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Probability Theory and Related Fields

Extreme Values and the Law of the Iterated Logarithm

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Summary. If X takes values in a Banach space B and is in the domain of attraction of a Gaussian law on B, then X satisfies the compact law of the iterated logarithm (LIL) with respect to a regular normalizing sequence $\{\gamma_n\}$ iff X satisfies a certain integrability condition. The integrability condition is equivalent to the fact that the maximal term of the sample $\{\|X_1\|, \|X_2\|, \ldots, \|X_n\|\}$ does not dominate the partial sums $\{S_n\}$, and here we examine the precise influence of these maximal terms and its relation to the compact LIL. In particular, it is shown that if one deletes enough of the maximal terms there is always a compact LIL with non-trivial limit set.

1. Introduction

The interplay between the maximal terms of the sample $\{\|X_1\|, \ldots, \|X_n\|\}$ and the partial sum $S_n = X_1 + \ldots + X_n$ has been studied in a variety of contexts by a number of authors. For example, the paper [8] by Feller deals with the LIL in this regard, and those of Mori [15, 16] examine this relationship in the setting of the strong law of large numbers. Here we study the compact LIL for Banach space valued random variables and the related maximal terms. One particular result we obtain shows that if one deletes enough of the maximal terms there is always a non-degenerate compact LIL when we are in the domain of attraction of a Gaussian law. Now we turn to some notation and discuss some background for our results.

Throughout, B will denote a real separable Banach space with topological dual B^* and norm $\|\cdot\|$. We assume $X, X_1, X_2, \ldots$ is a sequence of independent identically distributed B-valued random variables on some probability space $(\Omega, \mathscr{F}, P)$ and let as usual $S_n = X_1 + \ldots + X_n$. We use Lt to denote the function $\max(1, \log_e t)$ and write $L_2 t$ for $L(Lt)$. We write $\mathscr{L}(X)$ to denote the law of X.

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The random variable X is said to satisfy the classical central limit theorem (CLT) if there is a mean-zero Gaussian random variable G with values in B such that

$$\mathscr{L}\left(\frac{S_n - nE\{X\}}{\sqrt{n}}\right) \xrightarrow[n\to\infty]{} \mathscr{L}(G).$$

More generally, we say X is in the domain of attraction of a mean-zero Gaussian random variable $G(DA(G))$ if there exists a normalization sequence $\{d_n\} \nearrow \infty$ and shifts $\{\delta_n\} \subseteqq B$ such that

$$\mathscr{L}\left(\frac{S_n - \delta_n}{d_n}\right) \xrightarrow[n\to\infty]{} \mathscr{L}(G).$$

We always assume the limiting Gaussian random variable non-degenerate.

We now turn to the law of the iterated logarithm (LIL). If $A \subseteqq B$, the distance from $x \in B$ to A is given by

$$d(x, A) = \inf_{y \in A} \|x - y\|.$$

If $\{x_n\}$ is a sequence in B, $C(\{x_n\})$ denotes the set of all x in B such that $\liminf_n \|x_n - x\| = 0$. We use the notation $\{x_n\} \twoheadrightarrow A$ if both $\lim_n d(x_n, A) = 0$ and $C(\{x_n\}) = A$. With this notation, we say that X satisfies the classical (compact) LIL if there is a non-random compact limit set $D \subseteqq B$ such that

$$\left\{\frac{S_n - nE\{X\}}{\sqrt{2nL_2 n}}\right\} \twoheadrightarrow D \quad \text{w.p.1}.$$

Let us point out that the use of the shifts $nE\{X\}$ in the definition of the LIL, as well as in the classical CLT, does not restrict the class of random variables satisfying those limit properties and is justified from the fact that the conditional compactness (resp. tightness) of $\left\{\frac{S_n - \delta_n}{\sqrt{2nL_2 n}}\right\}$ $\left(\text{resp. } \left\{\frac{S_n - \delta_n}{\sqrt{n}}\right\}\right)$ for any shifts $\{\delta_n\}$ implies the existence of $E\{X\}$ and that $\left\{\frac{nE\{X\} - \delta_n}{\sqrt{2nL_2 n}}\right\}$ $\left(\text{resp. } \left\{\frac{nE\{X\} - \delta_n}{\sqrt{n}}\right\}\right)$ is conditionally compact.

If $E\{f^2(X)\} < \infty$ and $E\{f(X)\} = 0$ for each f in B^* we define the covariance function of X to be

$$T(f, g) = E\{f(X) g(X)\} \qquad (f, g \in B^*).$$

The covariance structure of X determines a Hilbert space $H_{\mathscr{L}(X)} \subseteqq B$ with unit ball $K_{\mathscr{L}(X)}$. Details on $H_{\mathscr{L}(X)}$ and $K_{\mathscr{L}(X)}$ can be found in [9], Lemma 2.1.

For a real-valued random variable X, it is well known that the three conditions $E\{X^2\} < \infty$, $X \in CLT$, and $X \in LIL$ are equivalent. This is not true in infinite dimensional spaces, but, however, the following result due to Goodman et al. [9] and Heinkel [10], after preliminary work by Pisier [17], provides a necessary and sufficient condition for a B-valued random variable satisfying the CLT to also satisfy the LIL.

Theorem 1.1. *Let X be a B-valued random variable such that $X \in CLT$. Then $X \in LIL$ with limit set $K_{\mathscr{L}(X)}$ iff $E\{\|X\|^2/L_2\|X\|\} < \infty$.*

This result has been recently refined in [14] when X is merely in the domain of attraction of a Gaussian law and the following result was proved.

Theorem 1.2. *Let X be a B-valued random variable in the $DA(G)$ where G is a centered Gaussian random variable. Then $E\{X\}$ exists and there is a strictly increasing continuous function $d: [0, \infty) \to [0, \infty)$ such that*

$$d(t) \sim \sqrt{t}\, Td(t)$$

where $T: [0, \infty) \to [0, \infty)$ is non-decreasing, slowly varying and

$$\mathscr{L}\left(\frac{S_n - nE\{X\}}{d(n)}\right) \xrightarrow[n\to\infty]{} \mathscr{L}(G).$$

Further, if we define $\alpha(t) = t/L_2 t$ and

$$\gamma(t) = \sqrt{2tL_2 t}\, Td\alpha(t),$$

then

$$\left\{\frac{S_n - nE\{X\}}{\gamma(n)}\right\} \twoheadrightarrow K \quad w.p.1. \text{ iff } \quad E\{\alpha^{-1} d^{-1} \alpha(\|X\|)\} < \infty$$

where $K = K_{\mathscr{L}(G)}$.

Here we use the notation $f(t) \sim g(t)$ as $t \to \infty$ if

$$\lim_{t\to\infty} \frac{f(t)}{g(t)} = 1,$$

and $fg(t)$ for the composition of f and g.

The importance of the integrability condition on $\|X\|$ in Theorems 1.1 and 1.2, which provides a necessary and sufficient condition for X to satisfy the compact *LIL* whenever X satisfies the central limit theorem, can be illustrated in several ways. First of all, some examples in [14] show that when it fails, K is possibly a proper subset of the cluster set $C\left(\left\{\frac{S_n - nE\{X\}}{\gamma(n)}\right\}\right)$ (in contrast with the situation of the classical *CLT* [6]), and, of course, it is well known that, in the notation of Theorem 1.2,

$$\lim_n \max_{1 \le j \le n} \|X_j\|/\gamma(n) = 0 \quad \text{w.p.1. iff} \quad E\{\alpha^{-1} d^{-1} \alpha(\|X\|)\} < \infty.$$

More subtle implications were examined by Feller in [8] and by Mori in [15], [16], but the thrust of all of this is that the asymptotic behavior of the maximal terms of the sample $\{\|X_1\|, \ldots, \|X_n\|\}$ parallels the behavior of the partial sum S_n. This suggests that removing the maximal terms from the sums might improve the *LIL* statement. More precisely, deleting maximal terms could lead to an *LIL* under weaker integrability conditions than those of Theorems 1.1 and 1.2, or even without any integrability assumption if sufficiently many extremal values are omitted. It is the purpose of this paper to make precise these intuitive remarks and to extend and clarify a portion of [8].

