

Persistent Random Walks in Random Environment

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Summary. Weak convergence of a class of functionals of PRWRE is proved. As a consequence CLT is obtained for the normed trajectory.

1. Introduction

In the present article we investigate the asymptotic behaviour of Persistent (or Physical) Random Walks (PRW) in Random Environment (RE) on \mathbb{Z}^d .

Suppose $\mathcal{U} \subset \mathbb{Z}^d$ is a finite symmetric subset of the lattice generating the group of translations on \mathbb{Z}^d : the set of possible steps of the random walker. At each site of the lattice ($z \in \mathbb{Z}^d$) a random scatterer is placed characterized by a stochastic matrix $\Gamma^{(z)} = (\gamma_{u,u'}^{(z)})_{u,u' \in \mathcal{U}}$, which we shall call the persistency (or scattering) matrix at z . The persistency matrices are random and the collection of them is the RE.

Given the environment a PRW is a Markov chain of order two on \mathbb{Z}^d with transition probabilities

$$P(X_{n+1} = z + u' \mid X_n = z, X_{n-1} = z - u) = \gamma_{u,u'}^{(z)}. \quad (1.1)$$

The model can be considered as a stochastic version of the Lorentz gas with finite horizon.

The following three conditions are imposed on the scattering matrices:

0. $(\Gamma^{(z)})_{z \in \mathbb{Z}^d}$ form a stationary and ergodic sequence of random matrices (under translations on \mathbb{Z}^d)

1. they are almost surely bistochastic
2. they satisfy almost surely a uniform Doeblin condition (condition (b) of the next section).

Comment: Condition 0. is natural. Condition 1. is the weakest symmetry condition on the scattering mechanism. In fact, from a physical point of view

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stronger conditions seem to be natural (e.g. $\gamma_{u,u'} = \gamma_{-u',-u}$ a.s.). Condition 2. is a technical one, and we think that it can be considerably weakened.

Under the above conditions we prove that the finite dimensional distributions of $\varepsilon^{1/2} X_{[t, t+1]}$ converge to those of a Brownian motion with a positive definite covariance matrix (in probability with respect to the environment).

For the one-dimensional case with nearest neighbour jumps the invariance principle has been proved in [6] using a “very one-dimensional” ad hoc argument, so this result can be considered as a generalization of that one. The study of this model was proposed by D. Szász.

Besides the intrinsic physical interest of the model, we think that the main mathematical interest of our result consists of the fact that it relies on a generalization to non-reversible Markov chains of a theorem recently announced by C. Kipnis and S.R.S. Varadhan on the asymptotics of additive functionals of reversible Markov chains [3].

The paper consists of two further sections and an Appendix. In Sect. 2 we give the exact mathematical formulation of the problem and state our result. In Sect. 3 the proof is given. The Appendix contains the sketch of proof of the generalization of the Theorem of Kipnis and Varadhan.

2. Exact Formulation and Main Result

Before entering into details we have to specify some notations.

Throughout this paper $D([0, 1])$ will denote the space of right-continuous real functions defined on $[0, 1]$, endowed with the Skorohod topology. The Wiener measure of variance $\sigma^2 \geq 0$ on $D([0, 1])$ is denoted by \mathcal{W}_σ .

Let $(\eta_n)_{n \in \mathbb{N}}$ be a Markov chain with state space Ω and trajectory space $\Omega^{\mathbb{N}}$ (endowed with the natural product σ -algebra). We shall denote by $\Pi^{(\omega)}$ the Markovian measure on $\Omega^{\mathbb{N}}$ conditioned to the initial state $\eta_0 = \omega$. If μ is a probability measure on Ω (that is: an initial distribution of the Markov chain), Π^μ will denote the Markovian measure on the space of trajectories conditioned to this initial distribution

$$\Pi^\mu = \int_{\Omega} \mu(d\omega) \Pi^{(\omega)}.$$

We say that the Markov chain η_n is stationary (ergodic) with respect to μ iff the measure Π^μ is stationary (ergodic) under the left shift of $\Omega^{\mathbb{N}}$. We denote by \mathcal{P} the transition kernel acting on the space of bounded, measurable functions defined on Ω . If \mathcal{P} is compatible with the μ -equivalence of measurable functions, \mathbf{P} and $\check{\mathbf{P}}$ will denote the transition operator and its adjoint acting on the $L_p(\Omega, \mu)$ spaces.

The following lemma is standard (see for example [4])

Standard Lemma

i) Let $\rho \in L_1(\Omega, \mu)$, $\rho \geq 0$ μ -a.s., $\int d\mu \rho = 1$. The Markov chain η_n is stationary w.r.t. $\rho d\mu$ if and only if

$$\check{\mathbf{P}} \rho = \rho$$

ii) Let η_n be stationary w.r.t. μ . It is also ergodic w.r.t. μ if and only if the only solutions in $L_\infty(\Omega, \mu)$ of the equation

$$P\varphi = \varphi$$

are the constant functions.

