mathrm{diam}(X)^\alpha \sup_{\Gamma_0}\inf_{\Gamma>\Gamma_0}\sum_{\sigma\in\Gamma}\prod_{n=1}^{|\sigma|}\mathrm{Lip}(S_{\sigma|n})^\alpha\leqq\mathscr{H}^\alpha(\psi(\mathscr{S})) \tag{3}$$

for μ^D-a.e. $\mathscr{S}$. Thus (1) follows from Theorem 6.11.

(ii)$\Rightarrow$(iii) is trivial.

(iii)$\Rightarrow$(i) follows immediately from Theorem 7.7.

7.9. Remark. It remains open whether the implication (i)$\Rightarrow$(iii) in Theorem 7.8 is true without the assumption that, for μ-a.e. $(S_0, \ldots, S_{N-1}) \in \mathrm{Con}(X)^N$, $\mathrm{Lip}(S_\rho) \geqq \delta$ $(\rho = 0, \ldots, N-1)$.

7.10. Examples. a) The following example is a random version of the classical Cantor set (see Mandelbrot [8], p. 210 and Falconer [4], Example 11.2).

Start with the unit interval I. Remove the first, second or last third of I at random with equal probability $\frac{1}{3}$. For each of the remaining two intervals of length 1/3 proceed in the same way. Continue this process. What remains of I is a random Cantor set.

In our language the above construction amounts to the following: Let T_0, T_1, T_2 be affine orientation preserving maps of $\mathbb{R}$ onto $\mathbb{R}$ mapping I onto the first, second and last third of I respectively.

Let $\mu = \frac{1}{3}\varepsilon_{(T_0, T_1)} + \frac{1}{3}\varepsilon_{(T_0, T_2)} + \frac{1}{3}\varepsilon_{(T_1, T_2)}$. Then the random Cantor set described above is a P_μ-random set.

It follows from Theorem 7.6 that the Hausdorff dimension of P_μ-a.e. $K \in \mathscr{K}(I)$ is equal to α, where

$$\begin{aligned}
1 &= \int \mathrm{Lip}(S_0)^\alpha + \mathrm{Lip}(S_1)^\alpha \, d\mu(S_0, S_1) \\
&= \tfrac{1}{3}(\mathrm{Lip}(T_0)^\alpha + \mathrm{Lip}(T_1)^\alpha) + \tfrac{1}{3}(\mathrm{Lip}(T_0)^\alpha + \mathrm{Lip}(T_1)^\alpha) \\
&\quad + \tfrac{1}{3}(\mathrm{Lip}(T_1)^\alpha + \mathrm{Lip}(T_2)^\alpha) \\
&= 2(\tfrac{1}{3})^\alpha,
\end{aligned}$$

hence $\alpha = \dfrac{\log 2}{\log 3}$.

Since $\mathrm{Lip}(S_0)^\alpha + \mathrm{Lip}(S_1)^\alpha = 1$ μ-a.e. it follows from Theorems 7.4 and 7.8 that $0 < \mathscr{H}^\alpha(K) < \infty$ for P_μ-a.e. $K \in \mathscr{K}(I)$.

b) The construction of the following random Cantor set is given by Mauldin-Williams [9] who also calculate its dimension. The construction is as follows: Choose a point (x, y) from $\Delta = \{(s, t) \in [0, 1]^2 \mid s < t\}$ at random with respect to normalized Lebesgue measure on Δ. Set $J_0 = [0, x]$ and $J_1 = [y, 1]$. Suppose J_σ has been defined for every $\sigma \in D(2)$ with $|\sigma| \leqq n$. If $J_\sigma = [a, b]$ choose (x, y) from Δ at random with respect to normalized Lebesgue measure on Δ and set $J_{\sigma * 0} = [a, x(b-a)]$, $J_{\sigma * 1} = [a + y(b-a), b]$. Then the corresponding random Cantor set is

$$K = \bigcap_{n \in \mathbb{N}} \bigcup_{\sigma \in \{0, 1\}^n} J_\sigma.$$

We translate this construction into our setting. Let

$$S_0^{x,y}, S_1^{x,y}\colon [0, 1] \to [0, 1]$$

be defined by

$$S_0^{x,y}(t) = x\,t$$

and

$$S_1^{x,y}(t) = y + (1-y)\,t.$$

Then we have $\mathrm{Lip}(S_0^{x,y}) = x$ and $\mathrm{Lip}(S_1^{x,y}) = 1 - y$. Let μ be the image of normalized Lebesgue on Δ w.r.t. the map $\Delta \to \mathrm{Con}(I)^2$, $(x, y) \to (S_0^{x,y}, S_1^{x,y})$.

According to Theorem 7.6 the Hausdorff dimension of P_μ-a.e. $K \in \mathscr{K}(I)$ is α, where

$$1 = \int \operatorname{Lip}(S_0)^\alpha + \operatorname{Lip}(S_1)^\alpha \, d\mu$$

$$= 2 \int_0^1 \int_x^1 [x^\alpha + (1-y)^\alpha] \, dy \, dx,$$

hence $\alpha = \frac{1}{2}(\sqrt{17} - 3)$.

Since obviously $\operatorname{Lip}(S_0)^\alpha + \operatorname{Lip}(S_1)^\alpha \neq 1$ for μ-a.e. (S_0, S_1) it follows from Theorem 7.8 that $\mathscr{H}^\alpha(K) = 0$ for P_μ-a.e. $K \in \mathscr{K}(I)$.

7.11. Remark. Mauldin-Williams [9] and Falconer [4] give a series of examples of random compact sets most of which can easily be translated into our setting. The random compact sets constructed in Examples 4.3, 4.4, 4.6, 4.7, 4.8 of Mauldin-Williams [9] and Example 11.4 of Falconer [4] all have α-dimensional Hausdorff measure 0 with probability one, if α is their Hausdorff dimension.

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