2. Statements of the Results

For $n \geqq 1$ and $1 \leqq j < n$, let

$$\mathscr{F}_n(j) = \#\{i : \|X_i\| > \|X_j\| \text{ for } 1 \leqq i \leqq n \text{ or } \|X_i\| = \|X_j\| \text{ for } 1 \leqq i \leqq j\};$$

here $\# D$ denotes the cardinality of the set D. If $\mathscr{F}_n(j) = k$, set $X_n^{(k)} = X_j$; i.e. X_j is the kth largest element of $X_1, \ldots, X_n$ when $\mathscr{F}_n(j) = k$. For any integer $r \geqq 1$ and $n \geqq r$ we set

$$^{(r)}S_n = S_n - \sum_{j=1}^{r} X_n^{(j)}.$$

When $r = 0$, $^{(0)}S_n$ is just S_n.

Our first theorem deals with the case where we only substract a finite number of maximal values from S_n, that is, in the preceding notation, when r is a fixed finite number. The integrability condition we need can be expressed using Lorentz spaces. For $1 \leqq p,\ q < \infty$, let $L_{p,q} = L_{p,q}(\Omega, \mathscr{F}, P)$ be the space of all real-valued random variables ξ such that

$$\int_0^\infty (t^p P\{|\xi| > t\})^{q/p} \frac{dt}{t} < \infty.$$

If $p = q$, $L_{p,p}$ is just L_p by the usual integration by parts fromula. If $q = \infty$, $L_{p,\infty}$ consists of all random variables ξ for which

$$\left(\sup_{t>0} t^p P\{|\xi| > t\}\right)^{1/p} < \infty.$$

Note that $L_{p,q_1} \subseteqq L_{p,q_2}$ whenever $q_1 \leqq q_2$.

Theorem 2.1. *Let X be a B-valued random variable in the $DA(G)$ where G is a mean-zero Gaussian random variable and let d, T, and γ be the functions constructed in Theorem 1.2. Then, for any integer r,*

$$\left\{\frac{^{(r)}S_n - nE\{X\}}{\gamma(n)}\right\} \twoheadrightarrow K \quad \text{w.p.1.} \tag{2.1}$$

iff

$$\alpha^{-1} d^{-1} \alpha(\|X\|) \in L_{1,r+1}. \tag{2.2}$$

For a better understanding of the integrability condition (2.2), we first provide the following equivalence (to be proved later): (2.2) holds iff

$$\mathscr{I}_{r+1}(X) = E\left\{\min_{1 \leqq j \leqq r+1} (\alpha^{-1} d^{-1} \alpha(\|X_j\|))^{r+1}\right\} < \infty.$$

The next corollary examines condition (2.2) further: that is, when X is in the $DA(G)$, (2.2) implies that it is possible to weaken the usual integrability condition $E\{\alpha^{-1}d^{-1}\alpha(\|X\|)\}<\infty$ of Theorem 1.2 as in (2.3) below. Precisely, we have the following

Corollary 2.2. *Let X be a B-valued random variable in the $DA(G)$ where G is a centered Gaussian random variable and let d, T, and γ be the functions constructed in Theorem 1.2. Assume that r is an integer and*

$$E\left\{\frac{\alpha^{-1}d^{-1}\alpha(\|X\|)}{(L_2\|X\|)^{\varrho}}\right\}<\infty \tag{2.3}$$

for some $\varrho\geqq 0$. Then (2.2) is satisfied for every $r>\varrho$ and therefore (2.1) holds. Further, if $T\alpha(t)\sim T(t)$ (for example if $X\in CLT$ where $T\equiv 1$ or T is sufficiently slowly varying), then condition (2.3) and $r\geqq\varrho$ suffice to imply (2.2).

Let us point out that even in this more general setting, the proof of Theorem 2.1 we present contains some significant simplifications in comparison with the earlier proofs of Theorems 1.1 and 1.2 and is particularly simple in case of Theorem 1.1.

We also note that when $r=\infty$, condition (2.2) reduces to

$$\sup_{t>0} tP\{\|X\|>\gamma(t)\}<\infty\,,$$

and this is always fulfilled when X is in the $DA(G)$ $\Big($since X in the $DA(G)$ with normalizing sequence $d(n)$ implies $\lim_{t\to\infty} tP\{\|X\|>d(t)\}=0$ [4], Corollary 2.12, and $\lim_{t\to\infty}\gamma(t)/d(t)=\infty$ [14], remark (V)$\Big)$. This suggests that by subtracting r_n maximal terms of the sample $\{\|X_1\|,\ldots,\|X_n\|\}$ from the sum S_n, where $r_n\to\infty$, could possibly yield LIL under the sole assumption that the central limit theorem holds. The following theorem describes this situation.

Theorem 2.3. *Let X be a B-valued random variable in the $DA(G)$ where G is a mean-zero Gaussian random variable and let d, T and γ be the functions constructed in Theorem 1.2. Then there exists a sequence $\{\xi_n\}$ of real numbers decreasing to 0 such that if $r_n=[\xi_n L_2 n]$, where $[\ ]$ is the integer part function, we have*

$$\left\{\frac{{}^{(r_n)}S_n-nE\{X\}}{\gamma(n)}\right\}\twoheadrightarrow K \quad w.p.1\,. \tag{2.4}$$

Note that the number of maximal terms we remove from S_n in Theorem 2.3 is very small in comparison to n and can be interpretated in the following way: among the maximal values, only those which are in norm larger than $d\alpha(n)$ influence the behavior of S_n since $L_2 n$ terms less than $d\alpha(n)$ can be dominated by the normalizing sequence $\gamma(n)\sim\sqrt{2}\,L_2 n\, d\alpha(n)$. In addition, the expected number of values larger than $d\alpha(n)$ is exactly

$$E\{\#\{1\leqq j\leqq n:\|X_j\|>d\alpha(n)\}\}=nP\{\|X\|>d\alpha(n)\}\,,$$

and since X is in the $DA(G)$, $\lim_{t\to\infty} tP\{\|X\|>d(t)\}=0$, so

$$E\{\#\{1\leqq j\leqq n:\|X_j\|>d\alpha(n)\}\}=o(L_2 n).$$

Hence, on the average, the number of values of $\{\|X_1\|,\ldots,\|X_n\|\}$ bigger than $d\alpha(n)$ behaves like $o(L_2n)$. These values are usually estimated by some integrability condition on $\|X\|$, but since we choose to proceed without integrability we remove them, and once they are removed, the *LIL* behavior of S_n follows directly from the *CLT* statement.

More general situations than the case of random variables in the domain of attraction of a Gaussian law can also be considered in this setting. These questions, as well as various related ones, will be studied in a forthcoming paper.

3. Proof of Theorem 2.1 and Corollary 2.2

The proofs of Theorems 2.1 and 2.3 follow the same pattern, we will detail the situation of Theorem 2.1 and provide in the next section the necessary modification to prove Theorem 2.3. We first concentrate on the necessity of condition (2.2) in Theorem 2.1 that we will deduce from the following lemmas.

Lemma 3.1. *Let r be an integer. The following are equivalent*

(3.1) $\alpha^{-1}d^{-1}\alpha(\|X\|)\in L_{1,r+1}$;

(3.2) $\mathscr{I}_{r+1}(X)<\infty$;

(3.3) *for some (all)* $\varrho>0$, $\mathscr{I}_{r+1}(\varrho X)<\infty$;

(3.4) *for some (all)* $\varrho>0$ *and* $\beta>1$,

$$\sum_k (n_kP\{\|X\|>\varrho\alpha^{-1}d\alpha(n_k)\})^{r+1}<\infty$$

where $n_k=[\beta^k]$, $k\geqq 1$.

