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Probability Theory and Related Fields

Extreme Values and a Gaussian Central Limit Theorem

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Summary. We examine the central limit theorem with Gaussian limit law for a sequence of independent, identically distributed, vector valued random variables whose partial sums can be centered and normalized to be tight with non-degenerate limit laws. These results apply to the situation when the sequence is in the domain of attraction of a non-degenerate stable law of index $p \in (0,2]$, and are achieved by eliminating the extreme values from the partial sums.

1. Introduction

In dealing with the law of the iterated logarithm (LIL) for a random variable in the domain of attraction of a Gaussian law a strong relationship between the maximal values of the sample $\{\|X_1\|, \dots, \|X_n\|\}$ and the asymptotic behavior of the partial sum S_n was obtained in [8], and in much more detail in [9]. In the process of this work we observed that there is also a related Gaussian central limit theorem which holds in a very broad setting. This is what we prove here.

A result of similar type was announced in [3] for real valued random variables in the domain of attraction of a stable law of index $p \in (0,2]$. The proof in [3] depends on a Brownian bridge approximation to the uniform empirical process in weighted supremum norms which is quite powerful, but this approach appears to be limited to the real valued case. We proceed using a more direct method, and obtain a result valid in any type 2 Banach space. The paper [3] contains some interesting references to related papers, but our results appear to be the first for the vector valued case and at the level of generality with which we proceed here.

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Throughout B is a real separable Banach space with topological dual B^* and norm $\|\cdot\|$. We also assume $X, X_1, X_2, \ldots$ are independent, identically distributed, B-valued random variables, and as usual $S_n = X_1 + \ldots + X_n$ for $n \geqq 1$. The law of X is denoted by $\mathscr{L}(X)$. A sequence of random variables $\{W_n\}$ is said to be tight if for each $\varepsilon > 0$ there is a compact set K_ε such that $\inf_n P(W_n \in K_\varepsilon) > 1 - \varepsilon$. The sequence $\{\mathscr{L}(W_n)\}$ converges weakly to $\mathscr{L}(W)$, and we write

$$\mathscr{L}(W_n) \xrightarrow{w} \mathscr{L}(W),$$

if $\lim_n E(f(W_n)) = E(f(W))$ for all bounded continuous f on the range space of $\{W_n\}$. We say a random variable is degenerate if its law is concentrated at a single point. Otherwise, it is said to be non-degenerate. Finally, a sequence of random variables $\{W_n\}$ is said to be stochastically compact if $\{W_n\}$ is tight and all weak limits of subsequences of $\{W_n\}$ are non-degenerate. The stochastically compact laws on $\mathbb{R}^1$ were studied by Feller in [5] and more recently in some work by Pruitt [10] and Griffin et al. [6]. Of course, since B is complete and separable it is well known that for B-valued $\{W_n\}$, tightness of $\{W_n\}$ implies every subsequence of $\{W_n\}$ contains a weakly convergent subsequence. Finally a Banach space is said to be of type 2 if for all integers n and independent centered random variables $Y_1, \ldots, Y_n$ we have

$$E\|Y_1 + \ldots + Y_n\|^2 \leqq A \sum_{j=1}^{n} E\|Y_j\|^2$$

for some finite constant A. We write $a_n \approx b_n$ if there is a $c \in (1, \infty)$ such that for all n sufficiently large

$$1/c < a_n/b_n < c,$$

and $a_n \sim b_n$ if $\lim_n a_n/b_n = 1$.

2. Statement of Results

The central limit theorems we obtain are contained in the following results.

For $n \geqq 1$ and $1 \leqq j \leqq n$, let

$$F_n(j) = \#\{i\colon \|X_i\| > \|X_j\| \text{ for } 1 \leqq i \leqq n \text{ or } \|X_i\| = \|X_j\| \text{ for } 1 \leqq i \leqq j\};$$

here $\# D$ denotes the cardinality of the set D. If $F_n(j) = k$, set $X_n^{(k)} = X_j$, i.e. $\|X_j\|$ is the k^{th} largest element of $\{\|X_1\|, \ldots, \|X_n\|\}$ when $F_n(j) = k$. For any $r > 0$, $n \geqq \xi r$, $\tau > 0$, $\xi > 0$, and positive function $d(t)$ defined on $[0, \infty)$ we define

$$(2.1) \qquad {}^{(\xi r)}_{\tau}S_n = S_n - \sum_{j=1}^{[\xi r]} X_n^{(j)} I\{\|X_n^{(j)}\| > \tau d([n/r])\}.$$

Here $[\cdot]$ denotes the greatest integer function. Hence ${}^{(\xi r)}_{\tau}S_n$ denotes the partial sum S_n with the $[\xi r]$ largest terms of the sample $\{\|X_1\|, \ldots, \|X_n\|\}$ deleted provided they exceed $\tau d([n/r])$ in norm. We also define

$$\delta_n(\tau, r) = \sum_{j=1}^{n} E(X_j I\{\|X_j\| \leqq \tau d([n/r])\}). \tag{2.2}$$

Now we can state our results.

Theorem 1. *Let $X, X_1, X_2, \ldots$ be independent, identically distributed, B-valued random variables where B is a type 2 Banach space, and assume:*

(2.3) *for the centerings $\{\bar{\delta}_n\}$ and positive normalizing constants $\{\bar{d}_n\}$ the sequence*

$$\{(S_n - \bar{\delta}_n)/\bar{d}_n\}$$

is tight, and

(2.4) *for some linear functional $h \in B^*$ the sequence*

$$\{h(S_n - \bar{\delta}_n)/\bar{d}_n\}$$

is stochastically compact.

