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An Ergodic Theorem for Interacting Systems with Attractive Interactions

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1. Introduction

Systems of infinitely many interacting Markov processes have recently received a lot of attention (see [1, 4, 7, 9]). With the exception of Dobrušin [1], no really satisfactory ergodic theorems have been proved for these systems, and even in [1] there is a strong restriction on the strength of the interaction. In this paper we replace the restriction on the strength of the interaction with the requirement that the interaction be in some sense attractive.

In order to define what is meant by an attractive interaction we need the following setup. Let I be a countable set and for each $i \in I$ let S_i be a finite set of real numbers. Each of the S_i will be the state space of one of our interacting Markov processes. $E = \prod_{i \in I} S_i$ will be the space of configurations. Thus if $\eta \in E$, the interpretation is that $\eta(i)$ is the state of the Markov process on the state space S_i. For each $\eta \in E$, $i \in I$, and $a \in S_i$ let $c(i, \eta, a) \geqq 0$ be the rate at which the particle on state space S_i goes from $\eta(i)$ to a when the entire configuration is η. To make this precise we let S_i have the discrete topology and give E the resulting product topology. Let $\mathscr{C}(E)$ be the Banach space of continuous functions on E with the uniform norm. Let $\mathscr{D} \subset \mathscr{C}(E)$ be the set of functions that only depend on finitely many coordinates. For $f \in \mathscr{D}$ let

$$\Omega f(\eta) = \sum_{i \in I} \sum_{a \in S_i} c(i, \eta, a)[f({}_i^a\eta) - f(\eta)], \tag{1.1}$$

where

$${}_i^a\eta(j) = \begin{cases} \eta(j) & \text{if } j \neq i \\ a & \text{if } j = i. \end{cases}$$

Under suitable conditions on $c(\cdot, \cdot, \cdot)$ (see Liggett [7]), the closure of the operator Ω is the infinitesimal generator of a standard Markov process η_t. Intuitively (1.1) says that if at time zero the configuration is η, then for small times t the probability that at time t the process at site i is in state $a(\neq \eta(i))$ is $c(i, \eta, a)\, t + o(t)$.

Note that the value of $c(i, \eta, \eta(i))$ has no effect on (1.1). Thus we can change the value of $c(i, \eta, \eta(i))$ without changing either the infinitesimal generator or its semigroup. We say that the interaction $c(\cdot, \cdot, \cdot)$ is *attractive* if $c(i, \eta, \eta(i))$ can be chosen in such a way that if $\eta, \varphi \in E$ are such that $\eta(i) \geqq \varphi(i)$ for all $i \in I$, then for all $j \in I$ and all $a \in S_j$

$$\begin{aligned} &\sum_{\substack{\alpha \leqq a \\ \alpha \in S_j}} c(j, \varphi, \alpha) \geqq \sum_{\substack{\alpha \leqq a \\ \alpha \in S_j}} c(j, \eta, \alpha), \quad \text{and} \\ &\sum_{\alpha \in S_j} c(j, \varphi, \alpha) = K(j), \end{aligned} \tag{1.2}$$

where $K(j)$ is a function which is independent of φ. If $c(\cdot,\cdot,\cdot)$ is attractive, we will assume that $c(i,\eta,\eta(i))$ has been chosen so that (1.2) is satisfied.

Since the actual choice of $c(i,\eta,\eta(i))$ is irrelevant, it seems that it should be possible to define attractive only in terms of the $c(i,\eta,a)$ for $a\neq\eta(i)$. This is in fact the case. It is easily seen that the following definition is equivalent to our original one. $c(\cdot,\cdot,\cdot)$ is attractive if conditions (i) and (ii) below hold.

(i) For all $j\in I$, $\sup_{\eta\in E} \sum_{\substack{\alpha\neq\eta(j)\\ \alpha\in S_j}} c(j,\eta,\alpha)<\infty$.

(ii) If $\eta,\varphi\in E$ are such that $\eta(i)\geqq\varphi(i)$ for all $i\in I$, then for all $j\in I$ and all $a, b\in S_j$ with $a<\varphi(j)$ and $b>\eta(j)$

$$\sum_{\substack{\alpha\leqq a\\ \alpha\in S_j}} c(j,\varphi,\alpha)\geqq\sum_{\substack{\alpha\leqq a\\ \alpha\in S_j}} c(j,\eta,\alpha)$$

and

$$\sum_{\substack{\beta\geqq b\\ \beta\in S_j}} c(j,\eta,\beta)\geqq\sum_{\substack{\beta\geqq b\\ \beta\in S_j}} c(j,\varphi,\beta).$$

The reason for the term attractive will become more obvious after reading the example in Section 4.