Given φ a real measurable function on Ω , we are generally interested in the asymptotics (as $\varepsilon \rightarrow 0$) under the measures $\Pi^{(\omega)}$ of the random functions:

$$X_\varepsilon^{(\varphi)}: \Omega^{\mathbb{N}} \rightarrow D([0, 1]); \quad X_\varepsilon^{(\varphi)}((\omega_n)_{n \in \mathbb{N}}, t) = \varepsilon^{1/2} \sum_{k=1}^{[t\varepsilon^{-1}]} \varphi(\omega_k). \tag{2.1}$$

We define the notion of weak convergence to be used in the present paper.

Definition. The finite-dimensional distributions of $X_\varepsilon^{(\varphi)}$ converge in μ -probability with respect to the initial state to those of \mathcal{W}_σ if and only if given any bounded, continuous real function f on $D([0, 1])$ depending only on a finite number of coordinates

$$\int_{\Omega} \mu(d\omega) \left| \int_{\Omega^{\mathbb{N}}} d\Pi^{(\omega)} f(X_\varepsilon^{(\varphi)}) - \int_{D([0, 1])} d\mathcal{W}_\sigma f \right| \rightarrow 0. \tag{2.2a}$$

Or, equivalently: for any $\delta > 0$

$$\mu(\{\omega \in \Omega: \left| \int_{\Omega^{\mathbb{N}}} d\Pi^{(\omega)} f(X_\varepsilon^{(\varphi)}) - \int_{D([0, 1])} d\mathcal{W}_\sigma f \right| > \delta\}) \rightarrow 0. \tag{2.2b}$$

We are ready now to turn to our concrete problem. Let $(\Omega_0, \mathcal{F}_0, \mu_0, \{\tau_z: z \in \mathbb{Z}^d\})$ be an ergodic dynamical system on the group $\{\tau_z: z \in \mathbb{Z}^d\}$, $\mathcal{U} \subset \mathbb{Z}^d$ a finite symmetric subset of the lattice which spans \mathbb{Z}^d ($|\mathcal{U}| = \nu$) and $\Gamma = (\gamma_{u,u'})_{u, u' \in \mathcal{U}}$ a stochastic-matrix-valued measurable function on Ω_0 . We shall call $\omega \in \Omega_0$ a realization of the environment, \mathcal{U} the set of possible steps of the random walker and $\Gamma(\omega)$ the scattering matrix corresponding to the environment ω . Consider the Markov chain (v_n, η_n) with state space

$$\Omega = \mathcal{U} \times \Omega_0 = \{(u, \omega): u \in \mathcal{U}, \omega \in \Omega_0\}.$$

$v_n \in \mathcal{U}$ is the n -th step of the random walker, $\eta_n \in \Omega_0$ is the environment seen by him after the n -th step. The transition kernel of this Markov chain is

$$(\mathcal{P}\varphi)(u, \omega) = \sum_{u' \in \mathcal{U}} \gamma_{u,u'}(\omega) \varphi(u', \tau_u \omega). \tag{2.3}$$

Using (i) of the Standard Lemma one can easily check that if

$$(a) \quad \sum_{u \in \mathcal{U}} \gamma_{u,u'} = 1 \quad \mu_0\text{-a.s.} \tag{2.4}$$

then this Markov chain is stationary with respect to the initial distribution.:

$$\mu = m \times \mu_0 \quad \text{where } m(\{u\}) = \nu^{-1} \quad u \in \mathcal{U}. \tag{2.5}$$

(Later on it will be shown that, assuming also condition (b) below, it is also ergodic.)

Our result is the following:

Theorem 1. *If, besides (a), Γ satisfies also condition*

$$(b) \quad \|\Gamma - \Pi\| \leq 1 - \delta \quad \text{a.s.,} \quad \text{where } \Pi_{u,u'} = \frac{1}{v} \tag{2.6}$$

then for any $\varphi \in L_2(\Omega, \mu)$ satisfying

$$\sum_{u \in \mathcal{Q}} \varphi(u, \cdot) = 0 \tag{2.7}$$

the finite dimensional distributions of $X_\varepsilon^{(\varphi)}$, defined by (2.1), converge to those of a Wiener process \mathcal{W}_σ with positive variance, in μ_0 -probability with respect to the initial environment.

Comments: 1. Condition (a) is that of bistochasticity of Γ , (b) is a Doeblin condition uniformly imposed on the scattering matrices. We think that (b) can be weakened.