Then, there is an increasing continuous function $d(t)$ defined on $[0, \infty)$ such that

$$d(n) \approx \bar{d}_n, \tag{2.5}$$

and for each positive sequence $\{r_n\}$ satisfying

$$\lim_n r_n = \infty, \tag{2.6}$$

$$\lim_n n/r_n = \infty, \tag{2.7}$$

and each $\tau > 0$ we have a number $\xi > 0$ such that

$$\left\{ \frac{{}^{(\xi r_n)}_{\tau}S_n - \delta_n(\tau, r_n)}{\sqrt{r_n}\, d(n/r_n)} \right\} \tag{2.8}$$

is a tight sequence in B with only centered Gaussian limits. Further, there is a $\tau_0 > 0$ such that for $\tau \geqq \tau_0$ the limits of (2.8) *are non-degenerate Gaussian.*

Remark. Using the result of Pruitt in [10] which characterizes the Levy measure of a limit law arising from stochastically compact normed sums, and assuming $B = \mathbb{R}^1$, one can simplify the proof of Theorem 1 somewhat and obtain non-degenerate Gaussian limits for all $\tau > 0$. The essential step in this regard is to use Pruitt's result to obtain (3.15) for all $\tau > 0$, and this is possible if $B = \mathbb{R}^1$.

We say X is in the domain of attraction of a non-degenerate stable law Z of index $p \in (0, 2]$ with respect to the centerings $\{\bar{\delta}_n\}$ and normalizations $\{\bar{d}_n\}$ if

$$\mathscr{L}((S_n - \bar{\delta}_n)/\bar{d}_n) \xrightarrow{w} \mathscr{L}(Z).$$

Stable laws and some of their immediate properties are described in [2] for the interested reader.

Theorem 2. *Let $X, X_1, X_2, \ldots$ be independent, identically distributed, B-valued random variables where B is a type 2 Banach space, and assume X is in the domain of attaction of a non-degenerate stable law Z of index $p \in (0,2]$ with respect to the centerings $\{\bar{\delta}_n\}$ and positive normalization sequence $\{\bar{d}_n\}$. Let r_n satisfy* (2.6) *and* (2.7), *$d(t)$ be as in Theorem* 1, *and assume $\tau > 0$. Then:*

(2.9) *$d(n) \sim \Gamma \bar{d}_n$ where Γ is a positive constant, and*

(2.10) *there is a $\xi > 0$ such that*

$$\mathscr{L}\left(\frac{{}^{(\xi r_n)}_{\tau}S_n - \delta_n(\tau, r_n)}{\sqrt{r_n}\, d(n/r_n)}\right) \xrightarrow{w} \mathscr{L}(G_\tau)$$

where G_τ is a non-degenerate centered Gaussian random variable for each $\tau > 0$. In addition, for $0 < p < 2$ and each $\tau > 0$, the sum of extreme terms

$$Z_n = \sum_{j=1}^{[\xi r_n]} X_n^{(j)} I(\|X_n^{(j)}\| > \tau d([n/r_n]))$$

is such that for some strictly positive constant c

$$\mathscr{L}((Z_n - \bar{\delta}_n + \delta_n(\tau, r_n))/(d(n)) \xrightarrow{w} \mathscr{L}(cZ). \tag{2.11}$$

3. Proof of Theorem 1

The first step of the proof is to define the function $d(t)$. There are two cases: they are $Eh^2(X) < \infty$, and $Eh^2(X) = \infty$ where $h \in B^*$ is as in (2.4).

If $Eh^2(X) < \infty$, then standard symmetrization arguments and (2.4) imply that $\bar{d}_n \approx \sqrt{n}$, so in this case we define

$$d(t) = \sqrt{t}.$$

Hence (2.3) implies $\{S_n/\sqrt{n}\}$ is shift tight and that $\{(S_n - S_n')/\sqrt{n}\}$ is actually weakly convergent to a Gaussian law. Here $\{S_n'\}$ is an independent copy of $\{S_n\}$. As a result it is known that $Ef^2(X - X') < \infty$ for all $f \in B^*$ and also from [2, p, 153] that $E\|X - X'\|^{2-\delta} < \infty$ for all $\delta > 0$ where X' is an independent copy of X. Thus EX exists. Further, for any finite dimensional subspace F of B, the semi-norm

$$q_F(x) = \inf_{y \in F} \|x - y\|$$

satisfies

$$E(q_F(S_n - nEX)/\sqrt{n}) \leqq E(q_F(S_n - S_n')/\sqrt{n}),$$

and a standard argument gives $\{(S_n - nE(X))/\sqrt{n}\}$ tight. Indeed, $\{(S_n - nEX)/\sqrt{n}\}$ is actually weakly convergent to a Gaussian law because its finite dimensional distributions converge and it is tight.