We are now in a position to state one of our main results.

(1.3) **Theorem.** *If the interaction $c(\cdot,\cdot,\cdot)$ is attractive and the Markov process η_t has only one stationary distribution, μ, then for all $f\in\mathscr{C}(E)$ and all $\eta\in E$,*

$$\lim_{t\to\infty} T_t f(\eta)=\int_E f(\xi)\,\mu(d\xi).$$

Here T_t is the semigroup whose infinitesimal generator is determined by (1.1).

Section 2 is devoted to a proof of this theorem. Section 3 is concerned with the question of uniqueness of the stationary distribution. This is a difficult problem and an important one in statistical mechanics. We give necessary and sufficient conditions for the stationary distribution to be unique in the case when the interaction is attractive. These conditions are hard to check, but they can be verified in some instances. Section 4 contains an example of one of these instances. This example also shows the difficulties that can arise when the stationary distribution is not unique.

This paper resulted from an attempt to understand Dobrušin's work [1], and the reader familiar with that work will recognize many of the techniques used here. For the sake of the exposition we have restricted ourselves to state spaces S_i which are finite subsets of the real numbers. It will be obvious that our results could be generalized to compact subsets of real numbers with only slight technical complications in the proofs.

2. The Ergodic Theorem

The essential idea is to compare the infinite system with a finite one. In order to do this we must first recall the construction used to prove the existence of the Markov process η_t (see [7]). Let $\{I_n\}$ be a sequence of finite subsets of I with

$I_n \subset I_{n+1}$ and $\bigcup_n I_n = I$. For each n define an operator Ω_n on $\mathscr{C}(E)$ by the formula

$$\Omega_n f(\eta) = \sum_{i \in I_n} \sum_{a \in S_i} c(i, \eta, a) [f({}_i^a\eta) - f(\eta)]. \tag{2.1}$$

Let $T_t^{(n)}$ be the semigroup $\exp\{t\Omega_n\}$. Then under suitable conditions on $c(\cdot, \cdot, \cdot)$ (see [7]) it can be proven that there exists a semigroup T_t such that for all $f \in \mathscr{C}(E)$ and all $t > 0$

$$\lim_{n \to \infty} \sup_{0 \le s \le t} \| T_s^{(n)} f - T_s f \| = 0. \tag{2.2}$$

T_t is the semigroup of the Markov process η_t. That is, its infinitesimal generator when restricted to $\mathscr{D}$ is given by (1.1).

(2.3) *Remark.* We will have several occasions to claim the existence of a positive contraction semigroup having a generator determined by expressions like (1.1) and obeying relations such as (2.1) and (2.2). Clearly some conditions on the c's are necessary for this to be true. Very general conditions for (1.1), (2.1) and (2.2) to hold have been determined by Liggett [7]. Rather than check Liggett's conditions every time we need results such as (1.1), (2.1) or (2.2), we will simply assume them to be true. Thus an unstated but implicit hypothesis for all of our theorems is that whenever expressions such as (1.1), (2.1) and (2.2) are needed, they are true. In specific examples this is usually easy to check using the results in [7].

We can always construct at least one stationary distribution for the Markov process η_t by using (2.2). This construction will be important for us so we give it in some detail.

For each n let

$$E_n = \prod_{i \in I_n} S_i \quad \text{and} \quad \bar{E}_n = \prod_{i \in I \setminus I_n} S_i.$$

Note that E_n has only finitely many elements. For each n we can write $E = E_n \times \bar{E}_n$, and if $\psi \in E_n$ and $\varphi \in \bar{E}_n$, we will write $[\psi, \varphi]$ for that element of E which is equal to ψ on I_n and equal to φ on $I \setminus I_n$. For each $\varphi \in \bar{E}_n$ we define an operator $\Omega_{n\varphi}$ on the Banach space, $\mathscr{C}(E_n)$, of functions on E_n by the formula

$$\Omega_{n\varphi} \bar{f}(\psi) = \Omega_n f([\psi, \varphi]),$$

where $\bar{f} \in \mathscr{C}(E_n)$ and f is the element of $\mathscr{C}(E)$ defined by $f([\psi, \varphi]) = \bar{f}(\psi)$. $\Omega_{n\varphi}$ is the infinitesimal generator of a Markov process on E_n, and since E_n is a finite set, there is at least one stationary distribution, $\bar{\mu}_{n\varphi}$, for the Markov process.

