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Probability Theory and Related Fields

A Propagation of Chaos Result for Burgers' Equation

A.S. Sznitman

Laboratoire de Probabilités, associé C.N.R.S. n° 224, Université Paris VI, 4, place Jussieu, F-75005 Paris, France

I. Introduction

In [15], McKean posed the problem of constructing a system of N interacting particles in $\mathbb{R}$ with generator

$$L=\frac{1}{2}\sum_i \frac{\partial^2}{\partial x_i^2}+\frac{1}{2(N-1)}\sum_{i<j}\delta(x^i-x^j)\left(\frac{\partial}{\partial x^i}+\frac{\partial}{\partial x^j}\right). \tag{1.1}$$

He conjectured that when the initial conditions are independent and u_0 distributed, and if one looks at the law at time t of the first k particles, k fixed, letting the number N of interacting particles be larger and larger, one restores asymptotically at time t the independence of our first k particles $(X_t^1, \ldots, X_t^k)$, and that their common limiting $(N\to\infty)$, distribution is given by the value at time t of the solution of Burgers' equation:

$$\frac{\partial u}{\partial t}=\frac{1}{2}\frac{\partial^2 u}{\partial x^2}-u\frac{\partial u}{\partial x}, \quad \text{with initial condition } u_0 \text{ at time 0.} \tag{1.2}$$

Such a type of phenomenon is called propagation of chaos (see Kac [12]). Several results concerning the questions of propagation of chaos and Burgers' equation have already been obtained, in Calderoni-Pulvirenti [2], where a smoothing procedure of the δ-function is used, in Gutkin-Kac [6], and Kotani-Osada [13], where the approach for the construction of the N-particle process and for the propagation of chaos result is rather analytical.

The approach presented here is probabilistic.

We consider a system of N particles satisfying:

$$dX_t^i=dB_t^i+\frac{c}{N}\sum_{j\neq i} dL^0(X^i-X^j)_t, \quad i=1,\ldots,N, \tag{1.3}$$
$$X_0^i=\mathbf{X}^i(0),$$

where $L^0(X^i-X^j)$ is the symmetric local time in 0 of X^i-X^j, B^i are independent Brownian motions, independent of the initial conditions $(X^i(0))$, with symmetric distribution u_N satisfying:

$$\bigcup_{\substack{i,j,k\\ \text{distinct}}} \{x^i = x^j = x^k\} \qquad \text{is } u_N\text{-negligible.} \tag{1.4}$$

Such a process was constructed with a probabilistic approach in Sznitman-Varadhan [21], where it is also shown that the process (1.3), is trajectorially approximated by the "smoothed" processes:

$$dX_t^{i,\alpha} = dB_t^i + \frac{c}{N} \sum_{j \neq i} \phi_\alpha(X_t^{i,\alpha} - X_t^{j,\alpha})\, 2\,dt, \tag{1.5}$$
$$X_0^{i,\alpha} = \mathbf{X}^i(0),$$

when α goes to zero, $\left(\phi_\alpha(\cdot) = \frac{1}{\alpha}\,\phi\left(\frac{\cdot}{\alpha}\right)\right.$ is an approximation of the Dirac measure$\Big)$.

This stability result links (1.3) with (1.1) (take $c = 1/4$), specially if one notices that (1.1) can be interpreted as the divergence type operator

$$L = \operatorname{div}(A \operatorname{grad}), \qquad A = \frac{1}{2}\, Id + \frac{H(x)}{4(N-1)}, \qquad \text{if } H_{ij}(x) = H(x^i - x^j),$$

$(H(t) = 1(t \geqq 0) - 1(t \leqq 0))$. This remark concerning the generator (1.1) was a key point noticed by Kotani-Osada [13].

Let us first introduce a

Definition. If E is a separable metric space, v a probability on E, a sequence (v_N) of symmetric probabilities on E^N is said to be v-chaotic, if for $\phi_1, \ldots, \phi_k$, continuous bounded functions on E,

$$\lim_{N \to \infty} \langle v_N, \phi_1 \otimes \ldots \otimes \phi_k \otimes 1 \otimes \ldots \otimes 1 \rangle = \prod_1^k \langle v, \phi_i \rangle. \tag{1.6}$$

In the following $M(E)$ will denote the set of probabilities on E. One can show (see Tanaka [23], Sznitman [20]), that being u-chaotic is equivalent to

$$\bar{X}_N = \frac{1}{N} \sum_1^N \varepsilon_{X^i} \qquad \text{(which is a } M(E) \text{ valued r.v. defined on } (E^N, v_N), \tag{1.7}$$
$$X_i \text{ are the canonical coordinates on } E^N),$$

converges in law towards the constant v. ($M(E)$ is endowed with the topology of weak convergence which allows us to define the convergence in law for the $M(E)$-valued sequence of r.v. $\bar{X}_N$).

In this work we obtain the fact that for u_N, u-chaotic, ($E = \mathbb{R}$; in the previous definition), and satisfying (1.4), then the laws P_N, on $C(R_+, R)^N$ of the processes (X^i) satisfying (1.3), with initial law u_N, are P-chaotic, (now $E = C(\mathbb{R}_+, \mathbb{R})$), where P is the law of the nonlinear process which describes the asymptotic ($N \to \infty$) individual behavior of the particles. Roughly speaking, this nonlinear process is obtained by considering (1.3), for say particle 1, and replacing the summation over the other particles by an integration over an independent copy of the process, namely:

$$X_t = X_0 + B_t + cE_Y[L^0(X-Y)_t], \tag{1.8}$$

where Y is an independent copy of X.

Section 2 gives a precise meaning to (1.8), and proves a weak uniqueness result for the solutions of (1.8). (This together with the results of Sect. 4 gives a weak existence and uniqueness result for (1.8)). We use Barlow-Yor's estimates [1], on the local time of a continuous semi martingale, which allow us to show that the law of X_t, has a density $u(t,x)\,dx$, with $u(t,x) \in L^2([0,T]\times R)$, for every $T>0$ (see also Krylov [14], Chap. 2, Melnikov [26]). As a consequence we obtain that

$$X_t = X_0 + B_t + \int_0^t 2cu(s, X_s)\,ds, \tag{1.9}$$

and also that $u(t,x)$ satisfies (weakly), Burgers' equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - 2c\,\frac{\partial (u^2)}{\partial x}. \tag{1.10}$$

Using the Cole Hopf transform [3], one can then show that u is the classical solution given by

$$\exp -4c\,F_t(x) = (\exp -4c\,F_0) * \rho_t(x),$$
$$\text{if } \rho_t(x) = \frac{1}{\sqrt{2\pi t}} \exp -\frac{x^2}{2t} \quad \text{and} \quad F_t(x) = \int_{-\infty}^{x} u(t,y)\,dy. \tag{1.11}$$

It is also easy to see that the law $\bar{m}$ of $(X_., B_.)$ on $C\times C_0$ is uniquely determined.

When the initial condition $u_0(dx) = u_0(x)\,dx$, with $u_0(x)$ bounded, we obtain a strong uniqueness (and existence result).

Let us now explain the general line of attack to the propagation of chaos result.

Instead of trying to get direct convergence estimates on the processes $(X_.^i)$, which is often difficult when one deals with local times, we rather try to obtain a tightness result on the laws of the empirical distributions

$$\bar{X}_N(\omega) = \frac{1}{N(N-1)} \sum_{i\neq j} \varepsilon_{(X^i(\omega),\, B_.^i(\omega),\, X_.^j(\omega),\, B_.^j(\omega),\, L_.^0(X^i - X^j)(\omega))}$$
$$\in M(C\times C_0 \times C \times C_0 \times C_0^+), \tag{1.12}$$

(C_0^+ is the set of continuous increasing functions on $\mathbb{R}_+$ with value 0 at time 0), and then show that the limit points of these laws inherit sufficiently many features of the approximating $\bar{X}_N$, so that one can prove they are concentrated on the probability on $C\times C_0\times C\times C_0\times C_0^+$, which is the image of $\bar{m}\otimes\bar{m}$ on $((C\times C_0)\times(C\times C_0))$, by: $((X^1, B^1), (X^2, B^2)) \to (X^1, B^1, X^2, B^2, L^0(X^1 - X^2))$, by (1.7) one then gets the result.

We first need a tightness result on (1.12); as one can see easily (see Sznitman [20]), this amounts to a tightness result on the law of $X^1_\cdot$, and on the law of $L^0(X^1-X^2)_\cdot$, (when N varies), and this is done in Sect. 3. There are several ways of trying to obtain estimates on (1.3). The first guess is that one should express the local times $L^0(X^i-X^j)$, using Tanaka's formula and then inject the result in (1.3), in this fashion, one can obtain estimates when c is small enough, but this method does not work for large c. One possible reason for that is that in a multidimensional way (1.3) has a structure near the one dimensional equation

$$X_t = X_0 + B_t + \alpha L^0(X)_t, \tag{1.13}$$

which admits no solution for $|\alpha|>1$ (see Harrison-Shepp [7]). One may also try to use the divergence structure of (1.1), and use related estimates (see for instance Nash [16], where estimates on the first moment are given with the explicit dependence of the constants on the dimension N, and also Kotani-Osada [13] and Osada [17]). Here we use a different method; taking advantage of the symmetry of the $(X^i_\cdot)$, we introduce the reordered process $Y^1_t \leqq \ldots \leqq Y^N_t$ (reordering of $X^1_t, \ldots, X^N_t$). It can be shown that $(Y^i_\cdot)$ satisfy the oblique reflection problem

$$\begin{aligned} Y^1_t &= Y^1_0 + W^1_t - \frac{1}{2}\left(1-\frac{2c}{N}\right)\gamma^1_t, \\ Y^k_t &= Y^k_0 + W^k_t - \frac{1}{2}\left(1-\frac{2c}{N}\right)\gamma^k_t + \frac{1}{2}\left(1+\frac{2c}{N}\right)\gamma^{k-1}_t, \quad 2\leqq k\leqq N-1, \\ Y^N_t &= Y^N_0 + W^N_t + \frac{1}{2}\left(1+\frac{2c}{N}\right)\gamma^{N-1}_t, \end{aligned} \tag{1.14}$$

where $W^1_t, \ldots, W^N_t$ are independent Brownian motions and

$$\gamma^i_t = \int_0^t 1(Y^i_s = Y^{i+1}_s)\, d\gamma^i_s, \quad 1\leqq i\leqq N-1, \ \gamma^i \text{ continuous increasing.} \tag{1.15}$$

We obtain estimates by comparison results with a normally reflected process in the convex set $\{x^1\leqq\ldots\leqq x^N\}$ (see Tanaka [22]) constructed on a perturbation of $W^1_\cdot, \ldots, W^N_\cdot$.

The last section studies the limit points of the laws of $\bar{X}_N$, it is not difficult to see that these limit points are concentrated on probabilities m on $C\times C_0 \times C\times C_0\times C_0^+$ such that $(X^1_\cdot, B^1_\cdot)$ and $(X^2_\cdot, B^2_\cdot)$ are m-independent (identically distributed), $B^1_\cdot, B^2_\cdot$ are independent Brownian motions, the law of X^1_0 (or X^2_0) under m is $u_0(dx)$ and m-a.s. $X^1_t = X^1_0 + B^1_t + A^1_t$, where $A^1_t = c\times E_m[A_t/(X^1_\cdot, B^1_\cdot)]$ (A denotes the C_0^+ valued coordinate on $C\times C_0\times C\times C_0\times C_0^+$). The main difficulty is to identify $A_\cdot$ as $L^0_\cdot(X^1-X^2)$.

Once this step is performed, one can apply the uniqueness results of Sect. 2.

As a consequence of this propagation of chaos result, one can see, using the fact that P_N is approximated by the laws P_N^α of $(X^{i,\alpha}_\cdot)$ (see (1.5)), when α goes to zero, that for sequences $\alpha(N)$ converging rapidly enough to zero, $P_N^{\alpha(N)}$ is also P-chaotic. Such a result for sequences $\alpha(N)$ converging slowly towards zero was obtained by Calderoni-Pulvirenti [2], see also Oelschlager [27] on related results.

In the case of initial conditions $u_N = u^{\otimes N}$, with $u(dx) = u(x)\,dx$, $u(x)$ bounded, our result also leads to a convergence in probability of $X^{i,N}_{\cdot}$ towards the nonlinear process $\bar{X}^i_{\cdot}$, "constructed on B^i and $X^i(0)$";

$$d\bar{X}^i_t = dB^i_t + 2cu(t, \bar{X}^i_t)\,dt,$$

$u(t,x)$ a solution of Burgers' equation with initial value u

$$\bar{X}^i_0 = X^i(0). \tag{1.16}$$

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II. A Uniqueness Result for the Nonlinear Process

In this section we will prove a uniqueness result for the law of the nonlinear process which is going to describe the limit individual behavior of the interacting system of particles we study. We are first going to prove some lemmas, in order to define precisely the quantity $E_Y[L^0_t(X-Y)]$ (Y independent copy of X), which appeared in the introduction.

Let $(\Omega, F, (F_t)_{t\geqq 0}, P)$ be a probability space endowed with a filtration $(F_t)_{t\geqq 0}$ (F is complete, (F_t) is right continuous, and each F_t contains the F negligible sets). We suppose that $(\Omega, F, (F_t), P)$ is endowed with an F_t-Brownian motion $(B_t)_{t\geqq 0}$. We are going to consider a continuous F_t-semimartingale $(X_t)_{t\geqq 0}$, satisfying:

$$X_t = X_0 + B_t + A_t, \tag{2.1}$$

where X_0 is F_0-measurable, and A_t is a continuous increasing process equal to zero at time zero, and such that A_t is integrable for every t.