Now assume $Eh^2(X) = \infty$. To define the function $d(t)$ we let

$$U(t) = E(h^2(X) \wedge t^2) \qquad (t \geqq 0). \tag{3.1}$$

Then the function

$$f(t)=\begin{cases} U(t)/t^2 & t>0 \\ P(|h(X)|>0) & t=0 \end{cases} \tag{3.2}$$

is positive, continuous, decreasing, and $\lim_{t\to\infty} f(t)=0$. Also f is strictly decreasing on $[a,\infty)$ where

$$a=\inf\{x\colon P(|h(X)|\leqq x)>0\}. \tag{3.3}$$

Thus $1/f(s)$ is strictly increasing on $[a,\infty)$ with range $[1/f(0),\infty)$, and we define the function $d(t)$ to be the inverse of $1/f(s)$ on $[1/f(0),\infty)$. That is,

$$d(t)=\begin{cases} \inf\{s>0\colon 1/t=f(s)\} & t\geqq 1/f(0) \\ a & 0\leqq t\leqq 1/f(0). \end{cases}$$

Then $d(t)$ is continuous, non-decreasing, and strictly increasing on $[1/f(0),\infty)$ with $\lim_{t\to\infty} d(t)=\infty$.

The remainder of the proof now proceeds via a sequence of lemmas. The first lemma contains some useful properties of the function $d(t)$ when $Eh^2(X)=\infty$.

Lemma 3.1. *If $Eh^2(X)=\infty$, the function $d(t)$ satisfies the following conditions:*

(3.4) (i) $d(t)=\sqrt{t}\,U^{1/2}(d(t))$ *for all* $t\geqq 1/f(0)$.
(ii) $d(n)\approx \bar{d}_n$ *and* $d(n+1)\approx d(n)$.

Proof. The definition of $d(t)$ implies (3.4-i) immediately. To prove (3.4-ii) we observe that since $Eh^2(X)=\infty$ and (2.4) holds, Theorem 2.5 of Jain-Orey [7] implies that there exist shifts $\{b_n\}$ such that

$$\{(h(S_n)-b_n)/d(n)\}$$

is stochastically compact. Of course, (2.4) holds by assumption, and since

$$(h(S_n)-b_n)/d(n)=\bar{d}_n/d(n)\cdot(h(S_n-\bar{\delta}_n))/\bar{d}_n+(h(\bar{\delta}_n)-b_n)/d(n),$$

it is impossible that $\overline{\lim_n}\,\bar{d}_n/d(n)=\infty$ or $\underline{\lim}_n\,\bar{d}_n/d(n)=0$, as a simple application of the convergence of types theorem implies either would destroy stochastic compactness. Hence $d(n)\approx\bar{d}_n$ as claimed.

The argument for $d(n+1)\approx d(n)$ is similar since

$$h(X_{n+1})/d(n+1)\xrightarrow{\text{prob}}0,$$

and

$$\begin{aligned}(h(S_{n+1})-b_{b+1})/d(n+1)&=(d(n)/d(n+1))(h(S_n)-b_n)/d(n)\\ &\quad+(b_n-b_{n+1}+h(X_{n+1}))/d(n+1).\end{aligned}$$

Lemma 3.2. *The condition* (2.3) *and* $d(n)\approx\bar{d}_n$ *together imply that for each* $\tau>0$ *and continuous semi-norm* q *we have*

$$(3.5)\quad \begin{aligned}&\text{(i)}\ \sup_{t>0} tP(q(X)>\tau d(t))=c(\tau,q)<\infty.\\ &\text{(ii)}\ \lim_{\lambda\to\infty}\sup_{t>0} tP(q(X)>\lambda d(t))=0.\end{aligned}$$

Proof. Since q is assumed to be a continuous semi-norm it suffices to prove the result when $q(\cdot)=\|\cdot\|$. Hence by applying Theorem 2.2 of [1] and a simple interpolation argument we obtain (3.5–i).

To prove (3.5–ii) we simply notice that (2.3) and Theorem 2.2 of [1] implies

$$\{n\mathcal{L}(X/d(n)|B_1^c)\}$$

is a tight sequence. Here $B_\tau^c=\{x\colon \|x\|>\tau\}$ and $(\mu|A)(E)=\mu(A\cap E)$ for E a Borel subset of B. Hence

$$\lim_{\lambda\to\infty}\sup_n n(P(\|X\|>\lambda d(n))=0$$

and a simple interpolation argument gives (3.5–ii).

Now let $\{r_n\}$ satisfy (2.6) and (2.7) and let $\tau>0$. We define for each integer n

$$u_j=u_j(n)=X_j I\{\|X_j\|\leqq\tau d([n/r_n])\}$$

and

$$v_j=X_j-u_j$$

for $1\leqq j\leqq n$, and let

$$U_n=\sum_{j=1}^n u_j$$

$$V_n=\sum_{j=1}^n v_j.$$

Then $E(U_n)=\delta_n(\tau,r_n)$ and for each $\xi>0$ we have

$$(3.6)\quad \begin{aligned}{}^{(\xi r_n)}_{\tau}S_n-\delta_n(\tau,r_n)&=(U_n-\delta_n(\tau,r_n))+V_n\\ &\quad-\sum_{j=1}^{[\xi r_n]}X_n^{(j)}I\{\|X_n^{(j)}\|>\tau d[n/r_n])\}.\end{aligned}$$

Lemma 3.3. *Let* $\{r_n\}$ *satisfy* (2.6) *and* (2.7) *and* $\tau>0$ *be fixed. Then there is a* $\xi>0$ *such that*

$$(3.7)\quad \left\|V_n-\sum_{j=1}^{[\xi r_n]}X_n^{(j)}I\{\|X_n^{(j)}\|>\tau d([n/r_n])\}\right\|\xrightarrow{\text{prob}}0.$$