E_n is isomorphic to $E_n \times \{\varphi\} \subset E$, and by means of the isomorphism we may identify $\bar{\mu}_{n\varphi}$ with a measure $\mu_{n\varphi}$ on E. Now for each n we pick an element $\varphi_n \in \bar{E}_n$ and form the measure $\mu_{n\varphi_n}$ on E. E is a compact metric space, and therefore there is a probability measure μ and a subsequence n' such that $\mu_{n'\varphi_{n'}}$ converges weakly to μ. It is not difficult to show using (2.2) (see [5]) that μ is a stationary measure for the Markov process, η_t, whose generator is determined by (1.1). This construction of a stationary measure for η_t will be called the standard construction, and the sequence $\{\mu_{n\varphi_n}\}$ will be called a standard sequence.

There are two choices of the configurations φ_n which play an important role. They are given by:

$$\begin{aligned}\varphi_{nu}(i)&=\max\{a\in S_i\}, \quad i\in I\setminus I_n, \quad \text{and}\\ \varphi_{nd}(i)&=\min\{a\in S_i\}, \quad i\in I\setminus I_n.\end{aligned} \tag{2.4}$$

For each n let ψ_{nu} and ψ_{nd} be the elements in E_n defined by

$$\begin{aligned}\psi_{nu}(i)&=\max\{a\in S_i\}, \quad i\in I_n\\ \psi_{nd}(i)&=\min\{a\in S_i\}, \quad i\in I_n,\end{aligned}$$

and let

$$\begin{aligned}\bar{\mu}_{nu}(\{\psi\})&=\lim_{t\to\infty}\exp\{t\,\Omega_{n\varphi_{nu}}\}\,\chi_{\{\psi\}}(\psi_{nu}), \quad \text{and}\\ \bar{\mu}_{nd}(\{\psi\})&=\lim_{t\to\infty}\exp\{t\,\Omega_{n\varphi_{nd}}\}\,\chi_{\{\psi\}}(\psi_{nd}).\end{aligned} \tag{2.5}$$

Here $\chi_{\{\psi\}}$ is the indicator function of $\{\psi\}\subset E_n$. The existence of the limits in (2.5) is guaranteed by the ergodic theorem for Markov processes with a finite state space.

Let μ_u (μ_d) be a stationary measure constructed by the standard method from the sequence $\{\mu_{nu}\}$ ($\{\mu_{nd}\}$). (It will soon be obvious that $\{\mu_{nu}\}$ converges weakly to μ_u and that it is not necessary to pass to a subsequence; however, we will not need this fact.)

(2.6) *Remark.* In order to prove statements such as

$$\lim_{t\to\infty} T_t f(\eta)=\int f(\xi)\,\mu(d\xi) \qquad \text{for all } f\in\mathscr{C}(E) \tag{2.7}$$

it suffices to prove (2.7) only for f's of the form

$$F_{i_1,\ldots,i_n,a_1,\ldots,a_n}(\eta)=\begin{cases}1 & \text{if } \eta(i_j)\leqq a_j, \quad j=1,\ldots,n\\ 0 & \text{otherwise.}\end{cases}$$

This is because convergence of the finite dimensional distributions is enough to imply weak convergence of measures on E.

(2.8) **Theorem.** *If the interaction $c(\cdot,\cdot,\cdot)$ is attractive (and see Remark (2.3)), then for all $\eta\in E$ and all $F_{i_1,\ldots,i_n,\ldots,a_n}$*

$$\begin{aligned}\int_E F_{i_1,\ldots,i_n,a_1,\ldots,a_n}(\xi)\,\mu_u(d\xi)&\leqq\liminf_{t\to\infty} T_t F_{i_1,\ldots,i_n,a_1,\ldots,a_n}(\eta)\\ &\leqq\limsup_{t\to\infty} T_t F_{i_1,\ldots,i_n,a_1,\ldots,a_n}(\eta)\\ &\leqq\int_E F_{i_1,\ldots,i_n,a_1,\ldots,a_n}(\xi)\,\mu_d(d\xi).\end{aligned} \tag{2.9}$$

Proof. Let $F_{i_1,\ldots,i_n,a_1,\ldots,a_n}=F$ be fixed and let $\varepsilon>0$ be given. We prove that

$$\int F(\xi)\,\mu_u(d\xi)-\varepsilon\leqq\liminf_{t\to\infty} T_t F(\eta). \tag{2.10}$$

The proof for the corresponding inequality involving the limit superior is similar.

