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Approximations of Positive Contractions on $L^\infty[0, 1]$ *

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1. Introduction and Results

This note is a continuation of [5]. We follow the notation and terminology of [5]. Let $(X, \mathscr{F}, \mu)$ denote the measure space consisting of the closed unit interval with Lebesgue measure. We shall abbreviate $(X, \mathscr{F}, \mu)$ by (X, μ) or X. Let $[L^\infty, L^p]$, $p=1$ or ∞, be the vector space of bounded linear operators from $L^\infty(X)$ into $L^p(X)$. We write $[L^p]$ for $[L^p, L^p]$. Let $\mathscr{C}$ be the set of positive operators $P \in [L^\infty]$ such that $P1=1$. Let $\mathscr{M}$ be the set of Markov operators on $L^\infty(X)$. Then $\mathscr{M} \subset \mathscr{C} \subset [L^\infty] \subset [L^\infty, L^1]$. Let Ψ be the set of Markov operators T_ϕ that are induced by nonsingular measurable point maps ϕ on X as $T_\phi f(x) = f(\phi(x))$. Let Ψ_i be the set of operators $T_\phi \in \Psi$ that are induced by injective maps ϕ. In the present note, since Theorems 1 and 2 of [5] are also valid for positive contractions P in $\mathscr{C}$, we prove the following sharper forms of approximation theorems.

Theorem 1. *$\mathscr{C}$ is a compact convex set and it is the closure of Ψ_i in the weak* operator topology of $[L^\infty]$.*

Theorem 2. *$\mathscr{C}$ is the closed convex hull of Ψ_i in the strong operator topology of $[L^\infty, L^1]$.*

Theorem 3. *$\mathscr{C}$ is the closed convex hull of Ψ in the strong operator topology of $[L^\infty]$.*

Theorem 4. *$\mathscr{M}$ is the sequential closure of Ψ_i in the* weak* *operator topology of $[L^\infty]$, and it is the sequential closure of the convex hull of Ψ_i in the strong operator topology of $[L^\infty, L^1]$.*

Note that if $P \in \mathscr{C}$ has a kernel in $L^1(X \times X)$, then it is necessarily a Markov operator. Thus, Theorems 3 and 4 of [5] do not extend to operators in $\mathscr{C}$.

2. Proofs

Proof of Theorem 1. An elementary argument shows that $\mathscr{S}_+ = \{P \in [L^\infty]: P \geqq 0 \text{ and } P1 \leqq 1\}$ is compact in the weak* operator topology. It is also easy to see that $\mathscr{C}$ is closed in $\mathscr{S}_+$ in this topology. Since Lemma 1 and Theorem 1 of [5] hold also for each $P \in \mathscr{C}$, we have $\mathscr{C} \subset$ w*-cl Ψ_i, the closure of Ψ_i in the weak* operator topology. By the compactness of $\mathscr{C}$, it follows $\mathscr{C} =$ w*-cl Ψ_i.

Proof of Theorem 2. On the set $\mathscr{C}$, the induced weak operator topology of $[L^\infty, L^1]$ is a Hausdorff topology and it is weaker than the induced weak* operator topology of $[L^\infty]$. Since $\mathscr{C}$ with the induced weak* operator topology is a compact

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Hausdorff space, the weak* operator topology of $[L^\infty]$ and the weak operator topology of $[L^\infty, L^1]$ coincide on $\mathscr{C}$. On the other hand, since a convex set in $[L^\infty, L^1]$ has the same closure in the weak operator topology and the strong operator topology [2, Corollary, p. 477], we see easily that $\mathscr{C}$ is the closed convex hull of Ψ_i in the strong operator topology of $[L^\infty, L^1]$.

Proof of Theorem 3. Note that Lemma 2 and Theorem 2 of [5] hold also for operators P in $\mathscr{C}$. Thus, $\mathscr{C} \subset$ s-cch Ψ, the closed convex hull of Ψ in the strong operator topology of $[L^\infty]$. Since $\mathscr{C}$ is also closed in the strong operator topology of $[L^\infty]$, we have at once $\mathscr{C} =$ s-cch Ψ.

Since $\mathscr{M} \neq \mathscr{C}$, it follows easily from Theorems 1 and 2 of [5] and Theorems 1 through 3 of the present paper that $\mathscr{M}$ is neither closed in the weak* operator topology of $[L^\infty]$ nor in the strong operator topology of $[L^\infty]$ or $[L^\infty, L^1]$, respectively.

Suppose $\mathscr{C}$ is endowed with a topology τ. Let $A \subset \mathscr{C}$. Following Granirer [3] we denote the τ-sequential closure of A in $\mathscr{C}$ by τ-seq. cl. A. Observe that $b \in \tau$-seq. cl. A if and only if $b = \tau$-lim a_n for some sequence $a_n \in A$. w*-seq. cl. A and s'-seq. cl. A denote the sequential closures of A in the induced weak* operator topology of $[L^\infty]$ and in the induced strong operator topology of $[L^\infty, L^1]$.

Proof of Theorem 4. It is easy to show that

$$\text{w*-seq. cl. } \Psi_i \subset \text{w*-seq. cl. } \mathscr{M} = \mathscr{M}$$

and

$$s'\text{-seq. cl.} (\text{ch } \Psi_i) \subset s'\text{-seq. cl. } \mathscr{M} = \mathscr{M}.$$

To complete the proof we use an argument given in the Introduction of [5]. Given P in $\mathscr{M}$, let $T \in [L^1]$ be such that $T^* = P$. Such an operator T exists uniquely. Let $Y = \{x\colon T1(x) > 0\}$ and let $d\mu' = T1\, d\mu$. The measure space $(Y, \mathscr{F}, \mu')$ will be denoted by (Y, μ'). Define the operator $T'\colon L^1(X, \mu) \to L^1(Y, \mu')$ by the formula $T'f = Tf/T1$ on Y. The adjoint P' of T' which maps $L^\infty(Y, \mu')$ into $L^\infty(X, \mu)$ satisfies the equality $P'g' = Pg$ provided that $g' \in L^\infty(Y, \mu')$, $g \in L^\infty(X, \mu)$, and $g' = g$ on Y. Then there exists an invertible measure preserving point map ξ from (X, μ) onto (Y, μ'). Let T_η be the operator induced by the map $\eta = \xi^{-1}$. If we put $\hat{P} = P' T_\eta$, then $\hat{P}\colon L^\infty(X, \mu) \to L^\infty(X, \mu)$ is a doubly stochastic operator. By the weak approximation theorem of J.R. Brown [1, p. 19], [4, p. 521], there exists a sequence $\{\theta_n\}$ of invertible measure preserving maps from (X, μ) onto itself such that T_{θ_n} converges to $\hat{P}$ in the weak* operator topology for $[L^\infty]$. Note that both μ and μ' are equivalent on the set Y and thus the map ξ is a nonsingular injection from (X, μ) into itself, that is, $T_\xi \in \Psi_i$. Thus, if we put $\phi_n = \xi \circ \theta_n$, then $T_{\phi_n} \in \Psi_i$. A simple calculation yields

$$|\langle f, (T_{\phi_n} - P)\, g\rangle| \to 0, \quad f \in L^1(X, \mu), \quad g \in L^\infty(X, \mu),$$

as $n \to \infty$. This proves $\mathscr{M} \subset$ w*-seq. cl. Ψ_i. Similarly, we prove $\mathscr{M} \subset s'$-seq. cl. (ch Ψ_i) by using the strong approximation theorem of Brown [1, p. 21], [4, p. 522].

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