One has the following:

Proposition 2.1. *Let $(\bar{\Omega}, \bar{F}, (\bar{F}_t), \bar{P})$ be a filtered probability space (also satisfying the "usual" conditions) and $(Y_t)_{t\geqq 0}$, be such that for $T>0$, $(Y-Y_0)_{t\wedge T}$ is in the space $H^1(\bar{\Omega})$ (see Dellacherie-Meyer [4], for the definition of H^1), and Y has the same law as the process $(X_t)_{t\geqq 0}$ (defined on Ω), then the formula*

$$C_t = \int_{\bar{\Omega}} L^0_t(X-Y)\,d\bar{P}, \tag{2.2}$$

where $L^0_{\cdot}(X-Y)$ denotes the symmetric local time in zero of the continuous semimartingale $X-Y$ on the product space $\Omega\times\bar{\Omega}$ defines a continuous increasing process, integrable for every t, which does not depend on the choice of Y (one can take a copy of Ω and X for instance). Moreover P a.s.

$$C_t(\omega) = \lim_n 2 \int_{\Omega} \int_0^t ds\, \phi_n(X_s(\omega) - X_s(\tilde{\omega}))\,dP(\tilde{\omega}), \qquad \forall\, t\geqq 0 \tag{2.3}$$

if ϕ is smooth positive symmetric around zero, $\int\phi = 1$, $\phi_n(\cdot) = n\phi(n\cdot)$. The convergence in (2.3) is dominated for $t\leqq T$ by an element $H_T(\omega)\in L^1(P)$.

Remark. We refer the reader to Jacod [10] for the definition of the continuous increasing process $L^0_{\cdot}(X-Y)$.

Proof. Let Y be as in the statement of Proposition 2.1, we know that $\langle Y\rangle_t$ is the limit in probability of $\sum_{k=1}^{2^n-1}(Y_{(k+1)/2^n}-Y_{k/2^n})^2$ (see Dellacherie-Meyer [4]), since Y has the same law as X, this last quantity converges towards t. So we can write $Y_t=Y_0+M_t+D_t$, where M_t is a continuous local martingale (in fact a Brownian motion), and D_t (by assumption on Y) has integrable variation on bounded intervals. (D is not necessarily increasing, for instance in the case where P is Wiener measure on $C(\mathbb{R}_+,\mathbb{R}^d)$, F_t the canonical filtation, $X=B$, if $\bar{\Omega}=\Omega$, $Y=X$ and $\bar{F}_t=\sigma(B_1)\vee F_t$, in this case $D_t=\int_0^{t\wedge 1}\frac{X_1-X_s}{1-s}\,ds$, and is not increasing, see Jeulin [11], p. 46).

Consider now $X-Y$ on the product space, one can find a version of the local time which for any $T>0$, is right continuous with left limits in the space variable, with values in $C(0,T)$, (time variable), see Yor [24]. Now, for $L(x,t,\omega,\bar{\omega})$ the symmetric local time of $X-Y$ in x at time t, we have

$$P\otimes\bar{P}-as,\quad \forall x\in\mathbb{R},\ \forall t\in\mathbb{R}_+, \tag{2.4}$$

$$L(x,t,\omega,\bar{\omega})=\lim_n 2\int_0^t \phi_n(X_s(\omega)-Y_s(\bar{\omega})-x)\,ds.$$

If we apply now formula (2.4) to $X-X_0$ and $Y-Y_0$, we find

$$\begin{aligned}L(x,t,\omega,\bar{\omega})&=\lim_n 2\int_0^t \phi_n(X_s-X_0-(Y_s-Y_0)+X_0-Y_0-x)\,ds\\ &=\underline{L}(Y_0-X_0+x,t,\omega,\bar{\omega}),\end{aligned} \tag{2.5}$$

if $\underline{L}(x,t,\omega,\bar{\omega})$ denotes the symmetric local time of $(X-X_0)-(Y-Y_0)$. By an inequality of Barlow-Yor [1], one has:

$$E_{P\otimes\bar{P}}[\sup_x \underline{L}(x,t,\omega,\bar{\omega})]\leqq CE_{P\otimes\bar{P}}[\sup_{s\leqq t}|B_s|+|M_s|+A_t+|D|_t]<\infty. \tag{2.6}$$

Defining $Z_t(\omega,\bar{\omega})=\sup_x \underline{L}(x,t,\omega,\bar{\omega})$, $P\otimes\bar{P}$ a.s. $\forall s\leqq t$, $L^0(X-Y)_s\leqq Z_t(\omega,\bar{\omega})$ and as a consequence of (2.6), by dominated convergence, $\int_{\Omega} L^0_t(X-Y)\,dP(\bar{\omega})$ defines a continuous increasing process integrable for every t. Since

$$2\int_0^t \phi_n(X_s-Y_s)\,ds=\int_{\mathbb{R}} L(x,t,\omega,\bar{\omega})\,\phi_n(x)\,dx=\int_{\mathbb{R}} \underline{L}(Y_0-X_0+x,t,\omega,\bar{\omega})\,\phi_n(x)\,dx;$$

we also find that $P\otimes\bar{P}$ a.s. $2\int_0^t \phi_n(X_s-Y_s)\,ds\leqq Z_t(\omega,\bar{\omega})$, using (2.4), by dominated convergence, we see that for P a.e. ω, $\forall s\leqq t$,

$$2\int d\bar{P}(\bar{\omega})\int_0^s \phi_n(X_u(\omega)-Y_u(\bar{\omega}))\,du=\int_0^s 2E_{P'}[\phi_n(X_u(\omega)-X_u(\omega'))]\,du,$$

converges to $C_s(\omega)=\int_{\Omega} L_s^0(X-Y)\,d\tilde{P}(\omega)$, and this a.s. convergence is dominated by the integrable $H_t(\omega)=E_{\tilde{P}}[Z_t(\omega,\tilde{\omega})]$. So this tells us that $C_t(\omega)$ does not depend on the special choice of Y, and that formula (2.3) is valid). □

Notation. Let $(X_t)_{t\geqq 0}$ be a process satisfying (2.1), we denote by $E_Y[L_t^0(X-Y)]$ the continuous integrable (on bounded time intervals) increasing process (C_t) defined by (2.2).

Our goal is now to give a more convenient formula than (2.3) for the computation of $E_Y[L_t^0(X-Y)]$. In view of (2.3), if we can show that X_t has a smooth density, $u(t,x)$, then a natural formula is $E_Y[L_t^0(X-Y)]=2\int_0^t u(s,X_s)\,ds$.

It turns out that condition (2.1) does not imply that the law of X_t is smooth for every t; the law of X_t can even have a purely atomic part. (If we dropped in (2.1) the assumption A increasing, one can simply build a counterexample with a Brownian bridge).

In the case of assumption (2.1), one has the following example due to S.R.S. Varadhan

Example 2.2. Consider a Brownian motion B_t, define the sequences $t_n=1-\frac{1}{2^n}$, $n\geqq 1$, $C_n=\frac{1}{n}$.

Set $\bar{A}_t=0$, $t\leqq t_1$, $\bar{A}_{t_{n+1}}=-B_{t_n}-C_n$, $n\geqq 1$, and $\bar{A}_t$ linear on each interval $[t_n,t_{n+1}]$, and $\bar{A}_1=-B_1$, then $\bar{A}_t$ is continous, consider the set:

$$D=\bigcap_{n\geqq 1}\{\bar{A}_{t_{n+1}}-\bar{A}_{t_n}\geqq 0\},$$

we have,

$$D=\bigcap_{n\geqq 2}\{B_{t_{n-1}}-B_{t_n}>C_n-C_{n-1}\}\cap\{-B_{t_1}-C_1\geqq 0\}.$$

Let us show that D has positive probability; this comes from the fact that for Y gaussian of variance one,

$$\prod_n P(Y>(C_n-C_{n-1})\times(\sqrt{2})^n)=\prod_n\left(1-P\left(Y\leqq-\frac{(\sqrt{2})^n}{n(n-1)}\right)\right)$$

and $\sum P\left(Y\leqq-\frac{(\sqrt{2})^n}{n(n-1)}\right)\leqq\sum C\times\frac{n(n-1)}{(\sqrt{2})^n}<\infty$.

One can then define $\tau=\inf\{t_n, \bar{A}_{t_{n+1}}-\bar{A}_{t_n}<0\}$, τ is a stopping time (because $\bar{A}_{t_{n+1}}$ is $\sigma(B_s, s\leqq t_n)$ measurable), and $A_t=\bar{A}_{t\wedge\tau}$ is a continuous increasing process which with positive probability satisfies $A_1=-B_1$. So $X_t=B_t+A_t$ has a positive probability of being zero at time 1. □

In spite of the previous example, we have the following smoothness result:

Proposition 2.3. *Let* $(X_t)_{t\geqq 0}$ *satisfy* (2.1), *then*

(i) *for* $f(t,x)$ *a continuous function on* $[0,T]\times\mathbb{R}$

$$Af=E_P\left[\int_0^T f(s,X_s)\,ds\right]\leqq K\left[\int_0^T\int_R f^2(s,x)\,ds\,dx\right]^{1/2}, \tag{2.7}$$

where the constant K *depends on* T *and the* H^1 *norm of* $X_{t\wedge T}$.

(ii) *if* $u(s,x)\in L^2([0,T]\times\mathbb{R})$ *is the density of* X_s, *then*

$$E_Y[L_T^0(X-Y)]=\int_0^T 2u(s,X_s)\,ds. \tag{2.8}$$

Proof. Define $\phi_\lambda(\cdot)=\frac{1}{\sqrt{2\pi\lambda}}\exp-\frac{x^2}{2\lambda}$, then we know from Proposition 2.1, that on the product space $\Omega\times\Omega$,

$$\sup_\lambda \int_0^T 2\times\phi_\lambda(X_s(\omega)-X_s(\omega'))\,ds\leqq Z_T(\omega,\omega')$$

which is integrable. As a consequence,

$$\begin{aligned}\sup_{\lambda>0} E_{P\otimes P}\left[\int_0^T \phi_\lambda(X_s(\omega)-X_s(\omega'))\,ds\right]\\ =\sup_{\lambda>0}\int_0^T ds\langle u_s(dx)\otimes u_s(dy),\phi_\lambda(x-y)\rangle\\ \leqq E_{P\otimes P}[Z_T(\omega,\omega')]\leqq K\times E_P[\sup_{s\leqq T}|B_s|+A_T],\end{aligned} \tag{2.9}$$

if $u_s(dx)$ is the law of X_s. Now,

$$\phi_\lambda(x-y)=\int\phi_{\lambda/2}(x-z)\,\phi_{\lambda/2}(z-y)\,dz$$

and the left member of (2.9) is $\sup_{\lambda>0}\int_0^T\int_R u_\lambda(s,z)^2\,ds\,dz$

$$\text{if } u_\lambda(s,z)=\phi_\lambda * u_s(z). \tag{2.10}$$

Now for $f(s,x)$ continuous on $[0,T]\times\mathbb{R}$, with compact support,

$$\begin{aligned}\left|E\left[\int_0^T f(s,X_s)\,ds\right]\right|&=\left|\int_0^T ds\int u_s(dx)f(s,x)\right|\\ &=\left|\lim_{\lambda\to0}\int_0^T \mathrm{d}s\int u_s^\lambda(x)f(s,x)\,dx\right|\\ &\leqq\sqrt{K\times E_P[\sup_{s\leqq T}|B_s|+A_T]}\times\left[\int_0^T ds\int dx f^2(s,x)\right]^{1/2}\end{aligned}$$

by Cauchy Schwarz inequality.

This shows that the positive measure of (2.7) defines an element of $L^2([0,T]\times\mathbb{R})$. Let us now compute $E_Y[L_t^0(X-Y)]=C_t(\omega)$. By (2.3), we know that

$$C_t(\omega)=\lim_n 2E_P\left[\int_0^t\phi_n(X_s(\omega)-X_s(\omega'))\,ds\right],\qquad \text{a.s. and in } L^1(P).$$

Using (2.7), denote by $u(s,x)$ the element of $L^2([0,T]\times\mathbb{R})$ which for almost every $s\in[0,T]$ is the density with respect to Lebesgue measure of $u_s(dx)$.

$$C_t(\omega) = \lim_n 2 \int_0^t u_{1/n}(s, X_s(\omega))\, ds, \qquad (u_\lambda \text{ is defined by (2.10)}),$$

now,

$$E\left[\left| C_t(\omega) - 2\int_0^t u(s, X_s)\, ds \right|\right]$$
$$\leqq \liminf 2E\left[\int_0^T |u_{1/n}(s, X_s(\omega)) - u(s, X_s(\omega))|\, ds\right]$$
$$= \liminf 2 \int_0^T ds \int_R dy\, |u_{1/n}(s, y) - u(s, y)|\, u(s, y)$$
$$\leqq \liminf 2\, \|u(s, y)\|_{L^2([0,T]\times \mathbb{R})} \times \left[\int_0^T ds \int dy\, |u_{1/n}(s, y) - u(s, y)|^2\right]^{1/2}.$$

We know that $u_{1/n}(s, y) \to u(s, y)$ in $L^2(\mathbb{R})$ for a.e. s and

$$\|u_{1/n}(s, y)\|^2_{L^2(\mathbb{R})} \leqq \|u(s, y)\|^2_{L^2(\mathbb{R})}$$

(see for instance Stein-Weiss [16]), as a consequence

$$\left[\int dy \int_0^T ds\, |u_{1/n}(s, y) - u(s, y)|^2\right]^{1/2}$$

goes to zero, and this proves that P-a.s. $\forall t \geqq 0$, $C_t(\omega) = 2\int_0^t u(s, X_s(\omega))\, ds$.

