Z. Wahrscheinlichkeitstheorie verw. Geb. 24, 339–340 (1972)

Existence of Most Powerful Tests for Undominated Hypotheses

Dieter Landers and Lothar Rogge

It is well known that for a dominated composite hypothesis against a simple alternative there exist most powerful level α-tests (see for instance Hájek [1], Schmetterer [3], Witting [4]). It is the purpose of this paper to show that the assumption of domination can be dispensed with.

Let $Q|\mathscr{A}$ be a probability measure (p-measure) and $\mathscr{L}_1(Q)$ be the system of all $\mathscr{A}$-measurable Q-integrable functions, endowed with the vector topology of convergence in the first mean, called Q-strong topology. We remark that in the corresponding weak topology—for short: Q-weak topology—a net f_α, $\alpha \in D$, in $\mathscr{L}_1(Q)$ converges to a function $f \in \mathscr{L}_1(Q)$ if and only if $\lim_{\alpha \in D} Q(f_\alpha g) = Q(fg)$ for every bounded $\mathscr{A}$-measurable function g.

Theorem. *Let $\mathfrak{P}|\mathscr{A}$ be an arbitrary family of p-measures and $Q|\mathscr{A}$ be a single p-measure. Then for every $\alpha \in [0, 1]$ there exists a most powerful test of level α for the composite hypothesis $\mathfrak{P}|\mathscr{A}$ against the simple alternative $Q|\mathscr{A}$.*

Proof. Let Φ be the family of all $\mathscr{A}$-measurable functions with values in $[0, 1]$. Let $\Phi_\alpha := \{\varphi \in \Phi: P(\varphi) \leqq \alpha \text{ for all } P \in \mathfrak{P}\}$ and $\beta := \sup\{Q(\varphi): \varphi \in \Phi_\alpha\}$. Then for each $n \in \mathbb{N}$ there exists $\varphi_n \in \Phi_\alpha$ with $Q(\varphi_n) \geqq \beta - \frac{1}{n}$. Since Φ is Q-weakly sequentially compact (see [4], p. 64) there exist a subsequence $\mathbb{N}_0 \subset \mathbb{N}$ and $\varphi_0 \in \Phi$ such that $(\varphi_n)_{n \in \mathbb{N}_0}$ converges Q-weakly to φ_0. Since closures of convex sets are convex and Φ_α is convex, the Q-strong closure of Φ_α and the Q-weak closure of Φ_α coincide (see [2], p. 154). As φ_0 is in the weak closure of Φ_α, φ_0 is in the strong closure of Φ_α. Hence there exist $\psi_n \in \Phi_\alpha$, $n \in \mathbb{N}$, converging Q-strongly to φ_0, whence there exists a subsequence ψ_n, $n \in \mathbb{N}_1$, converging pointwise to φ_0 except on some Q-null set $N \in \mathscr{A}$. Let $\psi_0 = \varphi_0 1_{\bar{N}}$. Since $P(\psi_n 1_{\bar{N}}) \leqq P(\psi_n) \leqq \alpha$ for all $n \in \mathbb{N}_1$, $P \in \mathfrak{P}$, and $\psi_n 1_{\bar{N}}$, $n \in \mathbb{N}_1$, converges pointwise to ψ_0, we obtain $P(\psi_0) \leqq \alpha$ for all $P \in \mathfrak{P}$. As $\varphi_0 = \psi_0$ Q-a.e. and $Q(\varphi_0) = \lim_{n \in \mathbb{N}_0} Q(\varphi_n) = \beta$, we obtain $Q(\psi_0) = \beta$. Consequently ψ_0 is a most powerful test of level α for $\mathfrak{P}|\mathscr{A}$ against $Q|\mathscr{A}$.

References

1. Hájek, J.: Nonparametric Statistics. San Francisco-Cambridge-London: Holden-Day 1969.
2. Kelley, J.L. and Namioka, I.: Linear Topological Spaces. Princeton-New York-London: D. Van Nostrand 1963.

3. Schmetterer, L.: Einführung in die Mathematische Statistik. Wien-New York: Springer 1966.
4. Witting, H.: Mathematische Statistik. Stuttgart: B. G. Teubner 1966.

Dieter Landers
Lothar Rogge
Mathematisches Institut der Universität
D-5000 Köln-Lindenthal 41
Weyertal 86–90
Federal Republic of Germany

(Received September 20, 1972)