Proof. Fix $\varepsilon>0$. Then

$$\begin{aligned}J_n&=P\left(\left\|V_n-\sum_{j=1}^{[\xi r_n]}X_n^{(j)}I\{\|X_n^{(j)}\|>\tau d([n/r_n])\}\right\|>\varepsilon\right)\\ &\leqq P(\text{at least } [\xi r_n]+1\ X_j\text{'s } (1\leqq j\leqq n)\\ &\qquad\text{satisfy } \|X_j\|>\tau d([n/r_n])).\end{aligned}$$

If $Eh^2(X)<\infty$, then $d(n)=\sqrt{n}$ and we know $\{(S_n-nE(X))/\sqrt{n}\}$ converges weakly to a Gaussian law. From this (3.5–i) can be improved to the well-known fact that

$$(3.8)\qquad \lim_t tP(q(X) > \tau\sqrt{t}) = 0$$

for all $\tau > 0$ and continuous semi-norms $q(\cdot)$.

Now let $p_n = P(\|X\| > \tau d([n/r_n]))$, $s_n = [\xi r_n] + 1$. Then we have for any $\xi > 0$ that

$$J_n \leqq \sum_{j=s_n}^{n} \binom{n}{j} p_n^j (1-p_n)^{n-j},$$

and hence by [4], p. 173, and Stirling's formula for all n sufficiently large

$$(3.9)\qquad \begin{aligned} J_n &\leqq n \binom{n-1}{s_n - 1} \int_0^{p_n} t^{s_n - 1}(1-t)^{n-s_n}\,dt \\ &\leqq n\, n^{[\xi r_n]} p_n^{[\xi r_n]} \int_0^{p_n} (1-t)^{n-s_n}\,dt / [\xi r_n]! \\ &\leqq \left(\frac{n p_n e}{[\xi r_n]}\right)^{[\xi r_n]} \frac{1}{[\xi r_n]^{1/2}}\, n/(n - [\xi r_n]). \end{aligned}$$

Since (3.8) holds when $Eh^2(X) < \infty$ there is a sequence $\{\varepsilon(n)\}$ such that $\lim_n \varepsilon(n) = 0$ and

$$nP\|X\| > \tau\sqrt{n}) = \varepsilon(n).$$

Hence, in this case,

$$p_n \leqq \varepsilon([n/r_n])/[n/r_n] \sim (r_n/n)\,\varepsilon([n/r_n])$$

as $\lim_n n/r_n = \infty$, and thus for all n sufficiently large

$$(3.10)\qquad J_n \leqq 2/[\xi r_n]^{1/2} \cdot (2 e r_n \varepsilon([n/r_n])/[\xi r_n])^{[\xi r_n]}.$$

Since $\xi > 0$ and $r_n \to \infty$ we have $\lim_n J_n = 0$ for each $\xi > 0$ as required provided $Eh^2(X) < \infty$.

If $Eh^2(X) = \infty$, then the proof is as above except we do not have (3.8), but only (3.5–i). This implies

$$p_n \leqq c(\tau, \|\cdot\|)/[n/r_n] \sim (r_n/n)\, c(\tau, \|\cdot\|).$$

Substituting this into (3.9) we have for all large n that

$$J_n \leqq 2/[\xi r_n]^{1/2} \cdot (2c(\tau, \|\cdot\|) r_n e/[\xi r_n])^{[\xi r_n]}.$$

Taking $\xi > 4c(\tau, \|\cdot\|)e$ we have $\lim_n J_n = 0$, and the lemma is proved.

Since (2.6), (2.7), and $d(n) \approx d(n+1)$ together imply that

$$\sqrt{r_n}\, d(n/r_n) \approx \sqrt{[r_n]}\, d([n/r_n]),$$

and $E(U_n) = \delta_n(\tau, r_n)$, part of the proof of Theorem 1 will follow by combining (3.6), Lemma 3.3, and by showing

(3.11)
$$\left\{\frac{U_n - E(U_n)}{\sqrt{[r_n]}\, d([n/r_n])}\right\}$$

is tight with only centered Gaussian limits.

To prove (3.11) is tight let $q_F(\cdot)$ be as before. Since B is of type two, we have for each q_F that

(3.12)
$$\begin{aligned} \mathrm{E}\left(q_F^2\left(\frac{U_n - E(U_n)}{\sqrt{[r_n]}\, d([n/r_n])}\right)\right) &\leqq A n E(q_F^2(u_1 - Eu_1))/([r_n] d^2([n/r_n])) \\ &\leqq A' E(q_F^2(u_1 - Eu_1)) \end{aligned}$$

where $A' = A \sup_{n \geqq 1} n/([r_n] d^2([n/r_n])) < \infty$ by (3.4–i). Now for each $\varepsilon > 0$, there exists a finite dimensional subspace F of B satisfying

(3.13)
$$E(q_F^2(u_1 - Eu_1)) < \varepsilon / A',$$

and by combining (3.12) and (3.13) we easily have (3.11) tight.