(Notice that the right member of the previous inequality also defines a continuous increasing process, because P-a.s. $\forall T \geqq 0$, $\int_0^T u(s, X_s(\omega))\, ds < \infty$). This proves Proposition 2.3. □

Remark 2.4. Let us notice that if we had chosen the right continuous version $L^{r,0}$ (respectively the left continuous version $L^{l,0}$) of the local time in zero, instead of the symmetric local time in zero, formula (2.3) would remain valid, when choosing ϕ with compact support in $\mathbb{R}_+$ (resp. $\mathbb{R}_-$) instead of being symmetric around zero, and the proof of Proposition 2.3 shows that

$$E_Y[L_t^0(X-Y)] = E_Y[L_t^{l,0}(X-Y)]$$
$$= E_Y[L_t^{r,0}(X-Y)] = \int_0^t 2u(s, X_s)\, ds. \quad \square \tag{2.8'}$$

We are now going to define the nonlinear process which is associated with the propagation of chaos result, we have in view. Let c be a strictly positive number, u_0 be a probability on R, define the probability u_t on $\mathbb{R}$, $(t>0)$, by

$$\exp -4c\, F_t(x) = (\exp -4c\, F_0) * \phi_t(x), \qquad t>0, \tag{2.11}$$

with $F_t(x) = \int_{-\infty}^x u_t(dy)$, $t \geqq 0$, and $\phi_t(z) = \frac{1}{\sqrt{2\pi t}} \exp -\frac{z^2}{2t}$.

We are going to study the laws of the processes $(X_t)_{t\geqq 0}$, defined on some probability space, satisfying (2.1) and such that:

$$X_0 \text{ has law } u_0 \text{ and } A_t = c E_Y[L_t^0(X-Y)]. \tag{2.12}$$

Denote by $S(u_0)$ the set of the laws on $C(\mathbb{R}_+, \mathbb{R})$ of these processes. We have the following weak uniqueness result:

Theorem 2.5. *The set $S(u_0)$ has at most one element, and if $P \in S(u_0)$, $X_t \circ P = u_t$,* where u_t is defined by (2.11).

Remark. As a consequence of the propagation of chaos result that we are going to prove, one has in fact $S(u_0)$ non-empty, and so reduced to a singleton.

Proof. Let X_t, on some filtered probability space satisfying the assumptions of the beginning of this section, satisfy (2.1) and (2.12). Denote by $u(s,x)$ $(\in L^2([0,T]\times\mathbb{R}),\ \forall T>0)$ the density of the law of X_s with respect to Lebesgue measure, by Proposition 2.3 we have:

$$X_t = X_0 + B_t + 2c \int_0^t u(s, X_s)\, ds. \tag{2.13}$$

Let $\phi(s,x)$ be a C^∞ function with compact support in $]0,T[\times\mathbb{R}$. By Ito's formula

$$\begin{aligned} 0 &= E[\phi(T, X_T) - \phi(0, X_0)] \\ &= E\left[\int_0^T \left(\frac{\partial\phi}{\partial s} + \frac{1}{2}\frac{\partial^2\phi}{\partial x^2} + 2cu\frac{\partial\phi}{\partial x}\right)(s, X_s)\, ds\right] \\ &= \int_0^T ds \int_R dx \left(\frac{\partial\phi}{\partial s} + \frac{1}{2}\frac{\partial^2\phi}{\partial x^2}\right) \times u + 2c\frac{\partial\phi}{\partial x} u^2. \end{aligned} \tag{2.14}$$

So (2.14) tells us that $u(s,x)$ satisfies Burgers' equation:

$$-\frac{\partial u}{\partial s} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - 2c\frac{\partial(u^2)}{\partial x} = 0 \tag{2.15}$$

in the distribution sense in $]0,T[\times\mathbb{R}$. So far we do not know if u is smooth. Define on $]0,T[\times\mathbb{R}$

$$F(t,x) = \int_{-\infty}^x u_t(y)\, dy \in [0,1]. \tag{2.16}$$

We have $\frac{\partial F}{\partial x} = u$ (in the distribution sense) and

$$\frac{\partial}{\partial x}\left(-\frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}\right) = 2c\frac{\partial}{\partial x}(u^2). \tag{2.17}$$

So $-\frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}$ and $2cu^2$ have the same space derivative in $]0,T[\times\mathbb{R}$, consequently their difference is a distribution invariant by translation in the x direction (see Schwartz [18], p. 55).

Let $\phi(t,x)$ be a test function, we have

$$\left\langle -\frac{\partial F}{\partial t}+\frac{1}{2}\frac{\partial^2 F}{\partial x^2}-2cu^2,\phi\right\rangle$$
$$=\int F(t,x)\left(\frac{\partial \phi}{\partial t}(t,x+z)+\frac{1}{2}\frac{\partial^2\phi}{\partial x^2}(t,x+z)\right)-2cu^2\,\phi(t,x+z)\,dt\,dx$$

for any z. Letting z go to $+\infty$, we have $\int dt\,dx\,F(t,x-z)\left(\frac{\partial\phi}{\partial t}(t,x)+\frac{1}{2}\frac{\partial^2\phi}{\partial x^2}(t,x)\right)$ which converges towards zero by bounded convergence, and $\int u^2(t,x-z)\,\phi(t,x)\,dt\,dx$ goes to zero because $u^2(t,x)\in L^1(]0,T[\times\mathbb{R})$.

As a consequence

$$-\frac{\partial F}{\partial t}+\frac{1}{2}\frac{\partial^2 F}{\partial x^2}-2cu^2=0 \tag{2.18}$$

in the distribution sense in $]0,T[\times\mathbb{R}$. Working now on the open set $(\varepsilon,T-\varepsilon)\times\mathbb{R}$, let $F_\lambda(t,x)$ be a regularization by convolution of F, on $(\varepsilon,T-\varepsilon)\times\mathbb{R}$ (in the time, space, variables). Set

$$W_\lambda(t,x)=\exp -4c\,F_\lambda(t,x)\quad \text{on}\ (\varepsilon,T-\varepsilon)\times\mathbb{R} \tag{2.19}$$

(this is the linearizing Cole-Hopf transform for Burgers' equation, see Cole [3]), then W_λ is smooth and $0\leqq W_\lambda\leqq 1$. Moreover we have:

$$\frac{\partial W_\lambda}{\partial t}=-4c\frac{\partial F_\lambda}{\partial t}W_\lambda,\qquad \frac{\partial W_\lambda}{\partial x}=-4c\frac{\partial F_\lambda}{\partial x}W_\lambda$$
$$\frac{\partial^2 W_\lambda}{\partial x^2}=16c^2\left(\frac{\partial F_\lambda}{\partial x}\right)^2 W_\lambda-4c\frac{\partial^2 F_\lambda}{\partial x^2}W_\lambda.$$

As a consequence,

$$\frac{\partial W_\lambda}{\partial t}-\frac{1}{2}\frac{\partial^2 W_\lambda}{\partial x^2}=4c\left(-\frac{\partial F_\lambda}{\partial t}+\frac{1}{2}\frac{\partial^2 F_\lambda}{\partial x^2}-2c\left(\frac{\partial F_\lambda}{\partial x}\right)^2\right)\times W_\lambda,$$

and because of (2.18) we have

$$-\frac{\partial F_\lambda}{\partial t}+\frac{1}{2}\frac{\partial^2 F_\lambda}{\partial x^2}-2c\left[\left(\frac{\partial F}{\partial x}\right)^2\right]_\lambda=0\quad \text{in}\ (\varepsilon,T-\varepsilon)\times\mathbb{R}$$

(the subscript $(v)_\lambda$ indicates the regularization by convolution). Consequently

$$\frac{\partial W_\lambda}{\partial t}-\frac{1}{2}\frac{\partial^2 W_\lambda}{\partial x^2}=8c^2\left(\left[\left(\frac{\partial F}{\partial x}\right)^2\right]_\lambda-\left(\frac{\partial F_\lambda}{\partial x}\right)^2\right)\times W_\lambda. \tag{2.20}$$

When λ goes to zero in (2.20), $\left[\left(\frac{\partial F}{\partial x}\right)^2\right]_\lambda=(u^2)_\lambda$, converges towards u^2 in $L^1((\varepsilon,T-\varepsilon)\times\mathbb{R})$, and since $(u_\lambda)^2-u^2=(u_\lambda-u)(u_\lambda+u)$, $(u_\lambda)^2$ converges towards u^2 in $L^1(\varepsilon,T-\varepsilon)\times\mathbb{R})$ as a consequence

$$\frac{\partial W}{\partial t}=\frac{1}{2}\frac{\partial^2 W}{\partial x^2}, \quad \text{in the distribution sense in } (\varepsilon, T-\varepsilon)\times \mathbb{R} \tag{2.21}$$
(and since ε was arbitrary, in $(0, T)\times \mathbb{R}$).

By hypoellipticity of $\frac{\partial}{\partial t}-\frac{1}{2}\frac{\partial^2}{\partial x^2}$ (see Hörmander [8]), $W(t, x)$ is a C^∞ function in $]0, T[\times \mathbb{R}$, which satisfies (2.21) and has values in $[0,1]$ so (see Friedman [5]),

$$W(t, x)=\frac{1}{\sqrt{2\pi(t-s)}}\int_{\mathbb{R}} W(s, y)\exp -\frac{(x-y)^2}{2(t-s)}\,dy, \qquad 0<s<t<T,$$

letting s to go zero, since $W(s, y)$ converges boundedly towards $W(0, y)$ except may be at points of discontinuity of $F(0,\cdot)$ (an at most denumerable set), we find

$$W(t, x)=\frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}} W(0, y)\exp -\frac{(x-y)^2}{2t}\,ds. \tag{2.22}$$

This proves the statement concerning the laws at fixed times of solutions of (2.1), (2.12). Now to prove that $S(u_0)$ contains at most one point, it is enough to check that any element of $S(u_0)$ induces the same law on the sigma fields $\sigma(W_t, t\geqq s)=F^{s,\infty}$, of the canonical space $C(\mathbb{R}_+, \mathbb{R})$ (W_t are the canonical coordinates), for any $s>0$. But,

$$X_t=X_s+B_t-B_s+\int_s^t 2cu(v, X_v)\,dv, \qquad t\geqq s, \tag{2.23}$$

and X_s has law $u(s,x)\,dx$ ($u(v,x)$ is given by (2.11)). Since $u(v,x)$, $v\geqq s$, is bounded and Lipschitz, one has uniqueness, in law for the solutions $X_t, t\geqq s$ of (2.23), this proves the theorem.

Remark 2.6. In the case where $u_0(dx)=u_0(x)\,dx$ and u_0 is bounded, one can see that $u(t,x)$, $t\geqq 0$, $x\in\mathbb{R}$, is bounded measurable, it follows from Zvonkin's results [25] (Theorem 4), that there is strong existence and strong uniqueness for the equation

$$X_t=X_0+B_t+\int_0^t 2cu(s, X_s)\,ds.$$

As a result one also has a strong existence (using part IV), and uniqueness result for the solutions of (2.1), (2.12). □

III. A Compactness Result

We consider now the system of interacting particles for which we want to obtain a propagation of chaos result. In this section we will prove a weak compactness result for the law of certain empirical measures.

We consider some filtered probability space $(\Omega, F, (F_t)_{t\geqq 0}, P)$ (F complete, $(F_t)_{t\geqq 0}$, right continuous, each F_t containing the P-negligible sets of F), endowed with F_t-Brownian motions $B^1_\cdot, \ldots, B^N_\cdot$, and real valued F_0-measurable random variables $(X_0^i)_{i\in[1,N]}$, with symmetric distribution u_N on $\mathbb{R}^N$, satisfying:

$$\text{The set } \bigcup_{\substack{i,j,k\\ \text{distinct}}} \{x^i=x^j=x^k\} \text{ is negligible for } u_N. \tag{3.1}$$

It was shown in Sznitman-Varadhan [21], that for $c>0$, one can construct a unique solution of

$$X_t^i = X_0^i + B_t^i + \frac{c}{N} \sum_{j\neq i} L^0(X^i - X^j)_t, \qquad i=1,\dots,N, \tag{3.2}$$

where $L^0(X^i-X^j)_.$ is the symmetric local time of X^i-X^j. The $(X_.^1,\dots,X_.^N)$ are expressed as measurable function of (X_0^i) and $(B_.^i)$, and as such the laws of solutions of (3.2) do not depend on the space where they are constructed. For $a=\{i,j\}$ (with the notational convention $i<j$), one defines

$$V_a = \sqrt{2}\,\frac{c}{N}\,(e_i+e_j), \qquad n_a = \frac{1}{\sqrt{2}}\,(e_i-e_j),$$

and one can construct $(T_n)_{n\geqq 0}$ ($\mathbb{R}_+$ valued strictly increasing sequences), $(X^n)_{n\geqq 0}$ ($\mathbb{R}^N$-valued), $(j_n)_{n\geqq 0}$ valued in the set J of pairs of elements of $[1,N]$; which satisfy:

$$T_0 = \inf(t\geqq 0,\ \exists\, a\in J,\ n_a\cdot(X_0+B_t)=0\},$$

$\{j_0\}=\{a\in J,\ n_a\cdot(X_0+B_{T_0})=0\}$ (it is a singleton a.s.), $X^0=X_0+B_{T_0}$, and for $n\geqq 0$

$$\tau_{n+1} = T_{n+1}-T_n = \inf\{s\geqq 0,\ \exists\, a\neq j_n(\omega),\ n_a\cdot(X^n+B_s^n+V_{j_n}\cdot L^0(n_{j_n}\cdot B^n)_s)=0\}$$

(with the notation $B_s^n = B_{T_n+s}-B_{T_n}$),

$$X^{n+1} = X^n + B^n_{\tau_{n+1}} + V_{j_n}\cdot L^0(n_{j_n}\cdot B^n)_{\tau_{n+1}},$$

$\{j_{n+1}\}=\{a\in J,\ X^{n+1}\cdot n_a=0\}$ (it is a singleton a.s.), it was shown in the reference quoted that a.s. the $(T_n)_{n\geqq 0}$ converge to $+\infty$ when n goes to infinity, and that one has:

$$X_{s\wedge T_0} = X_0 + B_{s\wedge T_0}, \qquad s\geqq 0,$$

$$X_{T_n+s\wedge\tau_{n+1}} = X^n + B^n_{s\wedge\tau_{n+1}} + V_{j_n(\omega)}\cdot L^0(n_{j_n}\cdot B^n)_{s\wedge\tau_{n+1}}, \qquad \text{for } s\geqq 0, \tag{3.4}$$

P a.s., $(L^0(B^n\cdot n_{j_n}))_s$ is the local time in zero of the F_{T_n}-Brownian motion $B_.^n\cdot n_{j_n}$).