Proof. The equivalence between (3.2) and (3.3) holds since T slowly varying implies that for each $\varrho\geqq 1$ there exists $u_0>0$ and $\lambda>0$ such that

$$\forall u\geqq u_0,\quad \alpha^{-1}d^{-1}\alpha(\varrho u)\leqq\lambda\alpha^{-1}d^{-1}\alpha(u).$$

Dealing with (3.4) we have:

$$\sum_k (n_kP\{\|X\|>\varrho\alpha^{-1}d\alpha(n_k)\})^{r+1}=\sum_k n_k^{r+1}E\left\{\prod_{j=1}^{r+1} I_{\{\|X_j\|>\varrho\alpha^{-1}d\alpha(n_k)\}}\right\}$$
$$=E\left\{\sum_k n_k^{r+1} I_{\{n_k\leqq \min_{1\leqq j\leqq r+1}\alpha^{-1}d^{-1}\alpha(\|X_j\|/\varrho)\}}\right\}$$

and this last expectation is clearly equivalent to $\mathscr{I}_{r+1}(X/\varrho)$. Finally (3.1) is equivalent to the preceding conditions since

$$\int_0^\infty t^r(P\{\alpha^{-1}d^{-1}\alpha(\|X\|)>t\})^{r+1}dt<\infty$$

iff

$$\sum_k \int_{2^k}^{2^{k+1}} t^r (P\{\alpha^{-1} d^{-1} \alpha(\|X\|) > t\})^{r+1} dt < \infty$$

iff

$$\sum_k (2^k P\{\alpha^{-1} d^{-1} \alpha(\|X\|) > 2^k\})^{r+1} = \sum_k (2^k P\{\|X\| > \alpha^{-1} d\alpha(2^k))^{r+1} < \infty .$$

To prove the necessity of (2.2) we will need the following lemma, a proof of which can be found in [15], Lemma 3.

Lemma 3.2. *For any integer r and any* $\varrho > 0$,

$$P\{\|X_n^{(r+1)}\| > \varrho\gamma(n) \text{ i.o.}\} = 0 \text{ or } 1$$

according as $\mathscr{I}_{r+1}(X) < \infty$ *or* $= \infty$.

The necessity of (2.2) then follows easily:

Lemma 3.3. *For any integer r, if*

$$\limsup_n \frac{\|{}^{(r)}S_n - nE\{X\}\|}{\gamma(n)} < \infty \quad \text{w.p.1}. \tag{3.5}$$

then (2.2) *holds.*

Proof. The case $r=0$ is known so we assume $r \geqq 1$. It is easy to see that

$$\|{}^{(r)}S_{n+1} - {}^{(r)}S_n\| = \min(\|X_{n+1}\|, \|X_n^{(r)}\|)$$

and thus by induction on n that

$$\max_{r \leqq j \leqq n} \|{}^{(r)}S_{j+1} - {}^{(r)}S_j\| = \|X_{n+1}^{(r+1)}\| .$$

By (3.5) and Lemma 3.2, the conclusion is immediate.

We now show that under condition (2.2), the conclusion of the theorem holds. Before turning to the main argument of the proof, let us state the following simple lemma which will prove to be useful.

Lemma 3.4. *Assume that*

$$\mathscr{L}\left(\frac{S_n - nE\{X\}}{d(n)}\right) \xrightarrow[n\to\infty]{} \mathscr{L}(G)$$

where d is the function of Theorem 1.2. Then, for every $\delta > 0$

$$\lim_n \frac{n}{d(n)} \|E\{XI_{\{\|X\| > \delta d(n)\}}\}\| = 0 . \tag{3.6}$$

Further, if $\{\varepsilon_j\}$ *is a Rademacher sequence independent of* $\{X_j\}$, $\left\{\sum_{j=1}^n \frac{\varepsilon_j X_j}{d(n)}\right\}$ *is tight.*

Proof. Since X is in the $DA(G)$ with normalizing constants $d(n)$, by Corollary 2.12 of [4], for each $\delta>0$

$$\mathscr{L}\left(\frac{S_n-nE\{XI_{\{\|X\|\leqq\delta d(n)\}}\}}{d(n)}\right)\xrightarrow[n\to\infty]{}\mathscr{L}(G).$$

By difference (3.6) holds. Now let q be a continuous seminorm on B; for each n

$$E\left\{q\left(\sum_{j=1}^{n}\varepsilon_jX_j\right)\right\}\leqq E\left\{q\left(\sum_{j=1}^{n}\varepsilon_j(X_j-E\{X_j\})\right)\right\}+\sqrt{n}\,q(E\{X\})$$
$$\leqq 2E\{q(S_n-nE\{X\})\}+\sqrt{n}\,q(E\{X\}).$$

Since $d(t)\sim\sqrt{t}\,Td(t)$, we already deduce from these inequalities the stochastic boundedness of $\left\{\sum_{j=1}^{n}\frac{\varepsilon_jX_j}{d(n)}\right\}$. Using convergence of moments [5], for every $\varepsilon>0$, we can find a finite dimensional subspace F of B such that if $q_F(\cdot)=d(\cdot,F)$,

$$\sup_n E\left\{q_F\left(\frac{S_n-nE\{X\}}{d(n)}\right)\right\}\leqq\frac{\varepsilon}{2}.$$

Defining $\bar{F}$ to be the finite dimensional subspace of B generated by F and $E\{X\}$, we get that

$$\sup_n E\left\{q_{\bar{F}}\left(\sum_{j=1}^{n}\frac{\varepsilon_jX_j}{d(n)}\right)\right\}\leqq\varepsilon.$$

The tightness of $\left\{\sum_{j=1}^{n}\frac{\varepsilon_jX_j}{d(n)}\right\}$ follows from its stochastic boundedness and this finite dimensional approximation.

The following proposition is the main point in the proof of Theorem 2.1. Anticipating, we notice that it also contains part of the proof of Theorem 2.3.

Proposition 3.5. *Let X be as in the theorem, r be an integer and assume* (2.2) *holds. Then, for every continuous semi-norm q on B satisfying*

$$E\left\{q\left(\sum_{j=1}^{n}\frac{\varepsilon_jX_j}{d(n)}\right)\right\}\leqq\tfrac{1}{5}\tag{3.7}$$

for all n large enough, we have

$$\limsup_n q\left(\frac{{}^{(r)}S_n-nE\{X\}}{\gamma(n)}\right)\leqq 3\sqrt{2}\quad w.p.l.\tag{3.8}$$

Proof. We will show that for every $\varepsilon>0$ there exists a $\beta>1$ such that if $n_k=[\beta^k]$, and $I(k)=(n_k,n_{k+1}]$, $k\geqq 1$,

$$\sum_k P\left\{\max_{n\in I(k)} q({}^{(r)}S_n-nE\{X\})>(3\sqrt{2}+\varepsilon)\gamma(n_k)\right\}<\infty.$$

Since q is continuous, there is a constant $M<\infty$ such that $q(x)\leqq M\|x\|$ for each x in B. Let $\varepsilon>0$ be fixed. We choose an integer m such that $l=2^m>3r+1$ ($m=1$ if

$r=0$!) and take $\tau=\tau(r,m,M,\varepsilon)$ such that

$$0<\tau<\frac{\varepsilon}{24M\cdot 3^m}.$$

Let also $\varrho>0$ be specified later. Define for every $k\geqq 1$ and $1\leqq j\leqq n_{k+1}$,

$$\begin{aligned} &u_j=u_j(k)=X_j I_{\{\|X_j\|\leqq\varrho d\alpha(n_k)\}},\\ (3.9)\qquad &v_j=v_j(k)=X_j I_{\{\varrho d\alpha(n_k)<\|X_j\|\leqq\tau\alpha^{-1}d\alpha(n_k)\}},\\ &w_j=w_j(k)=X_j I_{\{\|X_j\|>\tau\alpha^{-1}d\alpha(n_k)\}},\end{aligned}$$

and for $1\leqq n\leqq n_{k+1}$:

$$\begin{aligned} &U_n=\sum_{j=1}^{n}(u_j-E\{u_j\}),\\ (3.10)\qquad &V_n=\sum_{j=1}^{n}(v_j-E\{v_j\}),\\ &W_n=\sum_{j=1}^{n}(w_j-E\{w_j\}).\end{aligned}$$

Clearly $S_n-nE\{X\}=U_n+V_n+W_n$; further

$$\max_{n\in I(k)} q({}^{(r)}S_n-nE\{X\})\leqq\max_{n\in I(k)} q(U_n)+M\max_{n\in I(k)}\|V_n\|+M\max_{n\in I(k)}\left\|W_n-\sum_{j=1}^{r}X_n^{(j)}\right\|.$$