To verify (3.11) has only centered Gaussian limits now follows easily by applying Theorem 2.10 of [1] to weakly convergent subsequences of (3.11) since

$$\|u_j(n) - Eu_j(n)\| \leqq 2d([n/r_n])$$

and $\lim_n r_n = \infty$.

Hence by Theorem 2.14 of [1], or Corollary 4.8 of [2], the proof of Theorem 1 will be complete when we show there is a $\tau_0 > 0$ such that $\tau \geqq \tau_0$ implies

(3.14)
$$\varliminf_n\, [n/r_n] \frac{E(h^2(u_1(n) - Eu_1(n)))}{d^2([n/r_n])} > 0$$

To obtain (3.14) we use the following lemma applied to the subsequence $m_n = [n/r_n]$.

Lemma 3.4. *Under the previous conditions there is a $\tau_0 \geqq 0$ such that $\tau \geqq \tau_0$ implies*

(3.15)
$$\Gamma(h, \tau) = \varliminf_m\, (m/d^2(m))\, E(h^2(\eta_m - E\eta_m)) > 0$$

where $\eta_m = X I\{\|X\| \leqq \tau d(m)\}$.

Proof. If $Eh^2(X) < \infty$, then $d(n)$ is defined to be $\sqrt{n}$ and the above limit is $Eh^2(X - EX)$ for each $\tau > 0$. Further, this limit is positive by (2.4), so (3.15) holds in this case.

If $Eh^2(X) = \infty$, then $d(n) \approx \bar{d}_n$ by (3.4), and (2.4) holds with $\{\bar{d}_n\}$ replaced by $\{d(n)\}$. We will assume $\Gamma(h, \tau) = 0$ for all $\tau > 0$ and produce a contradiction.

Since $Eh^2(X) = \infty$ we have $(Eh(\eta_m))^2 = o(Eh^2(\eta_m))$ and hence $\Gamma(h, \tau) = 0$ for each $\tau > 0$ implies

$$\varliminf_m \frac{m}{d^2(m)} Eh^2(\eta_m) = 0.$$

Thus there is a subsequence $\{m_r\colon r \geqq 1\}$ such that

$$\mathscr{L}(h(S_{m_r} - \bar{\delta}_{m_r})/d(m_r)) \xrightarrow{w} \mathscr{L}(Z),$$

where Z is non-degenerate by (2.4), and for all $\tau > 0$

$$\lim_r \frac{m_r}{d^2(m_r)} Eh^2(\eta_{m_r}) = 0. \tag{3.16}$$

Let $\{X'_j\colon j \geqq 1\}$ be an independent copy of $\{X_j\colon j \geqq 1\}$. Then

$$\mathscr{L}(h(S_{m_r} - S'_{m_r})/d(m_r)) \xrightarrow{w} \mathscr{L}(Z - Z')$$

where $S'_n = \sum_{j=1}^{n} X'_j$ and Z' is an independent copy of Z.

Now (3.5) implies that for each $\varepsilon > 0$ there is a $\tau_0 > 0$ such that $\tau \geqq \tau_0$ implies

$$\overline{\lim_n}\, mP(\|X\| > \tau d(m)) < \varepsilon/2. \tag{3.17}$$

Thus for $\delta > 0$ a continuity point of the distribution of $|Z - Z'|$ we have

$$\begin{aligned} P(|Z - Z'| \geqq \delta) &= \lim_r P(|h(S_{m_r} - S'_{m_r})| > \delta d(m_r)) \\ &\leqq \overline{\lim_r}\, E\left(\left[\sum_{j=1}^{m_r} (h(X_j) I(\|X_j\| \leqq \tau d(m_r)) \right.\right. \\ &\qquad \left.\left. - h(X'_j) I(\|X'_j\| \leqq \tau d(m_r))\right]^2\right) \Big/ \delta^2 d^2(m_r) \\ &\qquad + \overline{\lim_r}\, 2 \sum_{j=1}^{m_r} P(\|X_j\| > \tau d(m_r)) \\ &\leqq \overline{\lim_r}\, 2 m_r E(\eta_{m_r}^2)/\delta^2 d^2(m_r) + \varepsilon \\ &= \varepsilon. \end{aligned} \tag{3.18}$$

Since $\varepsilon > 0$ was arbitrary we have $P(\{|Z - Z'| > \delta) = 0$ for each $\delta > 0$ which is a continuity point of the distribution of $Z - Z'$. Hence $Z - Z' = 0$ with probability one, which contradicts the fact that Z is non-degenerate. Thus $\Gamma(h, \tau_0) > 0$ for some $\tau_0 > 0$, and hence for $\tau \geqq \tau_0$ we have $\Gamma(h, \tau) > 0$ because

$$(Eh(\eta_m))^2 = o(Eh^2(\eta_m))$$

when $Eh^2(X) = \infty$, and $Eh^2(\eta_m)$ increases when τ increases. This completes the proof of the lemma and the theorem.

4. Proof of Theorem 2

First we need to verify that $d(n) \sim \Gamma \bar{d}_n$ for some $\Gamma \in (0, \infty)$. This follows from the following lemma.