2. Let $\vec{\varphi}$ be a finite-dimensional vector-valued function on the state-space satisfying componentwise the conditions of Theorem 1. Applying componentwise the martingale approximation of the proof we readily obtain an appropriate vector-valued martingale approximation of the process $X_\varepsilon^{(\vec{\varphi})}$. Thus the theorem holds word by word for vector-valued $\vec{\varphi}$'s too. Choosing

$$\vec{\varphi}(u, \omega) = u \in \mathbb{R}^d \tag{2.8}$$

we obtain weak convergence of the finite-dimensional distributions of the normed trajectories of PRWRE.

Theorem 1 will be proved using the following generalization to non-reversible Markov chains of a recent result of Kipnis and Varadhan [3].

Theorem KV. *Let η_n be a Markov chain with state space Ω , stationary and ergodic with respect to the initial distribution μ . \mathbf{P} is the transition operator acting on $L_2(\Omega, \mu)$ $\mathbf{R}_\lambda = (\mathbf{I} - \lambda \mathbf{P})^{-1}$, $0 \leq \lambda < 1$, its resolvent. Let $\varphi \in L_2(\Omega, \mu)$, with $\int d\mu \varphi = 0$, satisfy*

$$(i) \quad (\varphi, \mathbf{R}_\lambda \varphi) \rightarrow \frac{1}{2}(\sigma^2 + \|\varphi\|^2) \quad \text{as } \lambda \nearrow 1, \tag{2.9}$$

$$(ii) \quad (1 - \lambda)(\mathbf{R}_\lambda \varphi, \mathbf{R}_{\lambda'} \varphi) \rightarrow 0 \quad \text{as } \lambda, \lambda' \nearrow 1 \tag{2.10}$$

Then the finite dimensional distributions of $X_\varepsilon^{(\varphi)}$ converge, in μ -probability with respect to the initial state, to those of \mathcal{W}_σ .

Comment. The theorem of *K&V* was stated originally for reversible Markov chains (that is: \mathbf{P} self-adjoint) and only a condition equivalent to (i) above was imposed on φ . If \mathbf{P} is self-adjoint, our second condition is implied by the first one. In [3] the theorem is proved by using essentially a form of the spectral theorem for selfadjoint operators and also tightness of the processes is obtained. We give a sketch of our proof – which uses resolvent calculus instead of spectral theorem – in the Appendix. Unfortunately, for the present moment, we are not able to prove tightness. Hence the fact that we formulate our theorems in terms of the finite dimensional distributions.

3. Proof of Theorem 1

This section is divided into four subsections. In the first one the formalism needed in the sequel is introduced. In the second one ergodicity of the Markov chain is proved. In the last two we verify conditions (i) and (ii) of Theorem KV for our concrete case.

3.1

Throughout this section we shall work in the Hilbert spaces $\mathcal{H} = L_2(\Omega_0, \mu_0)$ and $\mathcal{K} = L_2(\Omega, \mu)$. (\cdot, \cdot) and $\langle \cdot, \cdot \rangle$ will denote scalar product in \mathcal{H} and \mathcal{K} respectively. Simple capital letters (sanserif) will be used for bounded linear operators on \mathcal{H} (e.g. \mathbf{E}), fat ones for those on \mathcal{K} (e.g. \mathbf{ID}). It is convenient to make the identification

$$\mathcal{K} = \bigoplus_{u \in \mathcal{U}} \mathcal{H}_u, \quad \mathcal{H}_u = \mathcal{H} \tag{3.1}$$

and to use the matrix notation

$$\varphi \in \mathcal{K}: \varphi = (\varphi_u)_{u \in \mathcal{U}}, \quad \varphi_u = \varphi(u, \cdot) \in \mathcal{H}_u \tag{3.2}$$

$$(\mathbf{A} \varphi)_u = \sum_{u' \in \mathcal{U}} \mathbf{A}_{u, u'} \varphi_{u'}. \tag{3.3}$$

We have

$$\langle \psi, \varphi \rangle = \frac{1}{v} \sum_{u \in \mathcal{U}} (\psi_u, \varphi_u) \tag{3.4}$$

We define now the operators needed throughout the proof. \mathbf{I} and \mathbf{II} will denote the identity on \mathcal{H} respectively \mathcal{K} . Given $z \in \mathbf{Z}^d$, \mathbf{D}_z is the shift by τ_z on Ω_0 ; for $u, u' \in \mathcal{U}$ $\Gamma_{u, u'}$ is the multiplication by $\gamma_{u, u'}$:

$$\varphi \in \mathcal{H} \quad (\mathbf{D}_z \varphi)(\omega) = \varphi(\tau_z \omega) \tag{3.5}$$

$$(\Gamma_{u, u'} \varphi)(\omega) = \gamma_{u, u'}(\omega) \varphi(\omega). \tag{3.6}$$

The \mathbf{D}_z 's are unitary and commute, the $\Gamma_{u, u'}$'s are positive contractions satisfying

$$\sum_{u \in \mathcal{U}} \Gamma_{u, u'} = \sum_{u' \in \mathcal{U}} \Gamma_{u, u'} = \mathbf{I}. \tag{3.7}$$

We shall need also

$$\mathbf{E} = \frac{1}{v} \sum_{u \in \mathcal{U}} \mathbf{D}_u. \tag{3.8}$$

Due to the symmetry of \mathcal{U} , \mathbf{E} is a selfadjoint contraction.