The result will therefore be proved if we show that for every $\beta>1$ and $\varrho>0$

$$(3.11)\qquad \sum_k P\left\{\max_{n\in I(k)}\left\|W_n-\sum_{j=1}^{r}X_n^{(j)}\right\|>\frac{\varepsilon}{3M}\gamma(n_k)\right\}<\infty,$$

$$(3.12)\qquad \sum_k P\left\{\max_{n\in I(k)}\|V_n\|>\frac{\varepsilon}{3M}\gamma(n_k)\right\}<\infty,$$

and for some $\beta>1$ and $\varrho>0$

$$(3.13)\qquad \sum_k P\left\{\max_{n\in I(k)} q(U_n)>\left(3\sqrt{2}+\frac{\varepsilon}{3}\right)\gamma(n_k)\right\}<\infty.$$

Let us first prove (3.11). We note that

$$W_n-\sum_{j=1}^{r}X_n^{(j)}=\sum_{j=1}^{n}w_j-\sum_{j=1}^{r}X_n^{(j)}-nE\{XI_{\{\|X\|>\tau\alpha^{-1}d\alpha(n_k)\}}\}$$

and hence

$$\max_{n\in I(k)}\left\|W_n-\sum_{j=1}^{r}X_n^{(j)}\right\|\leqq\max_{n\in I(k)}\left\|\sum_{j=1}^{n}w_j-\sum_{j=1}^{r}X_n^{(j)}\right\|+n_{k+1}\|E\{XI_{\{\|X\|>\tau\alpha^{-1}d\alpha(n_k)\}}\}\|.$$

Since $d(t)$ is continuous and strictly increasing we choose $m=m(k)$ such that $d(m)=\alpha^{-1}d\alpha(n_k)$. Now, since T is slowly varying, it is easily seen that

$$\lim_k \frac{n_k}{\gamma(n_k)} \cdot \frac{d(m)}{m} = 0,$$

so that we deduce from (3.6) that

$$\lim_k \frac{n_{k+1}}{\gamma(n_k)} \|E\{XI_{\{\|X\|>\tau\alpha^{-1}d\alpha(n_k)\}}\}\| = 0.$$

Note that the limit in (3.6) holds as $n\to\infty$ through the integers, but it is also true if n is a continuous variable. Therefore, to prove (3.11), it suffices to show that

$$\sum_k P\left\{\max_{n\in I(k)} \left\|\sum_{j=1}^n w_j - \sum_{j=1}^r X_n^{(j)}\right\| > \frac{\varepsilon}{6M}\gamma(n_k)\right\} < \infty .$$

But now, for k large enough

$$\text{(3.14)}\quad P\left\{\max_{n\in I(k)} \left\|\sum_{j=1}^n w_j - \sum_{j=1}^r X_n^{(j)}\right\| > \frac{\varepsilon}{6M}\gamma(n_k)\right\}$$
$$\leqq P\{\text{at least } r+1\ X_j\text{'s } (1\leqq j\leqq n_{k+1}) \text{ satisfy } \|X_j\|>\tau\alpha^{-1}d\alpha(n_k)\}$$

since on the complement of this set

$$\max_{n\in I(k)} \left\|\sum_{j=1}^n w_j - \sum_{j=1}^r X_n^{(j)}\right\| \leqq r\tau\alpha^{-1}d\alpha(n_k) \leqq \frac{\varepsilon}{6M}\gamma(n_k)$$

for $\tau\leqq\varepsilon/6Mr$ and $\gamma(n_k)\sim\sqrt{2}\,\alpha^{-1}d\alpha(n_k)$. It follows then from (3.14) that

$$P\left\{\max_{n\in I(k)} \left\|\sum_{j=1}^n w_j - \sum_{j=1}^r X_n^{(j)}\right\| > \frac{\varepsilon}{6M}\gamma(n_k)\right\} \leqq (n_{k+1}P\{\|X\|>\tau\alpha^{-1}d\alpha(n_k)\})^{r+1}$$

and (3.11) holds by Lemma 3.1.

Next we turn to the proof of (3.12). We first show that

$$\text{(3.15)}\qquad \lim_k \frac{E\{\|V_{n_{k+1}}\|\}}{\gamma(n_k)} = 0 .$$

Since X is in the $DA(G)$, [5] and $\lim_n \gamma(n)/d(n)=\infty$ together imply

$$\lim_n \frac{E\{\|S_n - nE\{X\}\|\}}{\gamma(n)} = 0 .$$

Using classical inequalities involving Rademacher random variables and a contraction principle, for each k

$$E\{\|V_{n_{k+1}}\|\} \leqq 2E\left\{\left\|\sum_{j=1}^{n_{k+1}} \varepsilon_j v_j\right\|\right\} \leqq 2E\left\{\left\|\sum_{j=1}^{n_{k+1}} \varepsilon_j X_j\right\|\right\}$$

and (3.15) follows from the inequalities in the proof of Lemma 3.4.

By Ottaviani's inequality, for every k,

$$P\left\{\max_{n\in I(k)}\|V_n\|>\frac{\varepsilon}{3M}\gamma(n_k)\right\}\leqq\frac{P\left\{\|V_{n_{k+1}}\|>\frac{\varepsilon}{6M}\gamma(n_k)\right\}}{1-\max_{n\in I(k)}P\left\{\|V_{n_{k+1}}-V_n\|>\frac{\varepsilon}{6M}\gamma(n_k)\right\}},$$

but for every $n\in I(k)$,

$$P\left\{\|V_{n_{k+1}}-V_n\|>\frac{\varepsilon}{6M}\gamma(n_k)\right\}\leqq\frac{6M}{\varepsilon\gamma(n_k)}E\{\|V_{n_{k+1}}-V_n\|\}\leqq\frac{6M}{\varepsilon\gamma(n_k)}E\{\|V_{n_{k+1}}\|\},$$

and hence by (3.15), for k sufficiently large,

$$P\left\{\max_{n\in I(k)}\|V_n\|>\frac{\varepsilon}{3M}\gamma(n_k)\right\}\leqq 2P\left\{\|V_{n_{k+1}}\|>\frac{\varepsilon}{6M}\gamma(n_k)\right\}.$$

Of course we also have that $P\left\{\|V_{n_{k+1}}\|\leqq\frac{\varepsilon}{12M}\gamma(n_k)\right\}\geqq\frac{1}{2}$ for k large and thus

(3.16)
$$\begin{aligned}&P\left\{\max_{n\in I(k)}\|V_n\|>\frac{\varepsilon}{3M}\gamma(n_k)\right\}\\&\leqq 4P\left\{\|V_{n_{k+1}}\|>\frac{\varepsilon}{6M}\gamma(n_k),\ \|V'_{n_{k+1}}\|\leqq\frac{\varepsilon}{12M}\gamma(n_k)\right\}\\&\leqq 4P\left\{\|\tilde V_{n_{k+1}}\|>\frac{\varepsilon}{12M}\gamma(n_k)\right\}\end{aligned}$$

where $\tilde V_{n_{k+1}}=V_{n_{k+1}}-V'_{n_{k+1}}$ and $V'_{n_{k+1}}=\sum_{j=1}^{n_{k+1}}(v'_j-E\{v'_j\})$ is an independent copy of $V_{n_{k+1}}$. Now, by J. Hoffman-Jorgensen's inequality [11] iterated m times, for some constant C,

$$P\left\{\|\tilde V_{n_{k+1}}\|>\frac{\varepsilon}{12M}\gamma(n_k)\right\}\leqq C\left(P\left\{\|\tilde V_{n_{k+1}}\|>\frac{\varepsilon}{12M\cdot 3^m}\gamma(n_k)\right\}\right)^l \tag{3.17}$$

since $\|v_j-v'_j\|\leqq 2\tau\alpha^{-1}d\alpha(n_k)\leqq 2\tau\gamma(n_k)\leqq\varepsilon\gamma(n_k)/12M\cdot 3^m$ for k sufficiently large. Using again (3.15), for k big enough

(3.18)
$$\begin{aligned}&P\left\{\|\tilde V_{n_{k+1}}\|>\frac{\varepsilon}{12M\cdot 3^m}\gamma(n_k)\right\}\\&\leqq 2P\left\{\|V_{n_{k+1}}\|>\frac{\varepsilon}{24M\cdot 3^m}\gamma(n_k)\right\}\\&\leqq 2P\left\{|\,\|V_{n_{k+1}}\|-E\{\|V_{n_{k+1}}\|\}|>\frac{\varepsilon}{48M\cdot 3^m}\gamma(n_k)\right\}.\end{aligned}$$