Lemma 4.1. *Let X be in the domain of attraction of a non-degenerate stable law Z of index $p\in(0,2]$ with normalizing sequence $\{\bar{d}_n\}$ and centerings $\{\bar{\delta}_n\}$. Let $d(t)$ be the function defined in Theorem 1 and suppose $h\in B^*$ is such that $h(Z)$ is non-degenerate. Then*

$$(4.1)\qquad \lim_n \frac{d(n)}{\bar{d}_n}=\begin{cases}\sigma_h(Z)/\sigma_h(X) & \text{when } Eh^2(X)<\infty \quad (p=2)\\ \sigma_h(Z) & \text{when } Eh^2(X)=\infty \quad (p=2)\\ \left(\dfrac{(2-p)p}{2(b_1+b_2)}\right)^{1/p} & \text{when } Eh^2(X)=\infty \quad (0<p<2)\end{cases}$$

where $\sigma_h^2(Y)=E(h^2(Y-EY))$ and $\dfrac{b_1+b_2}{p}=\lim_n nP(|h(X)|>\bar{d}_n)$ is a positive finite constant.

Proof. If $Eh^2(X)<\infty$, then $p=2$ since $h(Z)$ is non-degenerate by assumption and $\bar{d}_n\sim c\sqrt{n}$ by convergence of types. Further, $d(t)=\sqrt{t}$ in this case so (2.9) holds. Now $nEh^2(X-EX)/\bar{d}_n^2\sim Eh^2(Z-EZ)$ since $h(X-EX)$ is in the domain of attraction of $h(Z-EZ)$ with respect to the normalizing sequence $\{\bar{d}_n\}$, and hence

$$(4.2)\qquad d(n)/\bar{d}_n=\sqrt{n}/\bar{d}_n\sim\sigma_h(Z)/\sigma_h(X)$$

in this situation.

If $Eh^2(X)=\infty$ and Z is Gaussian $(p=2)$, then the construction of $d(t)$ implies

$$(4.3)\qquad \mathcal{L}(h(S_n-nEX)/d(n))\xrightarrow{w}N(0,1),$$

and, of course, replacing the centering by expectations (which is possible in the case Z is Gaussian) we have

$$(4.4)\qquad \mathcal{L}(h(S_n-nEX)/\bar{d}_n)\xrightarrow{w}\mathcal{L}(h(Z-EZ))=N(0,\sigma_h^2(Z)).$$

Hence by convergence of types (4.3) and (4.4) give $\lim_n \dfrac{d(n)}{\bar{d}_n}=\sigma_h(Z)$.

If $Eh^2(X)=\infty$ and Z is stable of index $p\in(0,2)$ then

$$(4.5)\qquad \mathcal{L}(h(S_n-\bar{\delta}_n)/\bar{d}_n)\xrightarrow{w}\mathcal{L}(h(Z))=c\,\mathrm{Pois}(\mu^h(b_1,b_2,p))$$

where μ^h is a measure on $\mathbb{R}^1$ such that

$$(4.6)\qquad d\mu^h(b_1,b_2,p)=\begin{cases}b_1x^{-1-p}dx & (x>0)\\ b_2|x|^{-1-p}dx & (x<0),\end{cases}$$

$$(4.7)\qquad \frac{(b_1+b_2)}{p}x^{-p}=\lim_n nP(|h(X)|>\bar{d}_nx)\qquad (x>0),$$

and $c\,\mathrm{Pois}(\mu^h(b_1,b_2,p))$ is the stable law of index p on $\mathbb{R}^1$ with Lévy measure μ^h and Fourier transform as in [2], Corollary 6.12-b. For details regarding (4.6) and (4.7) the interested reader can consider [2], p. 79–88.

Further, Corollary 6.17–ii of [2] immediately implies that if ε is independent of X and $P(\varepsilon=\pm 1)=1/2$, then the symmetric random variable $h(\varepsilon X)$ is in the domain of attraction of $h(\varepsilon Z)$ where, in the notation of (4.6) and (4.7),

$$\mathscr{L}(h(\varepsilon Z))=c\,\mathrm{Pois}\,\mu^h\left(\frac{b_1+b_2}{2}, \frac{b_1+b_2}{2}, p\right).$$

Further, $b_1+b_2\in(0,\infty)$ as $h(Z)$ is non-degenerate.

Now let $\varepsilon, \varepsilon_1, \varepsilon_2, \ldots$ be independent identically distributed and independent of $X, X_1, X_2, \ldots$. Wet $T_n=\sum_{j=1}^{n} \varepsilon_j X_j$, and consider $\{h(T_n)/d(n)\}$. Since $h(\varepsilon X)$ is in the domain of attraction of $h(\varepsilon Z)$ with $h(\varepsilon Z)$ non-degenerate, Theorem 6.17–ii of [2] implies

$$\Phi(t)=E(h^2(X)I(|h(X)|\leqq t))=E(h^2(\varepsilon X)I(|h(\varepsilon X)|\leqq t)$$

is regularly varying of order $2-p$ and

$$\lim_{t\to\infty}\frac{t^2}{\Phi(t)}P(|h(\varepsilon X)|>t)=(2-p)/p. \tag{4.8}$$

Hence the function $U(t)$ of (3.1) satisfies

$$U(t)/t^2=\Phi(t)/t^2+P(|h(\varepsilon X)|>t)\sim\frac{\Phi(t)}{t^2}(2/p),$$

and by definition of $d(t)$ we have

$$1=t\,U(d(t))/d^2(t)\sim 2/p\cdot t\Phi(d(t))/d^2(t).$$

Since $\Phi(t)$ is regularly varying of order 2-p we have for all $x>0$ that

$$t\Phi(d(t)x)/d^2(t)\sim t\Phi(d(t))x^{2-p}/d^2(t)\sim p/2x^{2-p} \qquad (0<p<2). \tag{4.9}$$