One can also prove that the laws Q_x of the solutions of (3.2) starting from $x\in E=\mathbb{R}^N\setminus \bigcup_{\substack{i,j,k\\ \text{distinct}}} \{x^i=x^j=x^k\}$, on $C(\mathbb{R}_+,E)$ form a strong Markov process with state space E, and one has the following "Brownian scaling" property:

$$\text{the image of } Q_x \text{ under } w\to\lambda w_./\lambda^2 \text{ is } Q_{\lambda x}. \tag{3.5}$$

Consider now $x=(x_1,\dots,x_N)\in E$ and $X_t=(X_t^1,\dots,X_t^N)_{t\geqq 0}$, the solution of (3.2) with initial condition x. We are first going to prove a lemma which shows that the increasing reordering, $(Y_t^1,\dots,Y_t^N)$, of $(X_t^1,\dots,X_t^N)$ satisfies an oblique type of reflection problem on the set $K=\{x^1 \leqq x^2 \leqq \ldots \leqq x^N\}$. Notice that:

$$Y_t^k = \sup\{\inf_{i\in A}(X_t^i),\ A \text{ subset of } [1, N], \text{ with card } A = N+1-k\}, \tag{3.6}$$

we can state

Lemma 3.1. *There exists N independent real valued F_t-Brownian motions $W_t^1, \ldots, W_t^N$ and $(N-1)$ continuous increasing F_t-processes $\gamma_t^1, \ldots, \gamma_t^{N-1}$, satisfying*

$$\gamma_t^i = \int_0^t 1(Y_s^i = Y_s^{i+1})\, d\gamma_s^i, \tag{3.7}$$

$$\begin{aligned} Y_t^1 &= Y_0^1 + W_t^1 - \tfrac{1}{2} a \gamma_t^1,\\ Y_t^k &= Y_0^k + W_t^k - \tfrac{1}{2} a \gamma_t^k + \tfrac{1}{2} b \gamma_t^{k-1}, \quad 2 \leqq k \leqq N-1,\\ Y_t^N &= Y_0^N + W_t^N + \tfrac{1}{2} b \gamma_t^{N-1}, \end{aligned} \tag{3.8}$$

$\left(\text{where } a = 1 - 2\dfrac{c}{N},\ b = 1 + \dfrac{2c}{N}\right)$.

Remark 3.2. As we shall see later, the normal reflected process in the convex $K = \{x^1 \leqq x^2 \leqq \ldots \leqq x^N\}$ constructed on $Y_0^i + W_t^i$ (see Tanaka [22]), satisfies (3.8), with $a = b = 1$.

Proof. Using Tanaka's formula and (3.6), we already know that $Y_t^k = Y_0^k + W_t^k + A_t^k$, where W_t^k are continuous local martingales and A_t^k is a finite variation process, $k \in [1, N]$. We are going to show by induction on n that for $n \geqq 0$,

$$\langle W^i, W^j \rangle_{t \wedge T_n} = (t \wedge T_n) \times \delta_{ij}, \tag{3.9}$$

$$A_{s \wedge T_n}^k = \int_0^{s \wedge T_n} [1(Y_s^k = Y_s^{k-1}) + 1(Y_s^k = Y_s^{k+1})]\, dA_s^k, \quad 1 \leqq k \leqq N, \tag{3.10}$$

(the expression where $Y_\cdot^0$ or $Y_\cdot^{N+1}$ appear are understood as being zero), and the first term of the right member of (3.10) is continuous and increasing, and

$$a \int_0^{t \wedge T_n} dA_s^k\, 1(Y_s^k = Y_s^{k-1}) = -b \int_0^{t \wedge T_n} dA_s^{k-1}\, 1(Y_s^k = Y_s^{k-1}), \tag{3.11}$$

for $2 \leqq k \leqq N$.

If we prove (3.9), (3.10), (3.11), defining

$$\gamma_t^k = \frac{2}{b} \int_0^t 1(Y_s^{k+1} = Y_s^k)\, dA_s^{k+1}, \quad 1 \leqq k \leqq N-1,$$

one easily obtains (3.7), (3.8).

Let us first show (3.9), (3.10), (3.11) for $n = 0$.

If $T_0 = 0$, this is immediate, else all components of the starting point x are distinct and there exists a unique permutation σ of $[1, N]$, such that $Y_0^k = x_{\sigma(k)}$; moreover, $Y_{t \wedge T_0}^k = x_{\sigma(k)} + B_{t \wedge T_0}^{\sigma(k)}$, $k \in [1, N]$, and our induction hypothesis is satisfied. So suppose we proved the induction hypothesis for $k \leqq n$, let us show that it is valid for $n+1$. We partition the space Ω in the F_{T_n}-measurable subsets

$A_{k,j,l}$, with $1 \leqq j < l \leqq N$, $1 \leqq k \leqq N-1$, defined by

$$A_{k,j,l} = \{j_n(\omega) = \{j, l\}\} \cap \{Y^k_{T_n} = Y^{k+1}_{T_n}\}. \tag{3.12}$$

On $A_{k,j,l}$ one can define an F_{T_n}-measurable permutation of $[1, N]$, σ (we omit the subscript k, j, l) such that

$$Y^m_{T_n} = X^{\sigma(m)}_{T_n}, \quad \text{and} \quad \sigma(k) = j, \qquad \sigma(k+1) = l. \tag{3.13}$$

Now on $A_{k,j,l}$, for $m \neq k$, $k+1$, we find

$$Y^m_{T_n + t \wedge \tau_{n+1}} - Y^m_{T_n} = B^{\sigma(m)}_{T_n + t \wedge \tau_{n+1}} - B^{\sigma(m)}_{T_n},$$

and

$$Y^k_{T_n + t \wedge \tau_{n+1}} - Y^k_{T_n} = \inf\Big(\bar{B}^{\sigma(k)}_{t \wedge \tau_{n+1}} + \frac{c}{N} L^0(\bar{B}^{\sigma(k+1)} - \bar{B}^{\sigma(k)})_{t \wedge \tau_{n+1}},$$

$$\bar{B}^{\sigma(k+1)}_{t \wedge \tau_{n+1}} + \frac{c}{N} L^0(\bar{B}^{\sigma(k+1)} - \bar{B}^{\sigma(k)})_{t \wedge \tau_{n+1}}\Big), \tag{3.14}$$

(with the notation $\bar{B}^m_{\cdot} = B^m_{T_n + \cdot} - B^m_{T_n}$, $m \in [1, N]$), and a similar formula for $k+1$ with sup instead of inf. Using Tanaka's formula and $x \wedge y = x - (x-y)_+$, we find on $A_{k,j,l}$:

$$\begin{aligned} Y^k_{T_n + t \wedge \tau_{n+1}} - Y^k_{T_n} = \int_0^{t \wedge \tau_{n+1}} & 1(\bar{B}^{\sigma(k)}_s < \bar{B}^{\sigma(k+1)}_s)\, d\bar{B}^{\sigma(k)}_s \\ & + 1(\bar{B}^{\sigma(k+1)}_s < \bar{B}^{\sigma(k)}_s)\, d\bar{B}^{\sigma(k+1)}_s \\ & - \tfrac{1}{2} a L^0(\bar{B}^{\sigma(k+1)} - \bar{B}^{\sigma(k)})_{t \wedge \tau_{n+1}}. \end{aligned} \tag{3.15}$$

similarly on $A_{k,j,l}$,

$$\begin{aligned} Y^{k+1}_{T_n + t \wedge \tau_{n+1}} - Y^{k+1}_{Tn} = \int_0^{t \wedge \tau_{n+1}} & 1(\bar{B}^{\sigma(k)}_s < \bar{B}^{\sigma(k+1)}_s)\, d\bar{B}^{\sigma(k+1)}_s \\ & + 1(\bar{B}^{\sigma(k+1)}_s < \bar{B}^{\sigma(k)}_s)\, d\bar{B}^{\sigma(k)}_s + \tfrac{1}{2} b L^0(\bar{B}^{\sigma(k+1)} - \bar{B}^{\sigma(k)})_{t \wedge \tau_{n+1}}. \end{aligned} \tag{3.16}$$

Now using (3.14), (3.15), (3.16), and the induction hypothesis, it is easy to write the (F_t)-semimartingale decomposition of the processes $Y^k_{t \wedge T_{n+1}}$, and one checks immediately, (3.9), (3.10), (3.11). □

In order to estimate the increasing processes (γ^i_t), $1 \leqq i \leqq N-1$, we are now going to obtain a comparison result with a certain normally reflected process. We first introduce some notations:

We set for $1 \leqq k \leqq N-1$,

$$D^k_t = \frac{1}{b^{k-1}} (Y^{k+1}_t - Y^k_t), \qquad H^k_t = \frac{1}{b^{k-1}} (W^{k+1}_t - W^k_t), \qquad C^k_t = \frac{1}{b^{k-1}} \gamma^k_t, \tag{3.17}$$

and we define the application F from the set

$$\{(v_{\cdot}, c_{\cdot}) \in C(\mathbb{R}_+, \mathbb{R}^{N-1}) \times (C^+_0)^{N-1},\ v(0)_i \geqq 0,\ i \in [1, N-1]\},$$

(where C^+_0 is the set of continuous increasing functions from $\mathbb{R}_+$ into $\mathbb{R}_+$ with value zero at time zero) into $(C^+_0)^{N-1}$:

$$F(v, c_\cdot)_t^1 = \sup_{s \leqq t} \left(-v_s^1 + \tfrac{1}{2}\alpha c_s^2\right)_+$$

$$F(v, c_\cdot)_t^k = \sup_{s \leqq t} \left(-v_s^k + \tfrac{1}{2}\alpha c_s^{k+1} + \tfrac{1}{2} c_s^{k-1}\right)_+, \quad 2 \leqq k \leqq N-2,$$

$$F(v, c_\cdot)_t^{N-1} = \sup_{s \leqq t} \left(-v_s^{N-1} + \tfrac{1}{2} c_s^{N-2}\right)_+, \tag{3.18}$$

where $\alpha = ab = 1 - \frac{4c^2}{N^2}$. We endow $C(\mathbb{R}_+, \mathbb{R}^{N-1})$ with the semi-norms $|v|_t = \sum_{i=1}^{N-1} \sup_{s \leqq t} |v_s^i|$, and we consider $(C_0^+)^{N-1}$ as included in $C(\mathbb{R}_+, \mathbb{R}^{N-1})$. We have

Lemma 3.3. (a) *For all v, c, c', one has for $t \geqq 0$,*

$$|F(v, c) - F(v, c')|_t \leqq \tfrac{1}{2}(1 + |\alpha|)\, |c - c'|_t, \tag{3.19}$$

(b) *If $c = F(v, c)$, $\bar{c} = F(\bar{v}, \bar{c})$ and $0 < \alpha < 1$, then*

$$|c - \bar{c}|_t \leqq \frac{2}{1-\alpha} |v - \bar{v}|_t, \quad \text{for } t \geqq 0 \quad \text{and} \quad v \leqq \bar{v} \Rightarrow \bar{c} \leqq c \tag{3.20}$$

(*where $v \leqq \bar{v}$ means for all i and t, $v_t^i \leqq \bar{v}_t^i$*).

(c) *P-a.s.*

$$(C_\cdot^k) = F((D_0^k + H_\cdot^k), C_\cdot^k). \tag{3.21}$$

Proof. (a) From (3.18) we see that for $t \geqq 0$,

$$\sum_{i=1}^{N-1} \sup_{s \leqq t} |F(v, c)_s^i - F(v, c')_s^i| \leqq \sum_i \tfrac{1}{2}(1 + |\alpha|) \sup_{s \leqq t} |c_s^i - c_s'^i|$$

which proves (3.19).

(b) If $0 < \alpha < 1$, the fixed points c and c' are obtained by iteration (by a contraction argument). Setting $c^1 = F(v, 0)$, $c^{k+1} = F(v, c^k)$ (resp. $\bar{c}' = F(\bar{v}, 0)$, $\bar{c}^{k+1} = F(\bar{v}, \bar{c}^k)$, we find

$$|c^1 - \bar{c}^1|_t \leqq |v - \bar{v}|_t, \qquad |c^{k+1} - \bar{c}^{k+1}|_t \leqq |v - \bar{v}|_t + \frac{(1 + |\alpha|)}{2} |c^k - \bar{c}^k|_t.$$

By induction one gets:

$$|c^{k+1} - \bar{c}^{k+1}|_t \leqq |\bar{v} - \bar{v}|_t \left(1 + \frac{1 + |\alpha|}{2} + \ldots + \left(\frac{1 + |\alpha|}{2}\right)^k\right) \leqq \frac{2}{1 - |\alpha|} |v - \bar{v}|_t,$$

letting k go to infinity, we obtain the first part of (3.20). For the second statement of (3.20), suppose $v \leqq \bar{v}$, then, $c^1 = F(v, 0) \geqq \bar{c}^1 = F(\bar{v}, 0)$, suppose that we know that $c^n \geqq \bar{c}^n$, then:

$$\begin{aligned} c_t^{n+1, k} &= \sup_{s \leqq t} \left(-v_s^k + \tfrac{1}{2}\alpha c_s^{n, k+1} + \tfrac{1}{2} c_s^{n, k-1}\right)_+ \\ &\geqq \sup_{s \leqq t} \left(-\bar{v}_s^k + \tfrac{1}{2}\alpha \bar{c}_s^{n, k+1} + \tfrac{1}{2} \bar{c}_s^{n, k-1}\right)_+ \\ &= \bar{c}_t^{n+1, k} \end{aligned}$$

(with the convention $\bar{c}_\cdot^{n, 0} = \bar{c}_\cdot^{n, N} = 0 = c_\cdot^{n, 0} = c_\cdot^{n, N}$). So by induction, $c^n \geqq \bar{c}_\cdot^n$ and we get $c \geqq \bar{c}$ in the limit.