Now $\|V_{n_{k+1}}\| - E\{\|V_{n_{k+1}}\|\}$ can be written as a sum of martingale difference sequence d_j, $1 \leqq j \leqq n_{k+1}$, such that (cf. [2])

$$E\{d_j^2\} \leqq 4E\{\|v_j - E\{v_j\}\|^2\} \leqq 16E\{\|v_j\|^2\},$$

and therefore, by Chebyschev's inequality,

$$(3.19)\qquad P\left\{|\,\|V_{n_{k+1}}\| - E\{\|V_{n_{k+1}}\|\}| > \frac{\varepsilon}{48M\cdot 3^m}\right\} \leqq 16\left(\frac{48M\cdot 3^m}{\varepsilon}\right)^2 \sum_{j=1}^{n_{k+1}} E\{\|v_j\|^2\}.$$

Bringing together (3.16), (3.17), (3.18), and (3.19) we finally get that for some constant C and all k large enough

$$P\left\{\max_{n\in I(k)} \|V_n\| > \frac{\varepsilon}{3M}\gamma(n_k)\right\} \leqq C\left(\frac{1}{(\gamma(n_k))^2}\sum_{j=1}^{n_{k+1}} E\{\|v_j\|^2\}\right)^l$$

and hence (3.12) will be proved if we show that under (2.2)

$$\Sigma = \sum_k \left(\frac{1}{(\gamma(n_k))^2}\sum_{j=1}^{n_{k+1}} E\{\|v_j\|^2\}\right)^l < \infty$$

for $l = 2^m > 3r+1$. To this aim, we need two elementary estimates and we assume $\tau \leqq 1$ for simplicity. First, for k large enough and some $C < \infty$

$$(3.20)\qquad E\{\|X\|^2 I_{\{\varrho d\alpha(n_k) < \|X\| \leqq \alpha^{-1} d\alpha(n_k)\}}\} \leqq C(T\alpha^{-1} d\alpha(n_k))^2 L_3 n_k$$

where $L_3 t$ denotes $L(L_2 t)$. Let us prove (3.20). By integration by parts,

$$E\{\|X\|^2 I_{\{\varrho d\alpha(n_k) < \|X\| \leqq \alpha^{-1} d\alpha(n_k)\}}\} \leqq (\varrho d\alpha(n_k))^2 P\{\|X\| > \varrho d\alpha(n_k)\} + \int_{\varrho d\alpha(n_k)}^{\alpha^{-1} d\alpha(n_k)} P\{\|X\| > t\} dt^2.$$

Using now that $\lim_{t\to\infty} tP\{\|X\| > d(t)\} = 0$ and $d(t) \sim \sqrt{t}\, Td(t)$, we get that for k large enough

$$\begin{aligned} E\{\|X\|^2 I_{\{\varrho d\alpha(n_k) < \|X\| \leqq \alpha^{-1} d\alpha(n_k)\}}\} &\leqq C\left((Td\alpha(n_k))^2 + \int_{\varrho d\alpha(n_k)}^{\alpha^{-1} d\alpha(n_k)} \frac{T^2(t)}{t^2} dt^2\right) \\ &\leqq C\left((Td\alpha(n_k))^2 + (T\alpha^{-1} d\alpha(n_k))^2 L\left(\frac{\alpha^{-1} d\alpha(n_k)}{\varrho d\alpha(n_k)}\right)\right) \end{aligned}$$

and (3.20) follows since $\alpha^{-1}(t) \sim tL_2 t$. Recall now that

$$(3.21)\qquad \frac{(\gamma(n_k))^2}{n_k} \sim 2L_2 n_k (Td\alpha(n_k))^2$$

and from [13], p. 546, since T is slowly varying, for every $\delta>0$ (to be specified later),

$$\frac{T\alpha^{-1}d\alpha(n_k)}{Td\alpha(n_k)} \leqq 2(L_2 n_k)^{\delta/4} \tag{3.22}$$

for k sufficiently large.

We turn to $\sum$ and in the estimates below, C will denote a constant possibly varying from line to line; also the sum over k in $\sum$ is assumed for k large enough. By (3.20)

$$\sum \leqq C\sum_k \left(\frac{n_k}{(\gamma(n_k))^2}\right)^l ((T\alpha^{-1}d\alpha(n_k))^2 L_3 n_k)^{l-r-1} (E\{\|X\|^2 I_{\{\varrho d\alpha(n_k)\leqq \|X\| \leqq \alpha^{-1}d\alpha(n_k)\}}\})^{r+1}$$

and by (3.21), (3.22) and independence:

$$\sum \leqq CE\left\{\|X_1\|^2 \ldots \|X_{r+1}\|^2 \sum_k \frac{(L_2 n_k)^{\delta(l-r-1)-l}}{(Td\alpha(n_k))^{2(r+1)}} \times I_{\{\varrho d\alpha(n_k)\leqq \|X_1\| \leqq \ldots \leqq \|X_{r+1}\| \leqq \alpha^{-1}d\alpha(n_k)\}}\right\}.$$

Using the properties of d and the fact that on the set

$$\{\varrho d\alpha(n_k)\leqq \|X_1\| \leqq \ldots \leqq \|X_{r+1}\| \leqq \alpha^{-1}d\alpha(n_k)\}$$

we have $\|X_j\| \leqq C\|X_1\| L_2\|X_1\|$ for $1<j\leqq r+1$, it is easily seen that

$$\begin{aligned}\sum &\leqq CE\left\{\|X_1\|^{2(r+1)} \frac{(L_2\|X_1\|)^{2r+\delta(l-r-1)-l}}{(1+T\alpha(\|X_1\|))^{2(r+1)}} I_{\{\|X_1\|\leqq \ldots \leqq \|X_{r+1}\|\}} \times \sum_k I_{\{\varrho d\alpha(n_k)\leqq \|X_1\| \leqq \alpha^{-1}d\alpha(n_k)\}}\right\} \\ &\leqq CE\left\{\left(\frac{\|X_1\|}{1+T\alpha(\|X_1\|)}\right)^{2(r+1)} (L_2\|X_1\|)^{2r+\delta(l-r-1)-l} L_3\|X_1\| I_{\{\|X_1\|\leqq \ldots \leqq \|X_{r+1}\|\}}\right\}.\end{aligned}$$

As $\alpha^{-1}d^{-1}\alpha(t) \sim t^2/L_2 t(T\alpha(t))^2$ and since $l>3r+1$, we choose $\delta>0$ so that $l-\delta(l-r-1)-2r>r+1$ and it follows that

$$\sum \leqq C\mathscr{I}_{r+1}(X)<\infty .$$

This establishes (3.12) and the proof of the proposition will therefore be complete if we show that (3.13) holds. For $k\geqq 1$, let $p_k=[L_2 n_k]$ and $s_k=n_{k+1}/p_k$. Note that

$$\gamma(n_k) \sim \sqrt{2}\alpha^{-1}d\alpha(n_k) \sim \sqrt{2}L_2 n_k d\alpha(n_k) \geqq \sqrt{2}p_k d\alpha(n_k);$$

therefore, for k sufficiently large

$$P\left\{\max_{n\in I(k)} q(U_n) > \left(3\sqrt{2}+\frac{\varepsilon}{3}\right)\gamma(n_k)\right\} \leqq P\left\{\max_{n\in I(k)} q(U_n) > \left(3\sqrt{2}+\frac{\varepsilon}{6}\right)\sqrt{2}p_k d\alpha(n_k)\right\}.$$

We set $A=\left(3\sqrt{2}+\frac{\varepsilon}{2}\right)\sqrt{2}$ for simplicity. By the maximal inequality for submartingales, for every $k\geqq 1$ and $\lambda>0$

$$\begin{aligned} P\left\{\max_{n\in I(k)} q(U_n)>Ap_k d\alpha(n_k)\right\} &\leqq \exp(-\lambda Ap_k)E\{\exp\lambda q(U_{n_{k+1}})/d\alpha(n_k)\} \\ &\leqq \exp(-\lambda Ap_k)E\left\{\prod_{m=1}^{p_k} e^{\lambda T_m}\right\} \\ &= \exp(-\lambda Ap_k)\prod_{m=1}^{p_k} E\{e^{\lambda T_m}\} \end{aligned}$$

where $T_m=q\left(\sum_{j=(m-1)s_k}^{ms_k} u_j-E\{u_j\}\right)\Big/ d\alpha(n_k)$ and $\sum_{j=s}^{t}=\sum_{j=[s]+1}^{[t]}$. We now recall an inequality due to A. de Acosta [1], Lemma 2.2, and apply it to our setting: for every $\lambda>0$, $t>0$, $m=1,\ldots,p_k$, and c such that $q(u_j-E\{u_j\})\leqq cd\alpha(n_k)$,

$$\begin{aligned} (3.23)\quad E\{e^{\lambda T_m}\} &\leqq e^{\lambda t}+e^{\lambda(t+c)} \\ &\times P\left\{\max_{(m-1)s_k<i\leqq ms_k} q\left(\sum_{j=(m-1)s_k}^{i}(u_j-E\{u_j\})\right)>td\alpha(n_k)\right\}E\{e^{\lambda T_m}\}. \end{aligned}$$