The transition operator of the Markov chain is

$$\mathbf{IP} = \mathbf{I} \Gamma \mathbf{ID} \tag{3.9}$$

with $\mathbf{I} \Gamma$ and \mathbf{ID} defined by

$$\mathbf{I} \Gamma_{u, u'} = \Gamma_{u, u'}; \quad \mathbf{ID}_{u, u'} = \delta_{u, u'} \mathbf{D}_u. \tag{3.10}$$

Consider also the orthogonal projection \mathbb{E} , which separates “the worst subspace” of \mathcal{H} :

$$\mathbb{E}_{u,u'} = \frac{1}{\nu} \mathbb{I}. \quad (3.11)$$

Conditions (a) and (b) are equivalent to the following

$$(a) \quad \mathbb{E} \mathbb{P} \mathbb{E} = \mathbb{P} \mathbb{E} \mathbb{E} = \mathbb{E} \quad (3.12)$$

$$(b) \quad \|\tilde{\mathbb{P}}\| \leq 1 - \delta \quad \text{where } \tilde{\mathbb{P}} \stackrel{\text{def}}{=} \mathbb{P} - \mathbb{E}. \quad (3.13)$$

We shall use the decomposition under this projection of the transition operator

$$\begin{aligned} \mathbb{E} \mathbb{P} \mathbb{E} &= \mathbb{E} \mathbb{D} \mathbb{E}; & \mathbb{E} \mathbb{P} (\mathbb{I} - \mathbb{E}) &= \mathbb{E} \mathbb{D} (\mathbb{I} - \mathbb{E}) \\ (\mathbb{I} - \mathbb{E}) \mathbb{P} \mathbb{E} &= \tilde{\mathbb{P}} (\mathbb{I} - \mathbb{E}) \mathbb{D} \mathbb{E} & (3.14) \\ (\mathbb{I} - \mathbb{E}) \mathbb{P} (\mathbb{I} - \mathbb{E}) &= \tilde{\mathbb{P}} (\mathbb{I} - \mathbb{E}) \mathbb{D} (\mathbb{I} - \mathbb{E}) \end{aligned}$$

defined by the matrix elements

$$\begin{aligned} (\mathbb{E} \mathbb{D} \mathbb{E})_{u,u'} &= \frac{1}{\nu} \mathbb{E} \\ (\mathbb{E} \mathbb{D} (\mathbb{I} - \mathbb{E}))_{u,u'} &= \frac{1}{\nu} (\mathbb{D}_{u'} - \mathbb{E}) = ((\mathbb{I} - \mathbb{E}) \mathbb{D} \mathbb{E})_{u',u} & (3.15) \\ ((\mathbb{I} - \mathbb{E}) \mathbb{D} (\mathbb{I} - \mathbb{E}))_{u,u'} &= \delta_{u,u'} \mathbb{D}_u - \frac{1}{\nu} (\mathbb{D}_u + \mathbb{D}_{u'}) + \frac{1}{\nu} \mathbb{E}. \end{aligned}$$

Two identities regarding the decompositions of the resolvent of a contraction will be used in Subjects 3.3 and 3.4. Let \mathbb{P} be an arbitrary contraction on a Hilbert space, \mathbb{P} an orthogonal projection and $\mathbb{R}_\lambda = (\mathbb{P} - \lambda \mathbb{P})^{-1}$ the resolvent of \mathbb{P} , ($|\lambda| < 1$). The following identities hold

$$\begin{aligned} (\mathbb{I} - \mathbb{P}) \mathbb{R}_\lambda (\mathbb{I} - \mathbb{P}) &= [\mathbb{I} - \lambda (\mathbb{I} - \mathbb{P}) \mathbb{P} (\mathbb{I} - \mathbb{P}) \\ &\quad - \lambda^2 (\mathbb{I} - \mathbb{P}) \mathbb{P} \mathbb{P} (\mathbb{I} - \lambda \mathbb{P} \mathbb{P})^{-1} \mathbb{P} \mathbb{P} (\mathbb{I} - \mathbb{P})]^{-1} (\mathbb{I} - \mathbb{P}) \quad (3.16) \end{aligned}$$

$$\mathbb{P} \mathbb{R}_\lambda (\mathbb{I} - \mathbb{P}) = \lambda [\mathbb{P} (\mathbb{I} - \lambda \mathbb{P} \mathbb{P})^{-1} \mathbb{P} \mathbb{P} (\mathbb{I} - \mathbb{P})] (\mathbb{I} - \mathbb{P}) \mathbb{R}_\lambda (\mathbb{I} - \mathbb{P}) \quad (3.17)$$

((3.16) is to be understood in the sense that “if the right hand side exists”.)