(c) By (3.8) we get

$$\begin{aligned} Y_t^2 - Y_t^1 &= Y_0^2 - Y_0^1 + W_t^2 - W_t^1 + \gamma_t^1 - \frac{a}{2}\gamma_t^2,\\ Y_t^3 - Y_t^2 &= Y_0^3 - Y_0^2 + W_t^3 - W_t^2 + \gamma_t^2 - \tfrac{1}{2}a\gamma_t^3 - \tfrac{1}{2}b\gamma_t^1,\\ Y_t^{k+1} - Y_t^k &= Y_0^{k+1} - Y_0^k + W_t^{k+1} - W_t^k + \gamma_t^k - \tfrac{1}{2}a\gamma_t^k - \tfrac{1}{2}b\gamma_t^{k-1},\\ Y_t^N - Y_t^{N-1} &= Y_0^N - Y_0^{N-1} + W_t^N - W_t^{N-1} + \gamma_t^{N-1} - \tfrac{1}{2}b\gamma_t^{N-2}, \end{aligned} \tag{3.22}$$

multiplying the k^{th} line of (3.22) by $b^{-(k-1)}$, we get

$$\begin{aligned} D_t^1 &= D_0^1 + H_t^1 + C_t^1 - \tfrac{1}{2}\alpha C_t^2,\\ D_t^2 &= D_0^2 + H_t^2 + C_t^2 - \tfrac{1}{2}\alpha C_t^3 - \tfrac{1}{2} C_t^1,\\ D_t^k &= D_0^k + H_t^k + C_t^k - \tfrac{1}{2}\alpha C_t^{k+1} - \tfrac{1}{2} C_t^{k-1},\\ D_t^{N-1} &= D_0^{N-1} + H_t^{N-1} + C_t^{N-1} - \tfrac{1}{2} C_t^{N-2}. \end{aligned} \tag{3.23}$$

We know that $C_t^k = \int_0^t 1(D_s^k = 0)\, dC_s^k$, and $D_t^k \geqq 0$, (by (3.7)), so the solution of Skorohod's problem (see Ikeda-Watanabe [9], p. 120) tells us that $C = F((D_0 + H_.), C)$, which proves (c). □

We introduce now the martingales $\bar{W}_t^k = W_t^1 + \sum_{l \leqq k-1} H_t^l$, $1 \leqq k \leqq N$, we have

$$\bar{W}_t^1 = W_t^1, \qquad \bar{W}_t^k = \sum_{2 \leqq l \leqq k-1} W_t^l \times \frac{1}{b^{l-1}}(b-1) + \frac{1}{b^{k-2}} W_t^k, \qquad 2 \leqq k \leqq N. \tag{3.24}$$

We now introduce the normally reflected diffusion at the boundary of the convex $K (= \{x^1 \leqq \ldots \leqq x^N\})$, based on $\bar{W}_t = (\bar{W}_t^1, \ldots, \bar{W}_t^N)$. More precisely (see Tanaka [22], Theorem 3.1), for $x \in K$, we consider the unique couple $(Z_., k_.) \in C(\mathbb{R}_+, K) \times C(\mathbb{R}_+, \mathbb{R}^d)$ such that

$$Z_t = x + \bar{W}_t + k_t, \tag{3.25}$$

and $k_.$ is of bounded variation, $|k|_t = \int_0^t 1(Z_s \in \partial K)\, d|k|_s$, and

$$k_t = \int_0^t v_s\, d|k|_s$$

where v_s is a unit vector in the cone of inward normals to the convex K, $d|k|_s$ a.s.

Remark 3.4. K is the intersection of the hyperplanes $H_i = \{x_i \leqq x_{i+1}\}$, $1 \leqq i \leqq N - 1$, if a point x belongs to $H_i \setminus \bigcup_{j \neq i} H_j$, the inward normal at x is simply $n_i = \frac{1}{\sqrt{2}}(e_{i+1} - e_i)$, moreover it is easy to see that when x belongs to $\bigcap_{i \in I} H_i$, $I \subset [1, N-1]$ (and not to $\bigcap_{i \in J} H_i$, for J containing strictly I), then the cone of inward normals of K at x is the same as the cone of inward normals of the convex set $\bigcap_{i\in I}\{n_i\cdot y\geqq 0\}$, at x, which by an easy argument on dual cones is simply

$$\hat{K}_I=\{n=\sum_{i\in I}\lambda_i n_i,\ \lambda_i\geqq 0,\ i\in I\}. \tag{3.26}$$

Notice that the system $\{n_i, i\in I\}$ is linearly independent and the λ_i in (3.26) are uniquely determined.

We now see from (3.25) that there exists continuous increasing processes k_s^i, $1\leqq i\leqq N-1$, such that

$$\begin{aligned} &Z_t^1=x^1+\bar{W}_t^1-\tfrac{1}{2}k_t^1,\\ &Z_t^l=x^l+\bar{W}_t^l-\tfrac{1}{2}k_t^l+\tfrac{1}{2}k_t^{l-1}, \quad 2\leqq l\leqq N-1,\\ &Z_t^N=x^N+\bar{W}_t^N+\tfrac{1}{2}k_t^{N-1}, \end{aligned} \tag{3.27}$$

with

$$k_t^i=\int_0^t 1(Z_s^i=Z_s^{i+1})\,dk_s^i. \quad \square \tag{3.28}$$

We now have

Proposition 3.5. *There exists constants $d>0$, $\bar{d}>0$, such that for $N>2c$, and any $x\in\mathbb{R}^N\setminus\bigcup_{\substack{i,j,k\\ \text{distinct}}}\{x^i=x^j=x^k\}$, starting point of (X_t), one has*

$$\textit{for } t>0, \quad E\left[\exp\frac{d}{\sqrt{t}}\sum_1^N (X_t^i-x^i)\right]\leqq \bar{d}^N. \tag{3.29}$$

Proof. Observe first by the scaling property that it is enough to prove (3.29) for $t=1$, moreover, since clearly the Brownian motion B_t satisfies a similar condition, it is sufficient to show that there exists $d, \bar{d}>0$ such that

$$E\left[\exp\frac{d}{N}\sum_{i\neq j}L^0(X^i-X^j)_1\right]\leqq \bar{d}^N.$$

By (3.8), we know that $\frac{c}{N}\sum_{i\neq j}L^0(X^i-X^j)_t$ is the bounded variation term of $Y_t^1+\ldots+Y_t^N$, that is $\frac{2c}{N}(\gamma_t^1+\ldots+\gamma_t^{N-1})$. Notice that $(\gamma_t^1+\ldots+\gamma_t^{N-1})\leqq b^N(C_t^1+\ldots+C_t^{N-1})$ by (3.17), and the sequence $b^N=\left(1+\frac{2c}{N}\right)^N$ is bounded (in N). By Lemma 3.3, we know that (C_t) is the fixed point of $F(v,\cdot)$, associated with $v=(D_0^k+H_\cdot^k)$, and C is inferior (in the sense of Lemma 3.3) to the fixed point (recall that $0<\alpha<1$), associated with $F(\bar{v},\cdot)$ if $\bar{v}=(H_\cdot^k)$, $(D_0^k=0)$, we also know that if we pick instead of $D_0^k=0$, the initial $D_0^k=\frac{1}{N}$, the corresponding fixed point $\bar{c}$ will satisfy $\bar{C}_t^1+\ldots+\bar{C}_t^{N-1}\geqq C_t^1+\ldots+C_t^{N-1}-\frac{N^2}{2c^2}$, by (3.20).

So we have

$$\frac{1}{N}(C_1^1+\ldots+C_1^{N-1})\leqq\frac{1}{N}(\bar{C}_1^1+\ldots+\bar{C}_1^{N-1})+\frac{N}{2c^2}.$$

As a consequence of these remarks, it is sufficient in order to obtain (3.29), to prove that when one starts with $y=\left(0, \frac{1}{N}, \ldots, \frac{N-1}{N}\right) \in K$, the fixed point $(C_\cdot)$ associated with $F(v, \cdot)$, where $v_\cdot = y + H_\cdot$ satisfies

$$\exists\, d, \bar{d}>0, \quad \forall N>2c, \quad E\left[\exp \frac{d}{N}(C_1+\ldots+C_{N-1})\right] \leqq \bar{d}^N. \tag{3.30}$$

Consider now $(Z_t)_{t \geqq 0}$, the normally reflected process on K, starting at y. By (3.27), (3.28), one gets that

$$\begin{aligned}
Z_t^2 - Z_t^1 &= \frac{1}{N} + H_t^1 + k_t^1 - \tfrac{1}{2} k_t^2 \\
Z_t^{l+1} - Z_t^l &= \frac{1}{N} + H_t^l + k_t^l - \tfrac{1}{2} k_t^{l+1} - \tfrac{1}{2} k_t^{l-1}, \quad 2 \leqq l \leqq N-2, \\
Z_t^N - Z_t^{N-1} &= \frac{1}{N} + H_t^{N-1} + k_t^{N-1} - \tfrac{1}{2} k_t^{N-2}, \\
k_t^l &= \int_0^t 1(Z_s^{l+1} = Z_s^l)\, dk_s^l, \quad 1 \leqq l \leqq N-1,
\end{aligned}$$

from this we deduce, as in (3.23), using Skorohod's problem that

$$\begin{aligned}
k_t^1 &= \sup_{s \leqq t} \left(-\frac{1}{N} - H_s^1 + \tfrac{1}{2} k_s^2\right)_+ \\
k_t^l &= \sup_{s \leqq t} \left(-\frac{1}{N} - H_s^l + \tfrac{1}{2} k_s^{l+1} + \tfrac{1}{2} k_s^{l-1}\right)_+, \quad 2 \leqq l \leqq N-2, \\
k_t^{N-1} &= \sup_{s \leqq t} \left(-\frac{1}{N} - H_s^{N-1} + \tfrac{1}{2} k_s^{N-2}\right)_+.
\end{aligned} \tag{3.31}$$

So we have on one hand $C = F_\alpha(y + H_\cdot, C)$ where C is obtained by iteration (since $0 < \alpha < 1$), and on the other hand $k = F_1(y + H_\cdot, k)$. But it is elementary to check by induction that for every element of the approximating sequence $(C_\cdot^n)$, one has $C_\cdot^n \leqq k_\cdot$, so that in the limit one gets $C \leqq k$. To prove Proposition 3.5, it will suffice to exhibit $d, \bar{d}>0$ such that for $N>2c$:

$$E\left[\exp \frac{d}{N}(k_1^1 + \ldots + k_1^{N-1})\right] \leqq (\bar{d})^N. \tag{3.32}$$

Following Tanaka [22] (Theorem 3.1), by (3.27) we have

$$\begin{aligned}
\sum_{i=1}^N |Z_t^i - Z_0^i|^2 = {} & 2 \sum_i \int_0^t (Z_s^i - Z_0^i)\, d\bar{W}_s^i \\
& + \sum_{i=1}^{N-1} \int_0^t [(Z_s^{i+1} - Z_0^{i+1}) - (Z_s^i - Z_0^i)]\, dk_s^i + \sum_{i=1}^N \int_0^t d\langle \bar{W}^i \rangle_s.
\end{aligned}$$

Notice that dk_s^i a.s. $Z_s^{i+1}=Z_s^i$, so that we obtain:

$$\sum_{i=1}^{N}|Z_t^i-Z_0^i|^2+\frac{1}{N}(k_t^1+\ldots+k_t^{N-1})$$
$$\leqq 2\sum_{i=1}^{N}\int_0^t (Z_s^i-Z_0^i)\,d\bar{W}_s^i+\text{const. } N\cdot t \tag{3.33}$$

(the constant in (3.33) is independent of N).

Consider now the covariance matrix $\frac{d}{dt}\langle \bar{W}^i, \bar{W}^j\rangle_t$, we have by (3.24),

$$0\leqq\frac{d}{dt}\langle \bar{W}^i, \bar{W}^j\rangle\leqq\text{const.}\left(\frac{1}{N}+\delta_{ij}\right)=a_{ij}.$$

Notice that a_{ij} has an operator norm uniformly bounded in N, if $\mathbb{R}^N$ is endowed with the usual euclidean norm. As a consequence,

$$\left\langle\sum_i\int_0^t (Z_s^i-Z_0^i)\,d\bar{W}_s^i\right\rangle\leqq\sum_{ij}\int_0^t |Z_s^i-Z_0^i|\,|Z_s^j-Z_0^j|\,a_{ij}\,ds$$
$$\leqq C\times\sum_i\int_0^t |Z_s^i-Z_0^i|^2\,ds. \tag{3.34}$$

Now, using (3.33), we see that in order to obtain (3.32), it is enough to estimate $\exp d\sum_{i=1}^{N}\int_0^1 (Z_s^i-Z_0^i)\,d\bar{W}_s^i$, and using the exponential martingale, we know that

$$E\left[\exp d\sum_i\int_0^t (Z_s^i-Z_0^i)\,d\bar{W}_s^i\right]$$
$$\leqq E\left[\exp 2d\sum_i\int_0^t (Z_s^i-Z_0^i)\,d\bar{W}_s^i-2d^2\left\langle\sum_i\int_0^{\cdot} (Z_s^i-Z_0^i)\,d\bar{W}_s^i\right\rangle_t\right]^{1/2}$$
$$\times E\left[\exp 2d^2\left\langle\sum_i\int_0^{\cdot} (Z_s^i-Z_0^i)\,d\bar{W}_s^i\right\rangle_t\right]^{1/2}$$
$$\leqq E\left[\exp d^2\,C\int_0^t\sum_i |Z_s^i-Z_0^i|^2\,ds\right]$$