Since μ_u is the weak limit of a subsequence of the sequence $\{\mu_{nu}\}$, there is an N such that

$$\int F(\xi)\,\mu_u(d\xi)-\varepsilon \leqq \int F(\xi)\,\mu_{Nu}(d\xi). \tag{2.11}$$

Thus to prove (2.10) it will suffice to prove the first inequality in (2.9) with μ_u replaced by μ_{Nu}. This is done by comparing the Markov process with state space E_N and infinitesimal generator $\Omega_{N\varphi_{Nu}}$ with the Markov process having state space E and infinitesimal generator Ω. We do this by coupling the two processes together. Toward this end we define

$$\tilde{S}_i=\begin{cases} S_i\times S_i & \text{if } i\in I_N,\\ \{\max\{a\in S_i\}\}\times S_i & \text{if } i\in I\setminus I_N,\end{cases}$$

and

$$\tilde{E}=\prod_{i\in I}\tilde{S}_i.$$

$\tilde{E}$ will be the state space of the coupled processes. We write the elements of $\tilde{E}$ as ordered pairs, (η_1,η_2), where $\eta_1=[\psi,\varphi_{Nu}]$ with $\psi\in E_N$, and $\eta_2\in E$. Since the first coordinate of an element of $\tilde{E}$ is uniquely determined by an element of E_N, it will be convenient to think of η_1 as both an element of E_N and an element of E which is equal to φ_{Nu} outside of I_N. The subscript 1 or 2 in η_1 or η_2 always refers to the first or second coordinate of an ordered pair and never refers to the configuration of the Markov process η_t at time 1 or 2.

In order to define the infinitesimal generator of the coupled processes we let $d(i;(\eta_1,\eta_2);(a,b))$ be defined for $i\in I$, $(\eta_1,\eta_2)\in\tilde{E}$ and $(a,b)\in\tilde{S}_i$ by the formula

$$d(i;(\eta_1,\eta_2);(a,b))=\begin{cases} c(i,\eta_2,b) \quad \text{if } i\notin I_N\\ \max\Big[0,\Big\{\min\Big(\sum\limits_{\substack{\alpha\leqq a\\ \alpha\in S_i}} c(i,\eta_1,\alpha),\ \sum\limits_{\substack{\beta\leqq b\\ \beta\in S_i}} c(i,\eta_2,\beta)\Big)\\ \qquad -\max\Big(\sum\limits_{\substack{\alpha< a\\ \alpha\in S_i}} c(i,\eta_1,\alpha),\ \sum\limits_{\substack{\beta< b\\ \beta\in S_i}} c(i,\eta_2,\beta)\Big)\Big\}\Big] \quad \text{if } i\in I_N.\end{cases} \tag{2.12}$$

For each n we define an operator $\mathscr{A}_n$ on $\mathscr{C}(\tilde{E})$ given by the equation

$$\mathscr{A}_n f(\eta_1,\eta_2)=\sum_{i\in I_n}\ \sum_{(a,b)\in\tilde{S}_i} d(i;(\eta_1,\eta_2);(a,b))\,[f({}_i^a\eta_1,{}_i^b\eta_2)-f(\eta_1,\eta_2)].$$

The following facts about $\mathscr{A}_n$ are easy but tedious to verify.

(2.13) If $f\in\mathscr{C}(\tilde{E})$ depends only on η_1 (i.e. there is a $g\in\mathscr{C}(E_N)$ such that $f(\eta_1,\eta_2)=g(\eta_1)$), then for all $n\geqq N$

$$\mathscr{A}_n f(\eta_1,\eta_2)=\Omega_{N\varphi_{Nu}}\, g(\eta_1),$$

and $\mathscr{A}_n f$ again depends only on η_1.