3.2

Ergodicity is proved by showing that (ii) of the “Standard Lemma” is fulfilled even with $L_2(\Omega, \mu)$ instead of $L_\infty(\Omega, \mu)$. We have to show that the only solutions of the equation

$$(\mathbb{E} + \tilde{\mathbb{P}}) \mathbb{D} \varphi = \varphi \quad (3.18)$$

are the constants. Due to the fact that \mathbb{D} is unitary, this is equivalent to:

$$(\mathbb{E} + \tilde{\mathbb{P}}) \psi = \mathbb{D}^* \psi; \quad \varphi = \mathbb{D}^* \psi. \quad (3.19)$$

Equation (3.19) can be fulfilled only if

$$\|\mathbb{E}\psi\|^2 + \|\tilde{\Pi}(\mathbb{I} - \mathbb{E})\psi\|^2 = \|(\mathbb{E} + \tilde{\Pi})\psi\|^2 = \|\psi\|^2 = \|\mathbb{E}\psi\|^2 + \|(\mathbb{I} - \mathbb{E})\psi\|^2$$

whence, due to (3.13), $\psi = \mathbb{E}\psi$, and Eqs (3.19) transform to

$$\psi = \mathbb{D}^* \psi; \quad \varphi = \psi. \tag{3.19'}$$

But the dynamical system $(\Omega_0, \mathcal{F}_0, \mu_0, \{\tau_z: z \in \mathbb{Z}^d\})$ is ergodic, thus the only solutions of (3.19') are the constants.

3.3

Condition (2.7) of Theorem 1 reads:

$$\varphi = (\mathbb{I} - \mathbb{E})\varphi. \tag{3.20}$$

We prove that if $\lambda \nearrow 1$

$$(\mathbb{I} - \mathbb{E})\mathbb{R}_\lambda(\mathbb{I} - \mathbb{E}) \xrightarrow{\text{st}} (\mathbb{I} - \mathbb{E})[\mathbb{I} - \tilde{\Pi}\mathbb{D}\mathbb{V}]^{-1}(\mathbb{I} - \mathbb{E}) \tag{3.21}$$

where $\xrightarrow{\text{st}}$ stands for strong convergence of linear operators and \mathbb{V} is a selfadjoint and unitary operator to be defined below. Hence, for φ satisfying (3.20), we have condition (i) of Theorem KV verified, with

$$\sigma^2 = 2\langle \varphi, (\mathbb{I} - \tilde{\Pi}\mathbb{D}\mathbb{V})^{-1}\varphi \rangle - \langle \varphi, \varphi \rangle > \frac{\delta}{2 - \delta} \langle \varphi, \varphi \rangle. \tag{3.22}$$

Let

$$\begin{aligned} \mathbb{V}_\lambda &= \lambda\mathbb{I} - (\mathbb{I} - \lambda\mathbb{D}^*)\mathbb{E}(\mathbb{I} - \lambda\mathbb{E}\mathbb{D}\mathbb{E})^{-1}\mathbb{E}(\mathbb{I} - \lambda\mathbb{D}) \\ \mathbb{K}_\lambda &= (\mathbb{D} - (1 - \lambda)(\mathbb{I} - \lambda\mathbb{D}^*)^{-1})(1 - \lambda)(\lambda\mathbb{I} - \mathbb{V}_\lambda) \\ \mathbb{U}_\lambda &= \mathbb{D}\mathbb{V}_\lambda + \mathbb{K}_\lambda \end{aligned} \tag{3.23}$$

\mathbb{U}_λ is the operator which appears in (3.16)

$$(\mathbb{I} - \mathbb{E})\mathbb{R}_\lambda(\mathbb{I} - \mathbb{E}) = (\mathbb{I} - \tilde{\Pi}\mathbb{U}_\lambda)^{-1}(\mathbb{I} - \mathbb{E}) \tag{3.24}$$