Now (4.9) implies the variances of the truncated variables go to zero as $x\downarrow 0$ and using (4.8) and (4.9) together we have

$$tP(|h(\varepsilon X)|>xd(t))\sim\frac{(2-p)}{2}x^{-p}=\lambda(c,p)([x,\infty)) \tag{4.10}$$

where $c>0$ and

$$d\lambda(c,p)(x)=\begin{cases} cx^{-1-p}dx & (x>0)\\ c|x|^{-1-p}dx & (x<0)\end{cases}$$

provided

$$(2-p)x^{-p}/2=2cx^{-p}/p \qquad (x>0) \tag{4.11}$$

or

$$(2-p)p/4=c. \tag{4.12}$$

Thus by the convergence theorem, Corollary 4.8–d of [2], we have by symmetry that

(4.13) $$\mathscr{L}(h(T_n)/\bar{d}_n) \xrightarrow{w} c\,\mathrm{Pois}(\lambda(c,p))$$

where $c=(2-p)p/4>0$.

Since

(4.14) $$\mathscr{L}(h(T_n)/\bar{d}_n) \xrightarrow{w} \mathscr{L}(h(\varepsilon Z)) = c\,\mathrm{Pois}\,\mu^h\left(\frac{b_1+b_2}{2}, \frac{b_1+b_2}{2}, p\right)$$

and the limit laws in (4.13) and (4.14) are non-degenerate the convergence of types theorem immediately implies that

(4.15) $$\lim_n \frac{d(n)}{\bar{d}_n} = \Gamma$$

for some $\Gamma \in (0, \infty)$. Further, for each $x>0$

(4.16) $$n\,P(|h(\varepsilon X)| > x\,\bar{d}_n) = n\,P\left(|h(\varepsilon X)| > d(n)\frac{\bar{d}_n}{d(n)}x\right),$$

and since (4.10) holds we have from (4.15) that

(4.17) $$nP\left(|h(\varepsilon X)| > d(n)\frac{\bar{d}_n}{d(n)}x\right) \sim nP\left(|h|(\varepsilon X)\Big| > \frac{d(n)}{\Gamma}x\right)$$
$$\sim \frac{(2-p)}{2}(x/\Gamma)^{-p}.$$

Combining (4.7), (4.16), and (4.17) we have for each $x>0$ that

$$\frac{(2-p)}{2}(x/\Gamma)^{-p} = \frac{(b_1+b_2)}{p}x^{-p}$$

and hence

$$\Gamma = \left(\frac{(2-p)p}{2(b_1+b_2)}\right)^{1/p}$$

where $(b_1+b_2)/p = \lim_n nP(|h(X)| > \bar{d}_n)$.

Hence (2.9) holds, and we turn to (2.10). Recalling (3.6) and Lemma 3.3 it suffices to show that for each $\tau > 0$

$$\mathscr{L}\left(\frac{U_n - EU_n}{\sqrt{r_n}\,d(n/r_n)}\right) \xrightarrow{w} \mathscr{L}(G_\tau)$$

for some non-degenerate centered Gaussian random variable G_τ.

Now the proof of Theorem 1 implies

$$\left\{\frac{U_n - EU_n}{\sqrt{r_n}\,d(n/r_n)}\right\}$$

is tight and the only possible limit laws are Gaussian, so it suffices to prove that for each $f \in B^*$ and $\tau > 0$

$$\left\{\frac{f(U_n - EU_n)}{\sqrt{r_n}\, d(n/r_n)}\right\}$$

has a unique limiting distribution which is non-degenerate if $f(Z)$ is non-degenerate. As in the proof of Theorem 1, this follows from an application of Corollary 4.8-a of [2] provided we show

$$\sigma_f^2 \equiv \lim_n \frac{E(f^2(U_n - EU_n))}{[r_n]\, d^2([n/r_n])}$$

exists for each $f \in B^*$, $\tau > 0$, and that $\sigma_f^2 > 0$ for each $f \in B^*$ such that $f(Z)$ is non-degenerate.

This is the result of the following lemma. It is possible to use the normalizations $d(n)$ in this lemma because of Lemma 4.1.

Lemma 4.2. *Let X be in the domain of attraction of a stable law Z of index $p \in (0,2]$ with respect to the normalizations $\{d(n)\}$ and centerings $\{\bar{\delta}_n\}$ where $d(t)$ is as in Theorem 1. Let μ denote the Levy measure of $\mathscr{L}(Z)$ when $0 < p < 2$. Then, for each $f \in B^*$ such that $f(Z)$ is non-degenerate, we have for each $\tau > 0$ that*

$$\Gamma(f, \tau) = \lim_m \frac{m}{d^2(m)} E f^2(\eta_m - E\eta_m) \tag{4.18}$$

exists and is positive where $\eta_m = XI\{\|X\| \leqq \tau d(m)\}$. In fact if $p = 2$ (Z Gaussian) we have

$$\Gamma(f, \tau) = E f^2(Z - EZ), \tag{4.19}$$

and if Z is stable of index $p \in (0,2)$, then

$$\Gamma(f, \tau) = \int_B f^2(x)\, d\mu_\tau(x) \tag{4.20}$$

where μ_τ denotes μ restricted to the ball $\{x\colon \|x\| \leqq \tau\}$.