(we use that for $x\geqq 1$, $\sqrt{x}\leqq x$). Using now the convexity of the exponential function, it will suffice to find $d, d'>0$, such that for $N>2c$

$$\int_0^1 E[\exp d\sum_i |Z_s^i-Z_0^i|^2]\,ds\leqq \bar{d}^N, \tag{3.35}$$

and we will obtain (3.22). By Ito's formula, we have:

$$\exp\lambda\sum_i \frac{|Z_t^i-Z_0^i|^2}{(t+1)}$$
$$\leqq 1+\sum_{i=1}^{N}\int_0^t 2\lambda\frac{(Z_s^i-Z_0^i)}{s+1}\exp\lambda\frac{|Z_s-Z_0|^2}{s+1}\,d\bar{W}_s^i$$
$$+\lambda\sum_{i=1}^{N-1}\int_0^t\frac{(Z_s^{i+1}-Z_0^{i+1})-(Z_s^i-Z_0^i)}{s+1}\exp\lambda\frac{|Z_s-Z_0|^2}{(s+1)}\,dk_s^i$$
$$+\frac{1}{2}\int_0^t\exp\lambda\frac{|Z_s-Z_0|^2}{(s+1)}\left(\frac{4\lambda^2}{(s+1)^2}\sum_{i,j}|Z_s^i-Z_0^i|\,|Z_s^j-Z_0^j|\,a_{ij}-2\lambda\frac{|Z_s-Z_0|^2}{(s+1)^2}\right)ds$$
$$+CN\int_0^t\exp\lambda\frac{|Z_s-Z_0|^2}{s+1}\,ds.$$

Observe now that for $\lambda<\frac{1}{2|a|}$, $4\lambda|Z_s^i-Z_0^i|\,|Z_s^j-Z_0^j|\,a_{ij}-2|Z_s-Z_0|^2\leqq 0$. As a consequence, we have

$$\exp\lambda\frac{|Z_t-Z_0|^2}{(t+1)}\leqq 1+CN\int_0^t\exp\lambda\frac{|Z_s-Z_0|^2}{(s+1)}\,ds+\text{local martingale.}$$

Calling $U_t=\exp\lambda\frac{|Z_t-Z_0|^2}{(t+1)}$, we have for any stopping time T such that the stopped local martingale and $U_{\cdot\wedge T}$ are bounded $E[U_{t\wedge T}]\leqq 1+c\cdot N\cdot\int_0^t E[U_{s\wedge T}]\,ds$, which by Gronwall's lemma implies $E[U_{t\wedge T}]\leqq\exp CNt$. Letting T go to infinity, we find using Fatou's lemma,

$$E\left[\exp\frac{1}{8|a|}\frac{|Z_t-Z_0|^2}{(t+1)}\right]\leqq e^{CNt}, \tag{3.36}$$

from which we deduce (3.35) immediately, and which finishes the proof of Proposition 3.5. □

We are now going to use Proposition 3.5 to obtain estimates on the individual components X_t^i.

Proposition 3.6. *There exist constants $d>0$, $\bar{d}>0$, such that for $N>2c$ and any $x\in\mathbb{R}^N\setminus\bigcup_{i,j,k \text{ distinct}}\{x^i=x^j=x^k\}$, starting point of X_t, one has:*

$$\textit{for } i\in[1,N],\ t>0,\qquad E\left[\exp\frac{d}{\sqrt{t}}(X_t^i-X_0^i)\right]\leqq\bar{d}. \tag{3.37}$$

Proof. Again by the scaling property it is enough to prove (3.37) for $t=1$, and one also sees that it is enough to show the existence of d, $\bar{d}>0$, such that for $i\in[1,N]$

$$E\left[\exp\frac{d}{N}\sum_{j\neq i}L^0(X^i-X^j)_1\right]\leqq\bar{d}$$

(i is fixed in the previous expression).

For notational simplicity we will choose $i=1$ in the proof.

Let $f(y)$ denote the function $(\text{Artg } y)_+$, $f'(y)$ the function equal to 0, $y<0$, 1/2, $y=0$,

$$\frac{1}{1+y^2}, \quad y>0,$$

$g(y)$ the function

$$-\frac{(y)_+}{(1+y^2)^2}.$$

Using Tanaka's formula (see Jacod [1]).

$$\begin{aligned}&\frac{1}{N}\sum_{j\neq 1} f(X_t^1-X_t^j)-f(X_0^1-X_0^j)\\&\quad=\frac{1}{N}\sum_{j\neq 1}\int_0^t f'(X_s^1-X_s^j)\,d(B_s^1-B_s^j)+\frac{c}{N^2}\sum_{\substack{j\neq 1\\k\neq 1}}\int_0^t f'(X_s^1-X_s^j)\,dL^0(X^1-X^k)_s\\&\qquad-\frac{c}{N^2}\sum_{j\neq 1}\sum_{k\neq j}\int_0^t f'(X_s^1-X_s^j)\,dL^0(X^j-X^k)_s\\&\qquad+\frac{1}{2}\sum_{j\neq 1}L^0(X^1-X^j)_t+\frac{1}{N}\sum_{j\neq 1}\int_0^t g(X_s^1-X_s^j)\,ds.\end{aligned}$$

As a consequence we get the fact that $\exp\frac{\lambda}{N}\sum_{j\neq 1}L^0(X^1-X^j)_t+2\lambda S_t-2\lambda^2 U_t$ is a martingale (exponential martingale), if

$$\begin{aligned}S_t=&-\frac{1}{N}\sum_{j\neq 1} f(X_t^1-X_t^j)-f(X_0^1-X_0^j)+\frac{c}{N^2}\sum_{\substack{j\neq 1\\k\neq 1}}\int_0^t f'(X_s^1-X_s^j)\,dL^0(X^1-X^k)_s\\&-\frac{c}{N^2}\sum_{j\neq 1}\sum_{k\neq j}\int_0^t f'(X_s^1-X_s^j)\,dL^0(X^j-X^k)_s\\&+\frac{1}{N}\sum_{j\neq 1}\int_0^t g(X_s^1-X_s^j)\,ds,\end{aligned}\tag{3.39}$$

$$U_t=\frac{2}{N^2}\sum_{j\neq 1}\int_0^t f'^2(X_s^1-X_s^j)\,ds+\frac{1}{N^2}\sum_{\substack{j\neq k\\>1}}\int_0^t f'(X_s^1-X_s^j)\,f'(X_s^1-X_s^k)\,ds.\tag{3.40}$$

If we choose $\lambda\in]0,1]$, we have

$$\begin{aligned}&E\left[\exp\frac{\lambda}{2N}\sum_{j\neq 1}L^0(X^1-X^j)_1\right]\\&\quad=E\left[\exp\frac{\lambda}{2N}\sum_{j\neq 1}L^0(X^1-X^j)_1+\lambda S_1-\lambda^2 U_1+\lambda^2 U_1-\lambda S_1\right]\\&\quad\leqq(E[\exp 2\lambda^2 U_1-2\lambda S_1])^{1/2}\\&\qquad\text{(using Cauchy-Schwarz and the exponential martingale)}\\&\quad\leqq\text{Const.}\times\left(E\left[\exp\frac{2\lambda c}{N^2}\sum_{j\neq k}L^0(X^j-X^k)_1\right]\right)^{1/2}.\end{aligned}\tag{3.41}$$

By (3.29), we know that $E\left[\exp \frac{d}{N}\sum_1^N X_1^i - x^i\right] \leqq \bar{d}$, this implies that for possibly different d, $\bar{d}$,

$$E\left[\exp \frac{d}{N^2}\, c \sum_{i \neq j} L^0(X^i - X^j)_1\right] \leqq \bar{d}. \tag{3.42}$$

As a consequence of (3.41), (3.42) we obtain (3.37). □

We are now going to derive estimates that will be useful in order to prove tightness for certain empirical measures.

Proposition 3.7. *Let u_N be a sequence of initial law for $(X_t^1, \dots, X_t^N)$, such that $\mathbb{R}^N \setminus \bigcup_{\substack{i,j,k \\ \text{distinct}}} \{x^i = x^j = x^k\}$ is u_N negligible for every N, there exists $C > 0$ such that for $N > 2c$:*

$$\forall\, i \in [1, N] \qquad E[|X_t^i - X_s^i|^4] \leqq C\,|t-s|^2 \tag{3.43}$$

$$\forall\, i \neq j \qquad E[|L^0(X^i - X^j)_t - L^0(X^i - X^j)_s|^4] \leqq C\,|t-s|^2 \tag{3.44}$$

Moreover if the u_N are symmetric, the laws of the processes $(X_\cdot^i)_{i \in [1,N]}$ are symmetric.

Proof. The last point can be seen for instance from the fact that the $(X_\cdot^i)$ are approximated, as shown in Sznitman-Varadhan [21], by the solution of

$$dX_t^{i,\alpha} = dB_t^i + \frac{2c}{N} \sum_{j \neq i} \frac{1}{\alpha}\, \phi\left(\frac{X_t^{i,\alpha} - X_t^{j,\alpha}}{\alpha}\right) dt \tag{3.45}$$

where ϕ is a mollifier, when α goes to zero.

Formula (3.43) is an immediate consequence of the markov property of X_t, and Proposition 3.6, together with the fact that $X_t^i - X_0^i \geqq W_t^i$. We also have:

$$E\left[\left(\frac{1}{N}\sum_{j \neq i_0} L^0(X^{i_0} - X^j)_t - \frac{1}{N}\sum_{j \neq i_0} L^0(X^{i_0} - X^j)_s\right)^4\right] \leqq C\,|t-s|^2.$$

Now using Barlow-Yor's estimates [1],

$$E[(L^0(X^{i_0} - X^{j_0})_t - L^0(X^{i_0} - X^{j_0})_s^4] \leqq CE\Bigg[|B_\cdot^{i_0} - B_\cdot^{j_0} - (B_s^{i_0} - B_s^{j_0})|_t^{*4}$$
$$+ \left(\frac{1}{N}\sum_{i_1 \neq i_0} L^0(X^{i_0} - X^{i_1})_s^t\right)^4 + \left(\frac{1}{N}\sum_{j_1 \neq j_0} L^0(X^{j_0} - X^{j_1})_s^t\right)^4\Bigg] \leqq C'\,|t-s|^2.$$

(We have also used the fact that one does not need to take into account the value of the semi martingale at time zero, in Barlow-Yor's estimates, as this was explained in the proof of Proposition 2.1), so we get Proposition 3.7. □

Remark. When u_N are symmetric probabilities, the estimates (3.43), (3.44), follows from a very general result of Osada [17], concerning fundamental solutions of divergence form equations. □

Let us now consider $M(H)$, the space of probability measures on

$$H=\bar{H}\times\bar{H}\times C_0^+(\mathbb{R}_+,\mathbb{R}), \tag{3.41}$$

where C_0^+ is the set of continuous increasing functions on $\mathbb{R}_+$ with value 0 at time 0, and

$$\bar{H}=\{(X_., B_.)\in C(\mathbb{R}_+,\mathbb{R})\times C_0(\mathbb{R}_+,\mathbb{R}), X_.-X_0-B_.\in C_0^+(\mathbb{R}_+,\mathbb{R})\}. \tag{3.42}$$

We will denote the canonical coordinates on H by $(X_.^1, B_.^1, X_.^2, B_.^2, A_.)$. Clearly H is a closed set of $C^2\times C^2\times C_0^+$, endowed with the product topology.

Let us consider the law $\bar{P}_N$ of the "empirical distribution" variables:

$$\omega\in\Omega\to\bar{X}_N(\omega)=\frac{1}{N(N-1)}\sum_{i\neq j}\delta_{(X^i,\,B^i,\,X^j,\,B^j,\,L\ (X^i-X^j)_.)}\in M(H). \tag{3.43}$$

In the following, we endow $M(H)$ with the topology of weak convergence.

Remark 3.7. The laws $\bar{P}_N$ also do not depend on the particular space where the solution of (3.2) is constructed, because $L^0(X^i-X^j)_.=\lim_n 2\int_0^{\cdot}\phi_n(X_s^i-X_s^j)\,ds$, and $X_.^i$ are measurable functions of X_0^i and $B_.^i$. □

Proposition 3.8. *Suppose that the law* u_N of the initial condition (X_0^i) *are* u-*chaotic* (*see* (1.6)), *and that the set* $\bigcup_{\substack{i,j,k\\ \text{distinct}}}\{x^i=x^j=x^k\}$ *is* u_N *negligible, then the laws* $\bar{P}_N$ *are tight.*

Proof. Using the result in Sznitman [20], the tightness of $(\bar{P}_N)$ is equivalent to the tightness of the intensity of the random measure $\bar{X}_N$ (defined by $\mu(f)=E[\langle\bar{X}_N,f\rangle]$). Using the symmetry and the fact that H is closed in the ambient space $C^2\times C^2\times C_0^+$, it is sufficient to check that, the laws of the processes $(X_.^1)$ and $(L^0(X^1-X^2)_.)$ are tight, this is a consequence of the fact that that u_N is u-chaotic (so (X_0^1) are tight), and Proposition 3.6. □

IV. The Propagation of Chaos Result

Our goal in this section is to consider the laws P_N of $(X_.^1, \dots, X_.^N)$ when the laws u_N of the initial conditions (X_0^i) are u-chaotic, and show that the sequence P_N is P-chaotic, if P is the unique element of $S(u)$ ($S(u)$ is the set of the laws of the processes satisfying (2.1), (2.12)), as a result of this section we will obtain that $S(u)$ is not empty.

We denote by $(G_t)_{t\geq 0}$, the canonical filtration on the space H (defined by (3.41)), we are going to prove

Theorem 4.1. *Let u be a probability on* $\mathbb{R}$. *The set $S(u)$ has a unique element P^u. If (u_N) is u-chaotic and u_N satisfies* (3.1) *for every N, then the laws P_N of $(X_.^1,\dots,X_.^N)$ satisfying* (3.2), *with initial conditions (X_0^i), u_N-distributed, are P^u-chaotic.*

Proof. We are going to show in several steps that the sequence of random probabilities $\frac{1}{N}\Sigma \varepsilon_{X^i}$ concentrates its mass on $S(u)$, since it is always possible to construct (u_N), u-chaotic with u_N satisfying (3.1) for every N (for instance $u_N = u_{\lambda(N)}^{\otimes N}$, for a suitable smoothing sequence $u_{\lambda(N)}$ of u), we will obtain that $S(u)$ is not empty.