(2.14) If $f\in\mathscr{C}(\tilde{E})$ depends only on η_2 (i.e. there is an $h\in\mathscr{C}(E)$ such that $f(\eta_1,\eta_2)=h(\eta_2)$), then

$$\mathscr{A}_n f(\eta_1,\eta_2)=\Omega_n h(\eta_2),$$

and $\mathscr{A}_n f$ again depends only on η_2.

(2.15) If $(\eta_1, \eta_2) \in \tilde{E}$ is such that $\eta_1(i) \geqq \eta_2(i)$ for all $i \in I_N$, then $d(i; (\eta_1, \eta_2); (a, b)) = 0$ unless $a \geqq b$.

In checking (2.13) and (2.14) one needs the second part of (1.2), and in checking (2.15) the first part of (1.2) is used. This is the only place that we need the hypothesis that $c(\cdot, \cdot, \cdot)$ is attractive.

Now let $\tilde{\mathscr{D}}$ be the functions in $\mathscr{C}(\tilde{E})$ which only depend on finitely many coordinates, and for $f \in \tilde{\mathscr{D}}$ let

$$\mathscr{A} f(\eta_1, \eta_2) = \sum_{i \in I} \sum_{(a,b) \in S_i} d(i; (\eta_1, \eta_2); (a, b)) [f({}_i^a\eta_1, {}_i^b\eta_2) - f(\eta_1, \eta_2)].$$

Then (see Remark (2.3)) the closure of $\mathscr{A}$ is the infinitesimal generator of a semigroup, $\tilde{T}_t$, $\mathscr{A}_n$ is the infinitesimal generator of a semigroup, $\tilde{T}_t^{(n)}$, and for all $f \in \mathscr{C}(\tilde{E})$ and all $t > 0$

$$\lim_{n \to \infty} \sup_{0 \leqq s \leqq t} \| \tilde{T}_s^{(n)} f - \tilde{T}_s f \| = 0. \tag{2.16}$$

Let U_t be the semigroup of operators on $\mathscr{C}(E_N)$ with infinitesimal generator $\Omega_{N\varphi_N u}$, and let $T_t^{(n)}$ be as in (2.1). Then if $f_1 \in \mathscr{C}(\tilde{E})$ is defined by

$$f_1(\eta_1, \eta_2) = F(\eta_1)$$

(recall that F was fixed at the beginning of the proof) it follows from (2.13) that for all $n \geqq N$ and all η_2

$$U_t F(\eta_1) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \Omega_{N\varphi_N u}^k F(\eta_1) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \mathscr{A}_n^k f_1(\eta_1, \eta_2) = \tilde{T}_t^{(n)} f_1(\eta_1, \eta_2). \tag{2.17}$$

Similarly if $f_2(\eta_1, \eta_2) = F(\eta_2)$, then from (2.14) it follows that for all η_1

$$T_t^{(n)} F(\eta_2) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \Omega_n^k F(\eta_2) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \mathscr{A}_n^k f_2(\eta_1, \eta_2) = \tilde{T}_t^{(n)} f_2(\eta_1, \eta_2). \tag{2.18}$$

Combining (2.2), (2.16), (2.17), and (2.18) we see that

$$\tilde{T}_t f_1(\eta_1, \eta_2) = U_t F(\eta_1) \quad \text{for all } t \geqq 0 \text{ and all } \eta_2 \in E; \quad \text{and} \tag{2.19}$$

$$\tilde{T}_t f_2(\eta_1, \eta_2) = T_t F(\eta_2) \quad \text{for all } t \geqq 0 \text{ and all } \eta_1 \in E. \tag{2.20}$$

In order to take advantage of (2.15) we let

$$A = \{(\eta_1, \eta_2) \in \tilde{E} : \eta_1(i) \geqq \eta_2(i) \text{ for all } i \in I\}$$

and

$$\mathscr{B} = \{f \in \mathscr{C}(\tilde{E}) : f(\eta_1, \eta_2) = 0 \text{ for all } (\eta_1, \eta_2) \in A\}.$$

By using (2.15) and induction on k it is easily seen that if $f \in \mathscr{B}$, then $\mathscr{A}_n^k f \in \mathscr{B}$ for all n and all k. Thus it follows from (2.16) that if $f \in \mathscr{B}$ then $\tilde{T}_t f \in \mathscr{B}$ for all $t \geqq 0$.