The crucial fact is that, as $\lambda \nearrow 1$

$$\mathbb{V}_\lambda \xrightarrow{\text{st}} \mathbb{V} = \mathbb{I} - (\mathbb{I} - \mathbb{D}^*)\mathbb{E}(\mathbb{I} - \mathbb{E}\mathbb{D}\mathbb{E})^{-1}\mathbb{E}(\mathbb{I} - \mathbb{D}) \tag{3.25}$$

which is selfadjoint and unitary. Hence

$$\mathbb{K}_\lambda \xrightarrow{\|\cdot\|} \mathbb{O}. \tag{3.26}$$

Besides (3.25)

$$\|\mathbb{V}_\lambda\| \leq 1. \tag{3.27}$$

(In fact equality holds.) (3.26) and (3.27) together imply that for λ sufficiently close to 1

$$\|\tilde{\Pi}\mathbb{U}_\lambda\| < 1 - \delta.$$

Thus $(\mathbb{II} - \mathbb{IE})\mathbb{R}_\lambda(\mathbb{II} - \mathbb{IE})$ is bounded, uniformly in λ . From this fact and (3.25) assertion (3.21) follows.

(3.25) is verified through the matrix elements, using the common spectral representation of Toeplitz operators (see Th. X.2.1. of [1]):

$$\begin{aligned}
 (\mathbb{V}_\lambda)_{u,u'} &= \delta_{u,u'} \lambda \frac{1}{\nu} (1 - \lambda \mathbb{E})^{-1} (1 - \lambda \mathbb{D}_{-u}) (1 - \lambda \mathbb{D}_u) \\
 &= \int_{[-\pi, \pi]^d} \mathcal{E}(d\theta) \left[\delta_{u,u'} \lambda \frac{(1 - \lambda e^{i\theta \cdot u'}) (1 - \lambda e^{-i\theta \cdot u})}{\sum_{v \in \mathcal{U}} (1 - \lambda \cos \theta \cdot v)} \right] \\
 &\xrightarrow{\text{st}} \int_{[-\pi, \pi]^d} \mathcal{E}(d\theta) \left[\delta_{u,u'} \frac{(1 - e^{i\theta \cdot u'}) (1 - e^{-i\theta \cdot u})}{\sum_{v \in \mathcal{U}} (1 - \cos \theta \cdot v)} \right] \stackrel{\text{def}}{=} \mathbb{V}_{u,u'} \quad (3.28)
 \end{aligned}$$

where $\mathcal{E}(\cdot)$ is the projection valued measure on $[-\pi, \pi]^d$ associated to Toeplitz operators. (The value of the ratio in the last expression is defined to be zero at $\theta = 0$!)

For verifying (3.27) we give

$$\begin{aligned}
 \mathbb{V}_\lambda^k &= \lambda^k \mathbb{II} - (\mathbb{II} - \lambda \mathbb{ID}^*) \mathbb{IE} (\mathbb{II} - \lambda \mathbb{IE} \mathbb{ID} \mathbb{IE})^{-1} \mathbb{IE} \mathbb{IB}_\lambda^{(k)} (\mathbb{II} - \lambda \mathbb{ID}) \\
 (\mathbb{IB}_\lambda^{(k)})_{u,u'} &= \delta_{u,u'} \frac{1}{2} (\mathbb{C}_\lambda - 1)^{-1} \{ [\lambda + 2(\mathbb{C}_\lambda - 1)]^k - \lambda^k \} \quad (3.29) \\
 \mathbb{C}_\lambda &= \frac{1 - \lambda^2}{2} (1 - \lambda \mathbb{E})^{-1}.
 \end{aligned}$$

Using the spectral representation of \mathbb{E} one finds that $\mathbb{IB}_\lambda^{(k)}$ – and consequently \mathbb{V}_λ^k – is bounded uniformly in k . Hence (3.27).

3.4

We verify condition (ii) of Theorem KV. Throughout this subsection $\varphi^{(\lambda)} = (\mathbb{II} - \mathbb{IE})\mathbb{R}_\lambda(\mathbb{II} - \mathbb{IE})\varphi$.

$$\begin{aligned}
 &(1 - \lambda) \langle \mathbb{R}_\lambda \varphi, \mathbb{R}_{\lambda'} \varphi \rangle \\
 &= (1 - \lambda) \langle \varphi^{(\lambda)} \varphi^{(\lambda')} \rangle + (1 - \lambda) \langle \mathbb{IE} \mathbb{R}_\lambda (\mathbb{II} - \mathbb{IE}) \varphi, \mathbb{IE} \mathbb{R}_{\lambda'} (\mathbb{II} - \mathbb{IE}) \varphi \rangle \quad (3.30)
 \end{aligned}$$

By the assertion of the last subsection the first term of the right hand side tends to zero as $\lambda, \lambda' \nearrow 1$. To evaluate the second term we want to use (3.17). For this purpose we give the matrix elements

$$[\mathbb{IE}(\mathbb{II} - \lambda \mathbb{IE} \mathbb{P} \mathbb{IE})^{-1} \mathbb{IE} \mathbb{P}(\mathbb{II} - \mathbb{IE})]_{u,u'} = \frac{1}{\nu} (1 - \lambda \mathbb{E})^{-1} (\mathbb{D}_u - \mathbb{E}). \quad (3.31)$$