Proof. Let $\{Y_{m,j}\colon 1 \leqq j \leqq m\}$ denote the infinitesimal triangular array defined by

$$Y_{m,j} = X_j / d(m) \qquad (1 \leqq j \leqq m).$$

Let

$$Y_{m,j,\tau} = Y_{m,j} I\{\|Y_{m,j}\| \leqq \tau\}$$

for $1 \leqq j \leqq n$, $\tau > 0$, and set

$$S_{m,\tau} = \sum_{j=1}^{m} Y_{m,j,\tau}.$$

Then, by Theorem 2.10 of [1] for each $\tau > 0$ such that $\mu\{x\colon \|x\| = \tau) = 0$, we have

$$\mathscr{L}(S_{m,\tau} - ES_{m,\tau}) \xrightarrow{w} \gamma * c_\tau \operatorname{Pois}(\mu_\tau). \tag{4.21}$$

Here γ is the Gaussian component of Z, and for any Levy measure λ on B the τ-centered Poisson measure associated with λ is $c_\tau \operatorname{Pois}(\lambda)$ having Fourier transform

(4.22) $\int_B e^{if(x)} dc_\tau \mathrm{Pois}(\lambda)(x) = \exp\{\int_B [e^{if(x)} - 1 - if(x) I\{x: \|x\| \leqq \tau\}] d\lambda(x)\}$

for $f \in B^*$. Of course, since μ is a Lévy measure the restricted measure μ_τ is also a Lévy measure. Further, when $p=2$, then $\mu=\delta_0$ (so $c_\tau \mathrm{Pois}(\mu_\tau)=\delta_0$) and $\gamma = \mathscr{L}(Z)$. If $p \in (0,2)$, then $\gamma=\delta_0$ and $\mu=\sigma \times \frac{dr}{r^{1+p}}$ $(0<r<\infty)$ where σ is a finite positive measure on $\{x: \|x\|=1\}$. In particular, $\mu(x: \|x\|=\tau)=0$ for each $\tau>0$. In addition, the proof of Theorem 2.10 of [1] implies for each $f \in B^*$ and $\tau>0$ that

$$\lim_m Ef^2(S_{m,\tau} - ES_{m,\tau}) = \int_B f^2(x) d\gamma(x) + \int_B f^2(x) d\mu_\tau(x).$$

Now

$$Ef^2(S_{m,\tau} - ES_{m,\tau}) = md^{-2}(m) Ef^2(\eta_m - E\eta_m)$$

for each $f \in B^*$, so the limit used in $\Gamma(f,\tau)$ exists. If $p=2$ (and $\gamma=\mathscr{L}(Z)$), then

$$\Gamma(f,\tau) = \int_B f^2(x) d\gamma(x) > 0$$

for all f such that $f(Z)$ is non-degenerate. If $0<p<2$, then for all f such that $f(Z)$ is non-degenerate we have

$$\Gamma(f,\tau) = \int_B f^2(x) d\mu_\tau(x) > 0$$

since $\mu_\tau = \sigma \times \frac{dr}{r^{1+p}}$ $(0<r\leqq\tau)$, and the closed subspace of B which supports $\mathscr{L}(Z)$ equals the closed subspace generated by the support of μ_τ for each $\tau>0$. Thus the proof of the lemma is complete.

To finish the proof of Theorem 2 we must verify (2.11). Since $Z_n = S_n - {}^{(\xi r_n)}S_n$ we first observe that

(4.23) $$(Z_n - \bar{\delta}_n + \delta_n(\tau, r_n))/d(n) = ((S_n - \bar{\delta}_n) - ({}^{(\xi r_n)}S_n - \delta_n(\tau, r_n)))/d(n).$$

Now $\mathscr{L}((S_n - \bar{\delta}_n)/\bar{d}_n \xrightarrow{\omega} \mathscr{L}(Z)$ by assumption, and by Lemma 4.1 we have $\lim_n \bar{d}_n/d(n) = c$ where c is a strictly positive constant, so

(4.24) $$\mathscr{L}((S_n - \bar{\delta}_n)/d(n)) \xrightarrow{w} \mathscr{L}(cZ).$$

Combining (4.23) and (4.24) we have (2.11) provided

(4.25) $$\mathscr{L}(({}^{(\xi r_n)}S_n - \delta(\tau, r_n))/d(n)) \xrightarrow{w} \delta_0.$$

Now (4.25) is immediate from (2.10) provided

(4.26) $$\lim_n \sqrt{r_n}\, d(n/r_n)/d(n) = 0.$$

To prove (4.26) observe that by [2], p. 90, and Lemma 4.1 we have

$$d(n) = n^{1/p} L(n)$$

where $L(t)$ is a continuous slowly varying function as $t \to \infty$. Hence by the representation theorem for slowly varying functions [2, p. 90] we have

$$\lim_n \sqrt{r_n}\, d(n/r_n)/d(n) = \lim_n r_n^{1/2 - 1/p} \exp\left\{ \int_{n/r_n}^{n} \frac{\varepsilon(s)}{s}\, ds \right\} \tag{4.27}$$

where $\lim_{s \to \infty} \varepsilon(s) = 0$. If $0 < p < 2$, (4.27) yields (4.26) and Theorem 2 is proved.

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