But now, for every $m=1,\ldots,p_k$ and k large enough, applying again the maximal inequality for submartingales,

$$\begin{aligned} (3.24)\quad & P\left\{\max_{(m-1)s_k<i\leqq ms_k} q\left(\sum_{j=(m-1)s_k}^{i} u_j-E\{u_j\}\right)>2\beta d\alpha(n_k)\right\} \\ &\leqq \frac{1}{2\beta d\alpha(n_k)}E\left\{q\left(\sum_{j=(m-1)s_k}^{ms_k} u_j-E\{u_j\}\right)\right\} \\ &\leqq \frac{1}{\beta d\alpha(n_k)}E\left\{q\left(\sum_{j=(m-1)s_k}^{ms_k} \varepsilon_j u_j\right)\right\} \\ &\leqq \frac{1}{\beta d\alpha(n_k)}E\left\{q\left(\sum_{j=1}^{[s_k]+1} \varepsilon_j X_j\right)\right\} \\ &\leqq \frac{1}{d([s_k]+1)}E\left\{q\left(\sum_{j=1}^{[s_k]+1} \varepsilon_j X_j\right)\right\} \end{aligned}$$

where the last inequality follows from the fact that

$$d([s_k]+1)\sim d(s_k)\sim d\left(\frac{n_{k+1}}{L_2 n_k}\right)\sim d(\beta\alpha(n_k))\sim\sqrt{\beta}\, d\alpha(n_k).$$

Setting $t=2\beta$, $c=2\varrho M$, and $\lambda=1/2(\beta+\varrho M)$ in (3.23) and using (3.7) and (3.24) we get

$$E\{e^{\lambda T_m}\}\leqq e^{2\lambda\beta}+\tfrac{1}{5}e^{2\lambda(\beta+\varrho M)}E\{e^{\lambda T_m}\}\leqq e^{2\lambda\beta+1}$$

and thus

$$P\left\{\max_{n\in I(k)} q(U_n)>Ap_k d\alpha(n_k)\right\}\leqq\exp(-p_k(\lambda A-2\lambda\beta-1)).$$

These probabilities are summable if $\lambda A - 2\lambda\beta - 1 > 1$, that is, since $\lambda = 1/2(\beta + \varrho M)$, if $A > 6\beta + 4\varrho M$. Recalling that $A = \left(3\sqrt{2} + \frac{\varepsilon}{6}\right)\sqrt{2}$, this is accomplished if we choose $\beta > 1$ and $\varrho > 0$ such that

$$\frac{\varepsilon}{6} > 3\sqrt{2}(\beta - 1) + 2\sqrt{2}\,\varrho M\,.$$

This finishes the proof of Proposition 3.5.

The following proposition will allow us to identify the limit set of $\left\{\frac{{}^{(r)}S_n - nE\{X\}}{\gamma(n)}\right\}$ with $K = K_{\mathscr{L}(G)}$.

Proposition 3.6. *Under the hypothesis of Theorem 2.1 and condition* (2.2), *if f is a linear functional on B such that* $(E\{f^2(G)\})^{1/2} = \sigma_f > 0$, *we have*

$$\limsup_n f\left(\frac{{}^{(r)}S_n - nE\{X\}}{\gamma(n)}\right) = \sigma_f \quad w.p.1\,. \tag{3.25}$$

Proof. We assume without loss of generality that $\|f\| = 1$. We need to recall [4], Corollary 2.12, that for every $\delta > 0$

$$\lim_{t\to\infty} \frac{t}{(d(t))^2} E\{f^2(XI_{\{\|X\| \leqq \delta d(t)\}} - E\{XI_{\{\|X\| \leqq \delta d(t)\}}\})\} = \sigma_f^2\,. \tag{3.26}$$

Let us first prove the upper bound

$$\limsup_n \left| f\left(\frac{{}^{(r)}S_n - nE\{X\}}{\gamma(n)}\right)\right| \leqq \sigma_f \quad \text{w.p.}1\,. \tag{3.27}$$

We set $n_k = [\beta^k]$, $k \geqq 1$, where $\beta > 1$ is to be specified later and define u_j, v_j, w_j, U_n, V_n, W_n as in (3.9) and (3.10) with $\varrho > 0$ and $\tau > 0$ also to be specified later. Taking into account points (3.11) and (3.12) of the proof of Proposition 3.5 (with some appropriate choice of $\tau > 0$), we have only to show that for every $\varepsilon > 0$ there exist $\beta > 1$ and $\varrho > 0$ such that

$$\sum_k P\left\{\max_{n\in I(k)} |f(U_n)| > (1+\varepsilon)\sigma_f \gamma(n_k)\right\} < \infty\,. \tag{3.28}$$

By the maximal inequality for submartingales, for every $\lambda > 0$,

$$P\left\{\max_{n\in I(k)} |f(U_n)| > (1+\varepsilon)\sigma_f\gamma(n_k)\right\} \leqq \exp(-\lambda(1+\varepsilon)\sigma_f\gamma(n_k))E\{\exp\lambda|f(U_n)|\}$$

and by the proof of the exponential inequality in Lemma 2.2 of [3], this probability is also less than or equal to

$$2\exp\left(-\lambda(1+\varepsilon)\sigma_f\gamma(n_k) + \frac{\lambda^2}{2}\sum_{j=1}^{n_{k+1}} E\{f^2(u_j - E\{u_j\})\}e^{\lambda c}\right)$$

where $c=2\varrho d\alpha(n_k)$. Recalling that $\gamma(n_k)\sim\sqrt{2}\,L_2n_kd\alpha(n_k)$ and

$$\frac{n_{k+1}}{(d\alpha(n_k))^2}E\{f^2(XI_{\{\|X\|\leqq\varrho d\alpha(n_k)\}}-E\{XI_{\{\|X\|\leqq\varrho d\alpha(n_k)\}}\})\}\sim\beta L_2n_k\sigma_f^2,$$

we choose $\lambda=\sqrt{2}/\sigma_f d\alpha(n_k)$, and thus, for any $\delta>0$ and k sufficiently large

$$\begin{aligned}&P\left\{\max_{n\in I(k)}|f(U_n)|>(1+\varepsilon)\sigma_f\gamma(n_k)\right\}\\&\leqq 2\exp(-(2+\varepsilon)L_2n_k+(\beta+\delta)L_2n_ke^{2\sqrt{2}\varrho/\sigma_f})\\&=2\exp(-L_2n_k(2+\varepsilon-(\beta+\delta)e^{2\sqrt{2}\varrho/\sigma_f})).\end{aligned}$$

We then choose δ, $\varrho>0$, and $\beta>1$ such that

$$2+\varepsilon-(\beta+\delta)e^{2\sqrt{2}\varrho/\sigma_f}\geqq 1+\frac{\varepsilon}{2}$$

and finally get that for k large enough

$$P\left\{\max_{n\in I(k)}|f(U_n)|>(1+\varepsilon)\sigma_f\gamma(n_k)\right\}\leqq 2\exp\left(-\left(1+\frac{\varepsilon}{2}\right)L_2n_k\right);$$

this proves (3.28) and therefore also (3.27).