We now define $f_3 \in \mathscr{B}$ by the formula

$$f_3(\eta_1, \eta_2) = \begin{cases} 1 & \text{if } \eta_1(i) < \eta_2(i) \quad \text{for some } i = i_1, \ldots, i_n \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$f_1(\eta_1, \eta_2) \leqq f_2(\eta_1, \eta_2) + f_3(\eta_1, \eta_2). \tag{2.21}$$

Since $f_3 \in \mathscr{B}$, the inequality (2.21) implies that for all $(\eta_1, \eta_2) \in A$ and all $t \geqq 0$

$$\tilde{T}_t f_1(\eta_1, \eta_2) \leqq \tilde{T}_t f_2(\eta_1, \eta_2). \tag{2.22}$$

Finally we notice that if η is an arbitrary element of E then $([\psi_{Nu}, \varphi_{Nu}], \eta) \in A$. Thus

$$\begin{aligned} \liminf_{t\to\infty} U_t F(\psi_{Nu}) &= \liminf_{t\to\infty} \tilde{T}_t f_1([\psi_{Nu}, \varphi_{Nu}], \eta) \\ &\leqq \liminf_{t\to\infty} \tilde{T}_t f_2([\psi_{Nu}, \varphi_{Nu}], \eta) = \liminf_{t\to\infty} T_t F(\eta). \end{aligned} \tag{2.23}$$

From (2.5) we know that

$$\int F(\zeta)\, \mu_{Nu}(d\zeta) = \lim_{t\to\infty} U_t F(\psi_{Nu}). \tag{2.24}$$

Combining (2.23) and (2.24) we get the first inequality in (2.9) with μ_u replaced by μ_{Nu}, and the proof is complete.

The next corollary is Theorem (1.3) in the introduction.

(2.25) **Corollary.** *If $c(\cdot,\cdot,\cdot)$ is an attractive interaction and the standard construction applied to any sequence $\{\mu_{n\varphi_n}\}$ always yields the same measure, μ, then for all $f \in \mathscr{C}(E)$ and all $\eta \in E$*

$$\lim_{t\to\infty} T_t f(\eta) = \int f(\xi)\, \mu(d\xi).$$

Proof. This follows immediately from Remark (2.6) and Theorem (2.8).

3. Uniqueness of the Stationary Distribution

In this section we want to find a necessary and sufficient condition for the Markov process η_t to have a unique stationary distribution. One necessary condition is obvious. That is, for any two standard sequences $\{\mu_{n\varphi_n}\}$, $\{\nu_{n\varphi'_n}\}$ and any $i \in I$, $a \in S_i$ we must have

$$\lim_{n\to\infty} \mu_{n\varphi_n}\{\eta : \eta(i) \leqq a\} = \lim_{n\to\infty} \nu_{n\varphi'_n}\{\eta : \eta(i) \leqq a\}.$$

It turns out that this condition is also sufficient.

(3.1) **Theorem.** *Let $c(\cdot,\cdot,\cdot)$ be an attractive interaction. Then a necessary and sufficient condition for there to be a unique stationary distribution for the Markov process η_t is that for all $i \in I$ and all $a \in S_i$*

$$\lim_{n\to\infty} \mu_{nu}\{\eta : \eta(i) \leqq a\} = \lim_{n\to\infty} \mu_{nd}\{\eta : \eta(i) \leqq a\}. \tag{3.2}$$

The proof is preceded by two lemmas. As we have already remarked the necessity is obvious; therefore, we only prove the sufficiency. In order to keep the notation to a minimum we do not distinguish between the measure $\bar{\mu}_{n\varphi}$ on the subsets of E_n and the measure $\mu_{n\varphi}$ on the Borel sets of E.

(3.3) **Lemma.** *Let* $i_1, \ldots, i_m \in I_n$ *and* $a_j \in S_{i_j}, j=1, \ldots, m$. *Then*

$$\begin{aligned}(3.4)\qquad &\mu_{nu}\{\psi \in E_n\colon \psi(i_j) \leqq a_j,\ j=1, \ldots, m\}\\ &\leqq \mu_{n\varphi}\{\psi \in E_n\colon \psi(i_j) \leqq a_j,\ j=1, \ldots, m\}\\ &\leqq \mu_{nd}\{\psi \in E_n\colon \psi(i_j) \leqq a_j,\ j=1, \ldots, m\}.\end{aligned}$$