Consider $\bar{P}_\infty$ a limit point of the laws of the random variables $\bar{X}_N$ defined by (3.42), using Proposition 3.8. We have

Proposition 4.2. *For $\bar{P}_\infty$ a.e. $m \in M(H)$, $(X_\cdot^1, B_\cdot^1)$ and $(X_\cdot^2, B_\cdot^2)$ are m-independent, $(B_\cdot^1, B_\cdot^2)$ is a two dimensional G_t-Brownian motion and the law of X_0^1 (or X_0^2) under m is u.*

Proof. First let us show that for $\bar{P}_\infty$ a.e. m, $(X_\cdot^1, B_\cdot^1)$ and $(X_\cdot^2, B_\cdot^2)$ are m-independent. Let $f(X, B)$ and $g(X, B)$ be two continuous bounded functions on $\bar{H}$. We define:

$$F(m) = \langle m, f(X^1, B^1)\, g(X^2, B^2)\rangle - \langle m, f(X^1, B^1)\rangle \langle m, g(X^2, B^2)\rangle$$

for $m \in M(H)$. F is a continuous bounded function on $M(H)$, and it is enough to show that

$$\bar{P}_\infty \text{ a.s. } F(m) = 0. \tag{4.1}$$

We have

$$\begin{aligned} E_{\bar{P}_\infty}[F(m)^2] &= \lim_{N_k} E_{P_{N_k}}\Bigg[\bigg(\frac{1}{N(N-1)} \sum_{i \neq j} f(X^i, B^i)\, g(X^j, B^j) \\ &\quad - \frac{1}{N}\sum_i f(X^i, B^i) \times \frac{1}{N}\sum_j g(X^j, B^j)\bigg)^2\Bigg] \\ &= 0, \end{aligned}$$

and this proves (4.1).

Now let us prove that for $\bar{P}_\infty$ a.e. m, $B_t = (B_t^1, B_t^2)$ is a two dimensional G_t-Brownian motion. Let us define for $0 \leqq s < t$

$$F(m) = \left\langle m, \left[f(B_t) - f(B_s) - \frac{1}{2}\int_s^t \Delta f(B_u)\, du\right] \times H(w)\right\rangle, \tag{4.2}$$

where f is C^2 with compact support, and

$$H(w) = \prod_{p=1}^{k} g_p(X_{s_p}^1, B_{s_p}^1, X_{s_p}^2, B_{s_p}^2, A_{s_p}), \tag{4.2$'$}$$

with $0 \leqq s_1 < \ldots < s_k \leqq s < t$, and g_p bounded continuous functions on $\mathbb{R}^5$. F is continuous bounded on $M(H)$, and it is enough to show that for functions F as in (4.2),

$$E_{\bar{P}_\infty}[F(m)^2] = 0. \tag{4.3}$$

Similarly with obvious notations

$$E_{\bar{P}_\infty}[F(m)^2] = \lim_{N_k} E_{P_{N_k}}\Big[\Big(\frac{1}{N(N-1)}\sum_{i\neq j} f(B^i_t, B^j_t) - f(B^i_s, B^j_s) - \int_s^t \frac{1}{2}\Delta f(B^i_u, B^j_u)\,du\Big)^2 H^{ij}(\omega)^2\Big]$$

$$= \lim_{N_k} E_{P_{N_k}}\Big[\Big(\frac{1}{N(N-1)}\sum_{i\neq j}\int_s^t \nabla f(B^i_u, B^j_u).\, dB^{i,j}_u \times H^{ij}(\omega)\Big)^2\Big]$$

$$\leq \limsup_{k\to\infty} \frac{c}{N_k} = 0,$$

(because the F_t-Brownian motions $B^1_\cdot, \ldots, B^N_\cdot$ are independent, and the H^{ij} are F_s-adapted); this proves (4.3). Let us now prove that $\bar{P}_\infty$-a.s., the law of X^1_0 (or X^2_0) under m is u. Introduce the continuous function $h: w\in H \to X^1_0(w)\in\mathbb{R}$. For $m\in M(H)$, let $h\circ m$ be the image of m under h, $m\to h\circ m$ is a continuous map from $M(H)$ into $M(\mathbb{R})$, as such $(h\circ m)\circ \bar{P}_{N_k}$ converges weakly towards $(h\circ m)\circ \bar{P}_\infty$. But $(h\circ m)\circ \bar{P}_{N_k}$ is the law of the random measure $\frac{1}{N_k}\sum_1^{N_k} \delta_{X^i_0(\omega)}$, which converges towards δ_u since u_N is u-chaotic. So, $(h\circ m)\circ \bar{P}_\infty = \delta_u$, that is, $\bar{P}_\infty$ a.s., $h\circ m = u$; this proves Proposition 4.2. □

We denote by $A^i_\cdot$, $i=1,2$, the continuous increasing processes defined on H by $A^i_t = X^i_t - X^i_0 - B^i_t$, $t\geq 0$ (see (3.42)). The next step is to prove:

Proposition 4.3. *For $\bar{P}_\infty$ a.e. m,*

$$A^1_t = c\times E_m[A_t/\sigma(X^1_\cdot, B^1_\cdot)] \tag{4.4}$$

(and $A^2_t = cE_m[A_t/\sigma(X^2, B^2)]$),

$$H_t = |X^1_t - X^2_t| - |X^1_0 - X^2_0| - \int_0^t \operatorname{sign}^+(X^1_s - X^2_s)\, dA^1_s - \int_0^t \operatorname{sign}^+(X^2_s - X^1_s)\, dA^2_s - A_t, \tag{4.5}$$

is a continuous supermartingale, and

$$D_t = |X^1_t - X^2_t| - |X^1_0 - X^2_0| - \int_0^t \operatorname{sign}^-(X^1_s - X^2_s)\, dA^1_s - \int_0^t \operatorname{sign}^-(X^2_s - X^1_s)\, dA^2_s - A_t \tag{4.6}$$

is a continuous submartingale. (sign$^+$ *and* sign$^-$ *are respectively the right continuous and left continuous version of the function sign*).

Proof. Let us first prove (4.4), let s_p, $p\in[1,l]$ be such that $0 = s_0 < s_1 < \ldots < s_l < \infty$, and let g_{s_p}, $p\in[1,l]$, be continuous functions on $\mathbb{R}^2$, bounded by 1, we define:

$$F(m) = \Big\langle m, (cA_t - A^1_t)\times \prod_1^k g_{s_p}(X^1_{s_p}, B^1_{s_p})\Big\rangle, \tag{4.7}$$

we want to show that

$$F(m)=0, \qquad \bar{P}_{\infty} \text{ a.s.} \tag{4.7)'$$

Consider $\varepsilon>0$, fixed, introduce $F_C(m)$ which is defined in the same way as $F(m)$, except that $A_t-A_t^1$ is replaced by $A_t\wedge C-A_t^1\wedge C$, $F_C(m)$ is bounded and continuous on $M(H)$, and we have

$$\begin{aligned}&\text{For } k\leqq\infty\ E_{\bar{P}_{N_k}}[|F(m)-F_C(m)|]\\ &\leqq E_{P_{N_k}}[|\langle m,(A_t-C)_+ +(A_t^1-C)_+\rangle|]\leqq\frac{K(t)}{C},\end{aligned} \tag{4.8}$$

because of Proposition 3.6 (with the notation $\bar{P}_{N_\infty}=\bar{P}_\infty$). We can take C large enough such that this last quantity is less than ε. So:

$$\begin{aligned}E_{P_\infty}[|F(m)|]&\leqq\varepsilon+E_{P_\infty}[|F_C(m)|]=\varepsilon+\lim_{N_k}E_{\bar{P}_{N_k}}[|F_C(m)|]\\ &\leqq 2\varepsilon+c\limsup_{N_k}E_{P_{N_k}}\left[\left|\frac{1}{N(N-1)}\sum_{i\neq j}\left(L_t^0(X^i-X^j)\right.\right.\right.\\ &\quad\left.\left.\left.-\frac{1}{N}\sum_{k\neq i}L_t^0(X^i-X_t^k)\right)\times\prod_{p=1}^{l}g_{s_p}(X_{s_p}^i,B_{s_p}^i)\right|\right]\\ &=2\varepsilon+c\limsup_{N_k}E_{P_{N_k}}\left[\left|\frac{1}{N}\sum_i\left[\left(\frac{1}{(N-1)}\sum_{j\neq i}L_t^0(X^i-X^j)\right.\right.\right.\right.\\ &\quad\left.\left.\left.\left.-\frac{1}{N}\sum_{k\neq i}L_t^0(X^i-X^k)\right)\times\prod_{p=1}^{l}g_{s_p}(X_{s_p}^i,B_{s_p}^1)\right]\right|\right]\\ &\leqq 2\varepsilon.\end{aligned}$$

Since ε was arbitrary, we have proved (4.7)'. Let us show now (4.5) (the proof of (4.6) is similar). Consider $0\leqq s<t$, and $G(w)$ as in (4.2)' with the supplementary assumption that the functions g_p are positive continuous and bounded by one. Define

$$F(m)=\langle m,(H_t-H_s)\times G(w)\rangle, \tag{4.9}$$

(by Proposition 3.6 one has the fact that $E_{\bar{P}_\infty}[\langle m,|H_t|\rangle]<\infty$), we want to prove that $F(m)$ is negative $\bar{P}_\infty$ a.s. It is enough to check that for any positive bounded continuous function $K(m)$ bounded by one

$$E_{\bar{P}_\infty}[F(m)\,K(m)]\leqq 0. \tag{4.10}$$

Fix $\varepsilon>0$, and define $F_C(m)=\langle m,(H_t^C-H_s^C)\times G(w)\rangle$, where

$$\begin{aligned}H_t^C&=h_C(|X_t^1-X_t^2|-|X_0^1-X_0^2|)\\ &\quad-\int_0^t \operatorname{sign}^+(X_s^1-X_s^2)\,d(A^1\wedge C)_s\\ &\quad-\int_0^t \operatorname{sign}^+(X_s^2-X_s^1)\,d(A^2\wedge C)_s-(A\wedge C)_t,\end{aligned} \tag{4.1}$$

(where $h_C(x)=x$ when $|x|\leqq C$ and $C\times\operatorname{sign}(x)$ otherwise). We can write $\operatorname{sign}^+(x)$ as the decreasing limit

$$\operatorname{sign}^+(x)=\lim_n \phi_n(x),$$

$$\begin{aligned}\phi_n(x)&=-1, \quad \text{on }]-\infty,-1/n],\\ &=1, \quad \text{on } \mathbb{R}_+, \text{(linear in between)}.\end{aligned}$$

The map

$$w\in H \to \int_0^t \phi_n(X_s^1(w)-X_s^2(w))\, d(A^1(w)\wedge C)_s \in \mathbb{R}$$

is continuous for every n, as well as $F_C^n(m)=\langle m, G(m)\times(H_t^{C,n}-H_s^{C,n})\rangle$, where $H_t^{C,n}$ is obtained by replacing sign^+ by ϕ_n in (4.11). So we have

$$F_C(m)=\lim_n \uparrow F_C^n(m), \tag{4.12}$$

(F_C is bounded lower semicontinuous).

Moreover, by Proposition 3.6 (as in (4.8)), we can pick C large enough so that

$$\forall k\leqq\infty, \quad E_{P_{N_k}}[|F(m)-F_C(m)|]\leqq\varepsilon.$$

So we have

$$E_{P_\infty}[KF(m)]\leqq E_{P_\infty}[KF_C(m)]+\varepsilon \tag{4.13}$$

(and because of (and 4.12))

$$\begin{aligned}
&\leqq \liminf_{N_k} E_{P_{N_k}}[KF_C(m)]+\varepsilon\\
&\leqq 2\varepsilon+\liminf_{N_k} E_{P_{N_k}}[KF(m)]\\
&=2\varepsilon+\liminf_{N_k} E_{P_{N_k}}\Bigg[K(\bar X_N)\times\Bigg(\frac{1}{N(N-1)}\sum_{i\neq j}(|X_t^i-X_t^j|-|X_s^i-X_s^j|)\\
&\quad -c\int_s^t \operatorname{sign}^+(X_u^i-X_u^j)\,\frac{1}{N}\sum_{i\neq k} dL^0(X^i-X^k)\\
&\quad -c\int_s^t \operatorname{sign}^+(X_u^j-X_u^i)\times\frac{1}{N}\sum_{j\neq k} dL^0(X^j-X^k)_u - L^0(X^i-X^j)_t\Bigg)\\
&\quad \times\prod_1^l g_p(X_{s_p}^i, B_{s_p}^i, X_{s_p}^j, B_{s_p}^j, L^0(X^i-X^j)_{s_p})\Bigg].
\end{aligned}$$

We know from the construction of the process $(X_.^1,\dots,X_.^N)$, that when i,j,k are distinct, P_N-a.s. for all $t\geqq 0$, (X_t^i, X_t^j, X_t^k) is not on the diagonal $\{(x,x,x,), x\in\mathbb{R}\}$ of $\mathbb{R}^3$, so that

$$\int_0^t \operatorname{sign}^+(X_s^i-X_s^j)\,dL^0(X^i-X^k)_s=\int_0^t \operatorname{sign}(X_s^i-X_s^j)\,dL^0(X^i-X^k)_s,$$