We next turn to the lower bound and prove that

$$\limsup_n f\left(\frac{{}^{(r)}S_n-nE\{X\}}{\gamma(n)}\right)\geqq\sigma_f \quad \text{w.p.l.} \tag{3.29}$$

The notations are as before, we just take $\varrho=1$ for simplicity. Let $\varepsilon>0$ be fixed and define for each k

$$E_k=\{f(U_{n_{k+1}}-U_{n_k})>(1-\varepsilon)\sigma_f\gamma(n_{k+1})\}$$

and

$$F_k=\{|f(U_{n_k})|\leqq\varepsilon\sigma_f\gamma(n_{k+1})\}.$$

We will show that there is a $\beta>1$ large enough so that

$$\sum_k P(F_k^c)<\infty \quad\text{and}\quad \sum_k P(E_k)=\infty;$$

we will then have by the Borel-Cantelli Lemma that w.p.1.

$$f(U_{n_{k+1}}-U_{n_k})>(1-\varepsilon)\sigma_f\gamma(n_{k+1}) \quad \text{i.o.}$$

and

$$|f(U_{n_k})|\leqq\varepsilon\sigma_f\gamma(n_{k+1}) \quad \text{eventually},$$

that is

$$f(U_{n_{k+1}})>(1-2\varepsilon)\sigma_f\gamma(n_{k+1}) \quad \text{i.o.}$$

But it follows from (3.11) and (3.12) that for some appropriate choice of $\tau>0$, for every $\beta>1$

$$\sum_k P\{|f(V_{n_{k+1}})|>\varepsilon\sigma_f\gamma(n_{k+1})\}<\infty$$

and

$$\sum_k P\left\{\left|f\left(W_{n_{k+1}}-\sum_{j=1}^{r} X_{n_{k+1}}^{(j)}\right)\right|>\varepsilon\sigma_f\gamma(n_{k+1})\right\}<\infty,$$

so that, w.p.1.,

$$f({}^{(r)}S_{n_{k+1}}-n_{k+1}E\{X\})>(1-4\varepsilon)\sigma_f\gamma(n_{k+1}) \quad \text{i.o.}$$

which gives (3.29).

Let us now prove that $\sum_k P(F_k^c)<\infty$ and $\sum_k P(E_k)=\infty$. That $\sum_k P(F_k^c)<\infty$ follows from the same argument we used in the proof of the upper bound (3.27) since $\gamma(n_{k+1})\sim\sqrt{\beta}\,\gamma(n_k)$ and we take $\beta\geqq\beta_0>1$ for some β_0 large enough. We turn to $\sum_k P(E_k)$. Let $s_{n_{k+1}}^2=E\{f^2(U_{n_{k+1}}-U_{n_k})\}$; using (3.26)

$$\gamma(n_{k+1})\sim\sqrt{\beta}\,\gamma(n_k)\sim\sqrt{2\beta}\,L_2n_k d\alpha(n_k)\sim\frac{1}{\sigma_f}\sqrt{\frac{2\beta}{\beta-1}}\sqrt{L_2n_k}\,s_{n_{k+1}}.$$

Thus, for k large enough,

$$P(E_k)\geqq P\left\{f(U_{n_{k+1}}-U_{n_k})>\left(1-\frac{\varepsilon}{2}\right)\sqrt{\frac{2\beta}{\beta-1}}\sqrt{L_2n_k}\,s_{n_{k+1}}\right\}.$$

We apply to this last probability Kolmogorov's lower exponential inequality [18], p. 262. Let $\gamma>0$ be such that

$$(1+\gamma)\left(1-\frac{\varepsilon}{2}\right)^2<1-\frac{\varepsilon}{2},$$

and $\varepsilon(\gamma)$ and $\pi(\gamma)$ be as in [18]. Of course

$$\left(1-\frac{\varepsilon}{2}\right)\sqrt{\frac{2\beta}{\beta-1}}\sqrt{L_2n_k}\geqq\varepsilon(\gamma)$$

for k sufficiently large and since

$$|f(u_j-E\{u_j\})|\leqq 2d\alpha(n_k)\sim\frac{2}{\sqrt{\beta-1}}\cdot\frac{s_{n_{k+1}}}{\sqrt{L_2n_k}}\cdot\frac{1}{\sigma_f}=cs_{n_{k+1}}$$

where c is defined by the equality above, we then have

$$c\left(1-\frac{\varepsilon}{2}\right)\sqrt{\frac{2\beta}{\beta-1}}\sqrt{L_2n_k}=\left(1-\frac{\varepsilon}{2}\right)\frac{2\sqrt{2\beta}}{\sigma_f(\beta-1)}\leqq\pi(\gamma)$$

for $\beta \geqq \beta_1 > 1$, β_1 large enough and hence by Kolmogorov's lower bound

$$P(E_k) \geqq \exp\left(-\frac{\beta}{\beta-1}\left(1-\frac{\varepsilon}{2}\right)^2 (1+\gamma)L_2 n_k\right)$$
$$\geqq \exp\left(-\left(1-\frac{\varepsilon}{4}\right)L_2 n_k\right)$$

if $\left(1-\frac{\varepsilon}{2}\right)\beta/(\beta-1) \leqq 1-\frac{\varepsilon}{4}$ which is achieved for $\beta \geqq \beta_2 > 1$. Taking finally $\beta \geqq \max(\beta_0, \beta_1, \beta_2)$, we see that $\sum_k P(E_k) = \infty$ and thus (3.29) holds. The proof of Proposition 3.6 is complete.

Using Propositions 3.5 and 3.6 we can now conclude the proof of Theorem 2.1. Let us choose via Lemma 3.4, for every $\varepsilon > 0$, a finite dimensional subspace F such that

$$\sup_n E\left\{q_F\left(\sum_{j=1}^n \frac{\varepsilon_j X_j}{d(n)}\right)\right\} \leqq \varepsilon .$$

Applying Proposition 3.5 to $q_F/5\varepsilon$ we get

$$\limsup_n q_F\left(\frac{{}^{(r)}S_n - nE\{X\}}{\gamma(n)}\right) \leqq 15\sqrt{2}\varepsilon \quad \text{w.p.}1.$$

Since it is also bounded, $\left\{\frac{{}^{(r)}S_n - nE\{X\}}{\gamma(n)}\right\}$ is relatively compact w.p. 1. Taking into account Proposition 3.6, Theorem 3.1 of [12] concludes the proof if B is infinite dimensional; if not, a standard finite dimensional argument (see, for example, the proof of Corollary 3.1 of [12]) can be used to establish Theorem 2.1.

We finish this section with the proof of Corollary 2.2. By Lemma 3.1, (2.2) is equivalent to

$$\sum_k (2^k P\{\|X\| > \alpha^{-1} d\alpha(2^k)\})^{r+1} < \infty .$$

Since X is in the $DA(G)$ and $d(t) \sim \sqrt{t}\, Td(t)$, for t large enough,

$$P\{\|X\| > t\} \leqq \frac{T^2(t)}{t^2},$$

and thus, for k sufficiently large and some constant C

$$2^k P\{\|X\| > \alpha^{-1} d\alpha(2^k)\} \leqq \frac{C}{L_2 2^k}\left(\frac{T\alpha^{-1} d\alpha(2^k)}{Td\alpha(2^k)}\right)^2 .$$

As was already noted, since T is slowly varying, for k large

$$\left(\frac{T\alpha^{-1} d\alpha(2^k)}{Td\alpha(2^k)}\right)^2 \leqq 2(L_2 2^k)^{1-\varrho/r}$$

when $r>\varrho$. But then, C being a constant possibly changing from line to line, we have

$$\begin{aligned}\sum_k (2^k P\{\|X\|>\alpha^{-1}d\alpha(2^k)\})^{r+1} &\leqq C\sum_k \frac{2^k}{(L_2 2^k)^{\varrho}} P\{\|X\|>\alpha^{-1}d\alpha(2^k)\}\\ &= CE\left\{\sum_k \frac{2^k}{(L_2 2^k)^{\varrho}} I_{\{2^k<\alpha^{-1}d^{-1}\alpha(\|X\|)\}}\right\}\\ &\leqq CE\left\{\frac{\alpha^{-1}d^{-1}\alpha(\|X\|)}{(L_2\|X\|)^{\varrho}}\right\}.\end{aligned}$$

The further conclusions of the corollary follow immediately from the preceding proof since we are assuming $T\alpha(t)\sim T(t)$ when $\varrho=r$.