Proof. We prove only the first inequality in (3.4). The proof of the second one is similar. Let $F(\psi) = F_{i_1, \ldots, i_m, a_1, \ldots, a_m}(\psi)$. It suffices to show that

$$(3.5)\qquad \int F(\psi)\, \mu_{nu}(d\psi) \leqq \int F(\psi)\, \mu_{n\varphi}(d\psi).$$

If U_t (V_t) is the semigroup of operators on $\mathscr{C}(E_n)$ whose infinitesimal generator is $\Omega_{n\varphi_{nu}}$ ($\Omega_{n\varphi}$), then it follows from the definitions of μ_{nu} and $\mu_{n\varphi}$ that

$$(3.6)\qquad \int F(\psi)\, \mu_{nu}(d\psi) = \lim_{t \to \infty} U_t F(\psi_{nu})$$

and

$$(3.7)\qquad \int F(\psi)\, \mu_{n\varphi}(d\psi) = \int V_t F(\psi)\, \mu_{n\varphi}(d\psi)$$

for all $t \geqq 0$. Thus by (3.5), (3.6), and (3.7) it suffices to show that

$$(3.8)\qquad U_t F(\psi_{nu}) \leqq V_t F(\psi) \quad \text{for all } t \geqq 0 \text{ and all } \psi \in E_n.$$

If the N used in the proof of Theorem (2.8) is taken to be the n here and η_2 in (2.18) is taken to be $[\psi, \varphi]$, then (3.8) follows from (2.17) and (2.18) just as (2.23) did in the proof of Theorem (2.8). The details are left to the reader.

(3.9) **Lemma.** *Under the conditions of Lemma* (3.3),

$$\begin{aligned}(3.10)\qquad &\mu_{nd}\{\psi(i_j) \leqq a_j, j=1, \ldots, m\} - \mu_{nu}\{\psi(i_j) \leqq a_j, j=1, \ldots, m\}\\ &\leqq \sum_{j=1}^{m} [\mu_{nd}\{\psi(i_j) \leqq a_j\} - \mu_{nu}\{\psi(i_j) \leqq a_j\}].\end{aligned}$$

Proof. Let $G(\psi) = \sum_{j=1}^{n} F_{i_j a_j}(\psi) - F_{i_1, \ldots, i_n, a_1, \ldots, a_n}(\psi)$. Then (3.10) is equivalent to

$$(3.11)\qquad \int G(\psi)\, \mu_{nu}(d\psi) \leqq \int G(\psi)\, \mu_{nd}(d\psi).$$

(3.11) is similar to (3.5) except we now have G instead of F. A careful look at the proof of (3.5) reveals that the essential property of F used there is that if $\psi_1, \psi_2 \in E_n$ are such that $\psi_1(i) \leqq \psi_2(i)$ for all $i \in E_n$, then $F(\psi_1) \geqq F(\psi_2)$. One easily checks that G has this property too and thus the proof of (3.11) is exactly like the proof of (3.5).

We can now prove Theorem (3.1). Corollary (2.25) tells us that it is sufficient to show that there is only one stationary distribution constructed by the standard method. Let $\{\mu_{n\varphi_n}\}$ be a standard sequence. We show that for all $F_{i_1, \ldots, i_m, a_1, \ldots, a_m}$, $\lim_{n \to \infty} \int F_{i_1, \ldots, i_m, a_1, \ldots, a_m}\, d\mu_{n\varphi_n} = \lim_{n \to \infty} \int F_{i_1, \ldots, i_m, a_1, \ldots, a_m}\, d\mu_{nu}$. This shows that all stationary distributions constructed by the standard method have the same finite dimensional distributions, and hence are the same.

From Lemmas (3.3) and (3.9) we know that for all sufficiently large n

$$
\begin{aligned}
(3.12)\qquad & \left|\int F_{i_1,\ldots,i_m,a_1,\ldots,a_m}\, d\mu_{n\varphi_n} - \int F_{i_1,\ldots,i_m,a_1,\ldots,a_m}\, d\mu_{nu}\right| \\
& \leqq \int F_{i_1,\ldots,i_m,a_1,\ldots,a_m}\, d\mu_{nd} - \int F_{i_1,\ldots,i_m,a_1,\ldots,a_m}\, d\mu_{nu} \\
& \leqq \sum_{j=1}^{m} \left[\int F_{i_j,a_j}\, d\mu_{nd} - \int F_{i_j,a_j}\, d\mu_{nu}\right].
\end{aligned}
$$

Finally the hypotheses of Theorem (3.1) imply that the last expression in inequality (3.12) converges to zero, and the theorem is proved.