Using these we find that the second term equals

$$\frac{\lambda \lambda'}{\nu^2} \sum_{u,u' \in \mathcal{U}} (\varphi_{-u}^{(\lambda)}) [(1 - \lambda)(1 - \lambda \mathbb{E})^{-1}] [(\mathbb{D}_u - \mathbb{E})(\mathbb{D}_{u'} - \mathbb{E})(1 - \lambda' \mathbb{E})^{-1}] \varphi_{u'}^{(\lambda')} \quad (3.32)$$

But

$$(1 - \lambda)(1 - \lambda E)^{-1} \xrightarrow{\text{st}} \int_{[-\pi, \pi]^d} \mathcal{E}(d\theta) \chi_{\{\theta=0\}}(\theta) = \mathcal{E}(\{0\}) \quad \text{as } \lambda \nearrow 1$$

$$(D_u - E)(D_{u'} - E)(1 - \lambda' E)^{-1} \xrightarrow{\text{st}} \int_{[-\pi, \pi]^d} \mathcal{E}(d\theta) f(\theta) \quad \text{as } \lambda' \nearrow 1$$

with the concrete form of $f(\theta)$ unimportant except the fact that it is bounded and $f(0)=0$. Thus the product of these two converges strongly to zero. This fact and the assertion of the last subsection implies the convergence to zero of the expression in (3.32).

Appendix

We give a sketch of proof of Theorem KV. Let

$$\varphi_\lambda = R_\lambda \varphi; \quad \delta_\lambda = \varphi - (I - P) R_\lambda \varphi. \tag{A.1}$$

The basic ingredients of the proof are the following three consequences of conditions (i) and (ii), which can be proved by standard arguments of resolvent calculus:

$$\|\varphi_\lambda\|^2 = o\left(\frac{1}{1-\lambda}\right) \quad \text{as } \lambda \nearrow 1 \tag{A.2}$$

$$\|\delta_\lambda\|^2 = o(1-\lambda) \quad \text{as } \lambda \nearrow 1 \tag{A.3}$$

$$(\delta_\lambda, \varphi_\lambda) \rightarrow 0 \quad \text{as } \lambda, \lambda' \nearrow 1. \tag{A.4}$$

Let $\bar{\mu}$ be the probability measure on $(\Omega \times \Omega, \mathcal{F} \times \mathcal{F})$ defined by

$$\bar{\mu}(A \times B) = \int_A \mu(d\omega) \mathcal{P}(\omega, B) \quad A, B \in \mathcal{F}.$$

Consider $\mathcal{V}_\lambda \in L_2(\Omega \times \Omega, \bar{\mu})$

$$\mathcal{V}_\lambda(\omega_1, \omega_2) = \varphi_\lambda(\omega_2) - P \varphi_\lambda(\omega_1). \tag{A.5}$$

By (A.4) \mathcal{V}_λ is Cauchy as $\lambda \nearrow 1$. Let

$$\mathcal{V}_\lambda \xrightarrow{L_2(\Omega \times \Omega, \bar{\mu})} \mathcal{V}. \tag{A.6}$$

We have

$$\int d\bar{\mu} \mathcal{V}^2 = \lim_{\lambda \nearrow 1} \int d\bar{\mu} \mathcal{V}_\lambda^2 = 2 \lim_{\lambda \nearrow 1} (\varphi, R_\lambda \varphi) - (\varphi, \varphi) \stackrel{\text{def}}{=} \sigma^2. \tag{A.7}$$

\mathcal{V}_λ was defined in such a way that for μ -almost all $\omega \in \Omega$ $(\mathcal{V}_\lambda(\eta_k, \eta_{k+1}))_{k \in \mathbb{N}}$ (and consequently $(\mathcal{V}(\eta_k, \eta_{k+1}))_{k \in \mathbb{N}}$) is a sequence of martingale differences on $(\Omega^{\mathbb{N}}, \Pi^{(\omega)})$. Let

$$M_n = \sum_{k=1}^n \mathcal{V}(\eta_k, \eta_{k+1}). \tag{A.8}$$

Applying to M_n Brown's martingale limit theorem (see [2]) one finds that for μ -almost all initial conditions of the Markov chain

$$\varepsilon^{1/2} M_{[\varepsilon^{-1} \cdot]} \Rightarrow \mathcal{W}_\sigma \tag{A.9}$$

(\Rightarrow stands for convergence in distribution).