4. Proof of Theorem 2.3

We first construct the sequence $\{\xi_n\}$ which will determine the number of maximal terms we remove. Let

$$\Lambda(t)=\sup_{s\geqq t} sP\{\|X\|>d(s)\}.$$

Since X is in the $DA(G)$, we know that $\lim_{t\to\infty}\Lambda(t)=0$. We eliminate the case $\Lambda(t)=0$ for some t since in this case X is bounded and Theorem 1.2 implies (2.4) holds with $\xi_n\equiv 0$. Choose then for $\{\xi_n\}$ any sequence of positive real numbers decreasing to 0 such that for every $\delta>0$

$$\lim_n \xi_n L\left(\frac{\xi_n}{\Lambda(\delta\alpha(n))}\right)=\infty. \tag{4.1}$$

As a possible choice for $\{\xi_n\}$ we suggest ξ_n defined for n large enough by

$$\xi_n=\left(L\left(\frac{1}{\Lambda(\sqrt{n})}\right)\right)^{-1/2}.$$

Take now any sequence $\{\xi_n\}$ decreasing to zero satisfying (4.1) and let it be fixed; since a sequence larger than ξ_n still satisfies (4.1), we may clearly assume $\xi_n\searrow 0$ and $r_n=[\xi_n L_2 n]\nearrow\infty$.

We will need the following analogue of Proposition 3.5.

Proposition 4.1. *Let X be as in the Theorem 2.3 and $r_n=[\xi_n L_2 n]$. Then, for every continuous semi-norm q on B satisfying*

$$E\left\{q\left(\sum_{j=1}^n \frac{\varepsilon_j X_j}{d(n)}\right)\right\}\leqq\frac{1}{5} \tag{4.2}$$

for n large enough, we have

$$\limsup_n q\left(\frac{{}^{(r_n)}S_n-nE\{X\}}{\gamma(n)}\right)\leqq 3\sqrt{2}\quad\text{w.p.1}. \tag{4.3}$$

Proof. We will prove that for every $\varepsilon>0$ there exists a $\beta>1$ such that

$$\sum_k P\left\{\max_{n\in I(k)} q({}^{(r_n)}S_n - nE\{X\}) > (3\sqrt{2}+\varepsilon)\gamma(n_k)\right\} < \infty .$$

Let M be such that $q(x)\leqq M\|x\|$ for each x in B. Fix $\varepsilon>0$ and let $\varrho>0$ be specified later. Define for every $k\geqq 1$ and $1\leqq j\leqq n_{k+1}$

$$u_j = u_j(k) = X_j I_{\{\|X_j\|\leqq \varrho d\alpha(n_k)\}},$$
$$w_j = w_j(k) = X_j - u_j$$

and for $1\leqq n\leqq n_{k+1}$,

$$U_n = \sum_{j=1}^{n} (u_j - E\{u_j\}),$$
$$W_n = \sum_{j=1}^{n} (w_j - E\{w_j\}).$$

Clearly

$$\max_{n\in I(k)} q({}^{(r_n)}S_n - nE\{X\}) \leqq \max_{n\in I(k)} q(U_n) + M \max_{n\in I(k)} \left\| W_n - \sum_{j=1}^{r_n} X_n^{(j)} \right\| .$$

The proposition will thus be proved if we show that for all $\beta>1$ and $\varrho>0$

$$\sum_k P\left\{\max_{n\in I(k)} \left\| W_n - \sum_{j=1}^{r_n} X_n^{(j)} \right\| > \frac{\varepsilon}{2M}\gamma(n_k)\right\} < \infty, \tag{4.4}$$

and for some $\beta>1$ and $\varrho>0$

$$\sum_k P\left\{\max_{n\in I(k)} q(U_n) > \left(3\sqrt{2}+\frac{\varepsilon}{2}\right)\gamma(n_k)\right\} < \infty . \tag{4.5}$$

To establish (4.5) we need just to recall (3.13) of the proof of Proposition 3.5. Hence it is enough to show (4.4). Now

$$\max_{n\in I(k)} \left\| W_n - \sum_{j=1}^{r_n} X_n^{(j)} \right\| \leqq \max_{n\in I(k)} \left\| \sum_{j=1}^{n} w_j - \sum_{j=1}^{r_n} X_n^{(j)} \right\| + n_{k+1} \| E\{X I_{\{\|X\|>\varrho d\alpha(n_k)\}}\} \| .$$

From (3.6) and the fact that $\gamma(t)\sim\sqrt{2}\,L_2 t d\alpha(t)$, we see that

$$\lim_k \frac{n_{k+1}}{\gamma(n_k)} \| E\{X I_{\{\|X\|>\varrho d\alpha(n_k)\}}\} \| = 0,$$

so that (4.4) reduces to

$$\sum_k P\left\{\max_{n\in I(k)} \left\| \sum_{j=1}^{n} w_j - \sum_{j=1}^{r_n} X_n^{(j)} \right\| > \frac{\varepsilon}{4M}\gamma(n_k)\right\} < \infty .$$

We estimate these probabilities in the following way: since $r_n \nearrow \infty$ for k large enough,

$$P\left\{\max_{n\in I(k)}\left\|\sum_{j=1}^{n} w_j - \sum_{j=1}^{r_n} X_n^{(j)}\right\| > \frac{\varepsilon}{4M}\gamma(n_k)\right\}$$

$$\leqq P\{\text{at least } r_{n_k}+1\ X_j\text{'s } (1\leqq j\leqq n_{k+1}) \text{ satisfy } \|X_j\| > \varrho d\alpha(n_k)\}$$

since on the complement of this set

$$\max_{n\in I(k)}\left\|\sum_{j=1}^{n} w_j - \sum_{j=1}^{r_n} X_n^{(j)}\right\| \leqq \varrho \max_{n\in I(k)} r_n d\alpha(n_k) \leqq \frac{\varepsilon}{4M}\gamma(n_k).$$

Writing s_k for r_{n_k},

$$P\left\{\max_{n\in I(k)}\left\|\sum_{j=1}^{n} w_j - \sum_{j=1}^{r_n} X_n^{(j)}\right\| > \frac{\varepsilon}{4M}\gamma(n_k)\right\} \leqq \sum_{j=s_k+1}^{n_{k+1}} \binom{n_{k+1}}{j} p_k^j (1-p_k)^{n_{k+1}-j}$$

where $p_k = P\{\|X\| > \varrho d\alpha(n_k)\}$. By Feller [7], p. 173, and Stirling's formula, for all k sufficiently large

$$\sum_{j=s_k+1}^{n_{k+1}} \binom{n_{k+1}}{j} p_k^j (1-p_k)^{n_{k+1}-j} = n_{k+1}\binom{n_{k+1}-1}{s_k}\int_0^{p_k} t^{s_k}(1-t)^{n_{k+1}-s_k-1}dt$$

$$\leqq 2(en_{k+1}p_k)^{s_k} s_k^{-(s_k+\frac{1}{2})}.$$

As $d(\varrho^2 t) \sim \varrho d(t)$, for k large

$$\varrho^2\alpha(n_k)p_k \leqq 2\Lambda(\varrho^2\alpha(n_k)/2)$$

and therefore

$$P\left\{\max_{n\in I(k)}\left\|\sum_{j=1}^{n} w_j - \sum_{j=1}^{r_n} X_n^{(j)}\right\| > \frac{\varepsilon}{4M}\gamma(n_k)\right\}$$

$$\leqq 2\left(\frac{2e}{\varrho^2}\cdot\frac{n_{k+1}}{n_k}\Lambda(\varrho^2\alpha(n_k)/2)L_2 n_k\right)^{s_k} s_k^{-(s_k+\frac{1}{2})}$$

$$\leqq 2\left(\frac{4e\beta}{\varrho^2}\cdot\frac{\Lambda(\varrho^2\alpha(n_k)/2)}{\xi_{n_k}}\right)^{s_k} s_k^{-\frac{1}{2}}.$$

Remembering (4.1), we have (4.4) and Proposition 4.1 is proved.

The following proposition is just a consequence of (4.4) and the proof of Proposition 3.6 and may be proved as there. $\{\xi_n\}$ is fixed as before.

Proposition 4.2. *Under the hypothesis of Theorem 2.3, for any linear functional f on B such that $(E\{f^2(G)\})^{1/2} = \sigma_f > 0$,*

$$\limsup_n f\left(\frac{{}^{(r_n)}S_n - nE\{X\}}{\gamma(n)}\right) = \sigma_f \quad w.p.1. \tag{4.6}$$

The conclusion of the proof of Theorem 2.3 now follows exactly as for Theorem 2.1. We omit the details.

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