4. The Stochastic Ising Model

Let Z be the integers and Z^3 be the three dimensional cubic lattice. We take Z^3 for the index set, and for each $i\in Z^3$ we let $S_i=\{-1,1\}$. Each $\eta\in E$ then represents a configuration of spins: $\eta(i)=1$ (-1) means that the spin at site i is up (down).

For $\eta\in E$ and $i\in Z^3$ define the energy, U, of the spin at site i in configuration η to be

$$(4.1)\qquad U(i,\eta)=-\sum_j \eta(i)\,\eta(j)-H\eta(i),$$

where the summation is over those j for which $|j-i|=1$, and H is a real number representing the external magnetic field.

Finally let β be a positive number and let

$$(4.2)\qquad c(i,\eta,a)=\begin{cases}\exp\{\beta U(i,\eta)\} & \text{if } a=-\eta(i)\\ 0 & \text{if } a=\eta(i).\end{cases}$$

It is easily checked that $c(\cdot,\cdot,\cdot)$ is attractive in the sense of (1.2).

Intuitively it is clear that each spin is more willing to stay as it is when it is lined up with most of its neighbors than when it is not. This is the reason for the term attractive. (For this example the term ferromagnetic might be more appropriate.)

For the sequence of subsets $\{I_n\}$ we take

$$I_n=\{(i_1,i_2,i_3)\in Z^3\colon |i_j|\leqq n, j=1,2,3\}.$$

Since the interaction has finite range it is very easy to verify the hypotheses in Remark (2.3) (see [6] or [7]).

In order to describe the measures $\bar\mu_{n\varphi}$ we first fix $\varphi\in\bar E_n$. From the definitions of $\bar\mu_{n\varphi}$ and $\Omega_{n\varphi}$ only a simple calculation is required (see [9]) to conclude that if $\psi\in E_n$, then

$$\bar\mu_{n\varphi}(\{\psi\})=\frac{1}{Z_{n\varphi}}\exp\{-\beta[-\sum_{\langle i,j\rangle}\psi(i)\,\psi(j)-\sum_{\langle k,l\rangle}\psi(k)\,\varphi(l)-H\sum_{i\in I_n}\psi(i)]\},$$

where the summation over $\langle i,j\rangle$ is over all pairs $i,j\in I_n$ with $|i-j|=1$, the summation over $\langle k,l\rangle$ is over all pairs k,l with $k\in I_n$, $l\in I\setminus I_n$ and $|k-l|=1$, and $Z_{n\varphi}$

is the normalizing constant which makes $\bar{\mu}_{n\varphi}$ a probability measure. The measures $\mu_{n\varphi}$ and their weak limits have been studied extensively, and the following facts are known (see [2] and [8]).

(4.3) If H in (4.1) is not zero or if β in (4.2) is sufficiently small, then $\{\mu_{nu}\}$ and $\{\mu_{nd}\}$ satisfy (3.2).

(4.4) If $H=0$ and β is large, then (3.2) does not hold.

Thus if we take β sufficiently large (i.e. strong enough interaction), we have a very striking example of the difficulties involved in proving ergodic theorems for interacting systems. For if we consider the interactions c_0 and c_ε given by (4.1) and (4.2), where in c_0 we take $H=0$ and in c_ε we take $H=\varepsilon$, then for small ε the interactions c_0 and c_ε seem to be very nearly the same; yet the asymptotic behavior of the corresponding Markov processes is entirely different. In particular for the process with interaction c_0 the distribution at time t is strongly dependent on the initial configuration of spins even for arbitrarily large t ((4.4) and Theorem (3.1)), while for the process with interaction c_ε the distribution at time t converges weakly to a given limit independent of the initial configuration. For the interaction c_0 there is still the possibility that even though the distribution at time t depends strongly on the initial configuration it nevertheless converges weakly to some limit which also depends on the initial configuration. However, it can be proven that there are initial configurations for which this does not happen, and the distribution does not converge to anything. Thus for the interaction c_0 there is no hope of proving any kind of ergodic theorem if the initial configuration is allowed to be chosen arbitrarily. For further discussion of this model the reader is referred to [3].

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