We have to show that the finite dimensional distributions of

$$\varepsilon^{1/2} \Delta_{[\varepsilon^{-1} \cdot]} \stackrel{\text{def}}{=} \varepsilon^{1/2} \left[\sum_{k=1}^{[\varepsilon^{-1} \cdot]} \varphi(\eta_k) - M_{[\varepsilon^{-1} \cdot]} \right] \tag{4.10}$$

converge to those concentrated at zero (in μ -probability w.r.t. the initial state – which in case of degenerated limiting distribution is equivalent to showing the same thing on the probability space $(\Omega^{\mathbb{N}}, \Pi^\mu)$!).

Using (A.1) and (A.8)

$$\Delta_{[\varepsilon^{-1} \cdot]} = \sum_{k=1}^{[\varepsilon^{-1} \cdot]} (\mathcal{V}_\lambda - \mathcal{V})(\eta_k, \eta_{k+1}) + \sum_{k=1}^{[\varepsilon^{-1} \cdot]} \delta_\lambda(\eta_k) + \varphi_\lambda(\eta_0) - \varphi_\lambda(\eta_{[\varepsilon^{-1} \cdot]}). \tag{A.11}$$

Notice that the r.h.s. in fact does not depend on λ , thus we can choose $\lambda = 1 - \varepsilon$. The estimation of the first three terms in (A.11) goes on the same way as in [3]. Thus the following three arguments are copies of similar ones from there and we have also tightness for them.

$$\begin{aligned} (1) \quad & \Pi^\mu \left(\sup_{0 \leq t \leq 1} \left| \sum_{k=1}^{[t\varepsilon^{-1}]} (\mathcal{V}_{1-\varepsilon} - \mathcal{V})(\eta_k, \eta_{k+1}) \right| > \frac{\alpha}{\sqrt{\varepsilon}} \right) \\ & \leq \frac{\varepsilon}{\alpha^2} E^\mu \left(\sum_{k=1}^{[t\varepsilon^{-1}]} (\mathcal{V}_{1-\varepsilon} - \mathcal{V})(\eta_k, \eta_{k+1}) \right)^2 \\ & = \frac{\varepsilon [t\varepsilon^{-1}]}{\alpha^2} \int d\bar{\mu} (\mathcal{V}_{1-\varepsilon} - \mathcal{V})^2 \rightarrow 0. \end{aligned} \tag{A.12}$$

E^μ denotes expectation w.r.t. Π^μ . We have used the martingale difference property of $(\mathcal{V}_{1-\varepsilon} - \mathcal{V})$ and (A.6).

$$\begin{aligned} (2) \quad & \Pi^\mu \left(\sup_{0 \leq t \leq 1} \left| \sum_{k=1}^{[t\varepsilon^{-1}]} \delta_{1-\varepsilon}(\eta_k) \right| > \frac{\alpha}{\sqrt{\varepsilon}} \right) \\ & \leq \Pi^\mu \left(\sum_{k=1}^{[t\varepsilon^{-1}]} \left| \delta_{1-\varepsilon}(\eta_k) \right| > \frac{\alpha}{\sqrt{\varepsilon}} \right) \\ & \leq \frac{\varepsilon}{\alpha^2} E^\mu \left(\sum_{k=1}^{[t\varepsilon^{-1}]} \left| \delta_{1-\varepsilon}(\eta_k) \right| \right)^2 \leq \left(\frac{\varepsilon [t\varepsilon^{-1}]}{\alpha} \right)^2 \frac{1}{\varepsilon} \|\delta_{1-\varepsilon}\|^2 \rightarrow 0. \end{aligned} \tag{A.13}$$

We have used the Schwarz inequality and (A.3).

$$(3) \quad \Pi^\mu \left(\left| \varphi_{1-\varepsilon}(\eta_0) \right| > \frac{\alpha}{\sqrt{\varepsilon}} \right) \leq \frac{\varepsilon}{\alpha^2} \|\varphi_{1-\varepsilon}\|^2 \rightarrow 0. \tag{A.14}$$

We have used (A.2).

The estimation of the finite-dimensional distributions of the last term is rather simple:

$$\begin{aligned}
 (4) \quad & \Pi^\mu \left(\max_{1 \leq i \leq m} \left| \varphi_{1-\varepsilon}(\eta_{[\varepsilon^{-1}t_i]}) \right| > \frac{\alpha}{\sqrt{\varepsilon}} \right) \\
 & \leq \sum_{i=1}^m \Pi^\mu \left(\left| \varphi_{1-\varepsilon}(\eta_{[\varepsilon^{-1}t_i]}) \right| > \frac{\alpha}{\sqrt{\varepsilon}} \right) \leq \frac{m\varepsilon}{\delta^2} \|\varphi_{1-\varepsilon}\|^2 \rightarrow 0. \tag{A.15}
 \end{aligned}$$

(A.2) has been used once more. But we cannot prove tightness of this term without the assumption of reversibility (that is: self-adjointness of P).

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