(because $dL^0(X^i-X^k)_.$ is supported by the set $\{s\,|\,X_s^i=X_s^k\}$). Consequently, the last expression of (4.13) can be rewritten

$$2\varepsilon+\liminf_{N_k} E_{P_N}\Big[K(\bar{X}_N)\times\Big(\frac{1}{N(N-1)}\sum_{i\neq j}(|X_t^i-X_t^j|-|X_s^i-X_s^j|)$$
$$-c\int_s^t \operatorname{sign}(X_u^i-X_u^j)\frac{1}{N}\sum_{i\neq k} dL^0(X^i-X^k)_u$$
$$-c\int_s^t \operatorname{sign}(X_u^j-X_u^i)\times\frac{1}{N}\sum_{j\neq k} dL^0(X^j-X^k)_u-L^0(X^i-X^j)_t\Big)$$
$$\times\prod_1^l g_p^{ij}\Big]+0\left(\frac{1}{N}\right),$$

with obvious notations. This last expression is equal to

$$2\varepsilon+\liminf_{N_k} E_{P_{N_k}}\Big[K(\bar{X}_N)\times\frac{1}{N(N-1)}\sum_{i\neq j}\Big(\int_s^t \operatorname{sign}(X_u^i-X_u^j)\,d(B_u^i-B_u^j)\times\prod_1^l g_p^{ij}\Big)\Big], \tag{4.14}$$

and the square of the L^2-norm of $\frac{1}{N(N-1)}\sum_{i\neq j}\int_s^t \operatorname{sign}(X_u^i-X_u^j)\,d(B_u^i-B_u^j)$ is less than:

$$\frac{(t-s)}{N^2(N-1)^2}\times C\times N^3=O\left(\frac{1}{N}\right)\qquad (C \text{ independent of } N).$$

As a consequence, $E_{\bar{P}_\infty}[KF(m)]\leqq 2\varepsilon$, since ε was arbitrary, we obtain (4.10), and this proves Proposition 4.3. □

Our purpose now is to show that A_t is in fact $L^0(X^1-X^2)_t$ for $\bar{P}_\infty$ a.e. m, and then to deduce from this fact that for $\bar{P}_\infty$ a.e. m, the law of X^1 (or X^2) is in $S(u)$. We have

Proposition 4.4. *For $\bar{P}_\infty$ a.e. m, A_t is the symmetric local time in zero of X^1-X^2, and the law of X^1 and X^2 is in $S(u)$.*

Proof. From (4.6), we know that for $\bar{P}_\infty$ a.e. m:

$$D_t=|X_t^1-X_t^2|-|X_0^1-X_0^2|-\int_0^t \operatorname{sign}^+(X_s^1-X_s^2)\,dA_s^1$$
$$-\int_0^t \operatorname{sign}^-(X_s^2-X_s^1)\,dA_s^2-A_t+2\int_0^t 1(X_s^1=X_s^2)\,dA_s^1; \tag{4.15}$$

is a submartingale. By Tanaka's formula,

$$D_t=\int_0^t \operatorname{sign}^+(X_s^1-X_s^2)\,d(B^1-B^2)_s+L^{l,0}(X^1-X^2)_t-A_t+2\int_0^t 1(X_s^1=X_s^2)\,dA_s^1,$$

($L^{l,0}(X^1-X^2)_\cdot$ is the left limit of the local time of X^1-X^2 in 0). So we find that

$$A_t+K_t=L^{l,0}(X^1-X^2)_t+2\int_0^t 1(X_s^1=X_s^2)\,dA_s^1, \tag{4.16}$$

where (K_t) is a continuous increasing, integrable (for fixed finite time) process. Let us denote by $\bar{m}$ the law of (X^1, B^1) on $\bar{H}$ (see (3.42)), and let us consider $E_m[L^{l,0}(X^1-X^2)_t/(X^1, B^1)]$, since (X^1, B^1) and (X^2, B^2) are m-independent with law $\bar{m}$, by Remark 2.4, we find:

$$E_m[L^{l,0}(X^1-X^2)_t/(X^1, B^1)] = \int_0^1 2u(s, X_s^1)\, ds,$$

where $u(s,x) \in L^2([0,T]\times\mathbb{R})$ for every $T>0$, is the density of the law of X_s under $\bar{m}$ for a.e. $s\in[0,T]$. Taking conditional expectations in (4.16) with respect to the σ-field $\sigma(X^1_\cdot, B^1_\cdot)$, we find using (4.4):

$$\frac{1}{c} A_t^1 + C_t^1 = \int_0^t 2u(s, X_s^1)\, ds + 2\int_0^t p(s, X_s^1)\, dA_s^1, \tag{4.17}$$

where $p(s,x) = \int 1(X_s = x)\, dm$, and C_t^1 is an m a.s. increasing continuous process adapted to $\sigma(X_s^1, B_s^1, s\leq t)$, (it is continuous as the sum of continuous processes).

Now we can write

$$dA_t^1 = 1\left(p(s, X_s^1) < \frac{1}{4c}\right) dA_t^1 + 1\left(p(s, X_s^1) \geq \frac{1}{4c}\right) dA_t^1,$$

as a consequence of (4.17), we have the fact that

$$\left(\frac{1}{c} - 2p(s, X_s^1)\right) \times 1\left(p(s, X_s^1) < \frac{1}{4c}\right) dA_s^1 \ll ds,$$

so that $1(p(s, X_s^1) < 1/4c)\, dA_s^1 \ll ds$. Let us now study the measure $1\left(p(s, X_s^1) \geq \frac{1}{4c}\right) dA_s^1$, its support is contained in the set

$$F = \left\{t \geq 0, \exists\, x\in\mathbb{R}, p(t,x) \geq \frac{1}{4c}\right\}.$$

F is a closed set (consider t_n converging towards t, $u_{t_n}(dx)$ are tight probabilities on $\mathbb{R}$, so there exists a compact set K such that for $n\leq\infty$, $u_{t_n}(K^c) < 1/8c$, so we can choose a sequence $a_n \in K$, satisfying $p(t, a_n) \geq \frac{1}{4c}$, extracting a subsequence we can suppose that a_n converges towards a, and then $u_t(\{a\}) \geq \frac{1}{4c}$, which implies $t\in F$).

So F is a closed set of zero Lebesgue measure. We are now going to show that F can at most be the set $\{0\}$. If F is not included in $\{0\}$, then we can find an open interval $I =]a, b[$, with $I \subset F^c$, and $b\in F$, $(b<\infty)$. Since $I\subset F^c$, on I we have $dA_t^1 = 1\left(p(t, X_t^1) < \frac{1}{4c}\right) dA_t^1$, so $1_I \times dA_t^1$ is absolutely continuous with respect to Lebesgue measure.

Exploiting the fact that H_t is a supermartingale, and

$$H_t = |X_t^1 - X_t^2| - |X_0^1 - X_0^2| - \int_0^t \operatorname{sign}^+(X_s^1 - X_s^2)\, dA_s^1$$
$$- \int_0^t \operatorname{sign}^-(X_s^2 - X_s^1)\, dA_s^2 - A_t - 2\int_0^t 1(X_s^1 = X_s^2)\, dA_s^2$$
$$= \int_0^t \operatorname{sign}^+(X_s^1 - X_s^2)\, d(B_s^1 - B_s^2) + L^{l,0}(X^1 - X^2)_t - A_t - 2\int_0^t 1(X_s^1 = X_s^2)\, dA_s^2,$$

we find

$$dA_t \geqq dL^{l,0}(X^1 - X^2)_t - 2 \times 1(X_s^2 = X_s^1)\, dA_s^2.$$

restricting our attention to I, since $dA_s^2 \ll ds$ on I, we obtain with (4.16) that $dA_t = dL^{l,0}(X^1 - X^2)_t$, on I (because $1(X_s^2 = X_s^1)\, dA_s^2 = 0 = 1(X_s^2 = X_s^1)\, dA_s^1$). Pick a^1 in I, for $t \geqq a^1$, on $\bar{H}$ define $\bar{X}_t$, $0 \leqq t < b - a^1$ to be X_{t+a^1}, adapted to the filtration translated to time a^1. Moreover we have $L^{l,0}(\bar{X}^1 - \bar{X}^2)_t = L^{l,0}(X^1 - X^2)_{t+a^1} - L^{l,0}(X^1 - X^2)_{a^1}$ (now on the space $\bar{H} \times \bar{H}$),

$$\bar{X}_t = \bar{X}_0 + \bar{B}_t + c\int_{\bar{H}} L_t^{l,0}(\bar{X} - \bar{X}(w'))\, d\bar{m}(w'), \qquad t \geqq 0$$

($\bar{B}_t = B_{t+a^1} - B_{a^1}$, on $\bar{H}$). We have obtained this for any measure m in the $\bar{P}_\infty$ set of measure 1, defined by Proposition 4.2, (4.4), (4.5), (4.6).

This means that $\bar{X}_t$ satisfies (2.1), (2.12), for $t \leqq b - a^1$, by Theorem 2.5, we find

$$\exp -4c\,\bar{F}_t(x) = (\exp -4c\,\bar{F}_0) * \phi_t(x) \tag{4.18}$$

where $\bar{F}_t(t) = \int_{-\infty}^{x} u_{t+a^1}(dy)$ ($u_t(dy)$ is the law of X_t under $\bar{m}$).

As a consequence of (4.18), for t near $b - a^1$, $\exp -4c\,\bar{F}_t(x)$ is uniformly (in t) continuous in x, bounded above and below (by 1 and $\exp -4c$ respectively), so this precludes the possibility that $\bar{F}_{b-a^1}$ is non-continuous, and as a consequence $b \notin F$ this shows that F is contained in $\{0\}$. As a consequence we can hold the previous reasoning with $a = 0$, and we find that $A_t = L^{l,0}(X^1 - X^2)_t = L^0(X^1 - X^2)_t$ (since $1(X_s^2 = X_s^1)\, dA_s^2 = 0 = 1(X_s^2 = X_s^1)\, dA_s^1$) and that

$$X_t = X_0 + B_t + 2c\int_0^t u(s, X_s)\, ds, \qquad \bar{m} \text{ a.s. on } \bar{H}, \tag{4.19}$$

where $u(s, x)$ is given by (2.11) and is the density of the law of X_s for $s > 0$, and we have obtained that the law of (X_t) under $\bar{m}$ lies in $S(u)$. This, together with Theorem 2.5 proves that $S(u)$ is reduced to $\{P\}$. Formula (4.19) implies that $\bar{m}$ is the image of P by the application.

$$C \ni x_. \to \left(x_., x_. - x_0 - \int_0^{\cdot} 2c\,u(s, X_s)\, ds\right) \in \bar{H},$$

and m is the image of $\bar{m} \otimes \bar{m}$ under the application $\bar{H} \times \bar{H} \ni ((X^1, B^1), (X^2, B^2)) \to (X^1, B^1, X^2, B^2, L_.^0(X^1 - X^2))$. This shows that $\bar{P}_\infty$ is concentrated on this probability independently of the subsequence N_k, we started with, and we have proved Theorem 4.1 as well as Proposition 4.4. □

Remark 4.5. As a consequence of Theorem 4.1, we have obtained that for the reordering $(Y_t^1, \ldots, Y_t^N)$ which satisfies the oblique reflection problem (3.7), (3.8), and has initial distribution $(Y_0^1, \ldots, Y_0^N) \circ u_N$, one has the fact that $\frac{1}{N} \sum \varepsilon_{Y_t^i}$ converges to $u(t, x)\,dx$.

In the case where we have proved trajectorial uniqueness, for the nonlinear process, we will obtain a "trajectorial propagation of chaos result", namely, suppose

$$u(dx) = u(x)\,dx, \qquad u \text{ is bounded} \tag{4.20}$$

and define $\bar{X}_t^i$ satisfying

$$\bar{X}_t^i = X_0^i + B_t^i + \int_0^t + 2cu(s, \bar{X}_s^i)\,ds, \tag{4.21}$$

(using Zvonkin's result [25]), (the X_0^i are independent and $u(dx)$ distributed).

Then we have

Proposition 4.6. *Suppose the* (X_0^i) *independent,* $u(x)$*-distributed, consider the processes* $(X^i_{\bullet})$ *defined by* (3.2), *then for any* $T > 0$,

$$\sup_{t \leq T} |X_t^i - \bar{X}_t^i| \quad \text{converges in probability towards zero.} \tag{4.22}$$

Proof. The proof is similar to the proof of Theorem 4.1, one now takes the random measures $\bar{Y}_N$

$$\frac{1}{N(N-1)} \sum \varepsilon_{(X^i_{\cdot}, B^i, \bar{X}^i, X^j_{\cdot}, B^j, \bar{X}^j, L^0_{\cdot}(X^i - X^j))} \in M(\bar{H} \times C \times \bar{H} \times C \times C_0^+)$$

whose laws are tight, and one then notices that for any limit point $\bar{P}_\infty$, $\bar{P}_\infty$-a.e. m will be such that m-a.s.

$$\bar{X}_t^1 = \bar{X}_0^1 + B_t^1 + \int_0^t 2cu(s, \bar{X}_s^1)\,ds$$

(and the same for $\bar{X}_t^2$), because for $t > s > 0$,

$$\left\{ \bar{X}_t^1 - \bar{X}_s^1 = B_t^1 - B_s^1 + \int_s^t 2cu(h, \bar{X}_h^1)\,dh \right\}$$

is closed in $\bar{H} \times C \times \bar{H} \times C \times C_0^+$, and supports the random measures $\bar{Y}_N$. As a consequence of the proof of Theorem 4.1, we see that for $\bar{P}_\infty$ a.e. u, X^1 and $\bar{X}^1$ are solutions (trajectorially) of (2.1), (2.12), so that by Remark 2.6, $X^1_{\bullet} = \bar{X}^1_{\bullet}$ m a.s. (and similarly $X^2 = \bar{X}^2_{\bullet}$ m-a.s.), as a consequence, the $\bar{Y}_N$ are converging in law to the Dirac measure concentrated on the law of $(\bar{X}^1_{\cdot}, B^1_{\cdot}, \bar{X}^1_{\cdot}, \bar{X}^2_{\cdot}, B^2_{\cdot}, \bar{X}^2_{\cdot}, L^0_{\cdot}(\bar{X}^1 - \bar{X}^2))$. Formula (4.22) is then the consequence of this convergence in law result applied to the continuous bounded function: $m \to \langle m, \sup_{s \leq T} |X_s^1 - \bar{X}_s^1| \wedge 1 \rangle$. □

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