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# A Result in Doeblin's Theory of Markov Chains Implied by Suslin's Conjecture

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**0.** Let $P(x, B)$ be a transition probability function on the measurable space $(X, \mathscr{B})$. We begin with a resumé of the terminology originated by Doeblin in [3] (see also [2] and [8]). The set $B \in \mathscr{B}$ is called *stochastically closed*, or *closed*, if it is non-void and if $P(x, B) = 1$ for all $x \in B$. The set $B \in \mathscr{B}$ is called *indecomposable* if it does not contain a disjoint pair of closed subsets. For each probability measure $\mu$ on $\mathscr{B}$ there is a probability space $(\Lambda, \mathscr{F}, P_\mu)$ and a sequence $X_0, X_1, \ldots, X_n, \ldots$ of $(\mathscr{F} - \mathscr{B})$ measurable functions on $\Lambda$ into $X$ for which

(a) $P_\mu(X_0 \in B) = \mu(B)$

(b) $P_\mu(X_{n+1} \in B \mid X_0, \ldots, X_n) = P(X_n, B) \quad P_\mu$ – a.s.

for each $B \in \mathscr{B}$ and $n = 0, 1, \ldots$. If $\mu$ is the unit mass concentrated at $x$ we write $P_x$ for $P_\mu$. For each $x \in X$ and $B \in \mathscr{B}$ we write $L(x, B)$ for $P_x\left(\bigcup_{n=1}^{\infty} \{X_n \in B\}\right)$ and $Q(x, B)$ for $P_x\left(\bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} \{X_m \in B\}\right)$. A set $B \in \mathscr{B}$ is called *inessential* if $Q(x, B) = 0$ for all $x \in X$, *essential* otherwise. If $B$ is essential, it is called *improperly essential* if it is the countable union of inessential sets, *absolutely essential* otherwise. The reader is referred to [2, 3, 4, and 8] for the theory whose basic vocabulary we have just introduced. (In [4] it is shown that if $B \in \mathscr{B}$ is not absolutely essential then there are sets $B_1, B_2, \ldots$ such that $\sum_{n=1}^{\infty} P^n(x, B_k) < \infty$ for each $k = 1, 2, \ldots$ and $x \in X$ with $B = \bigcup_{k=1}^{\infty} B_k$.)

Consider the following condition:

($\mathscr{D}$) There is a countable disjoint family $\{C_n : n = 1, 2, \ldots\}$ of closed, indecomposable, and absolutely essential sets such that $I = X - \sum_{n=1}^{\infty} C_n$ is either inessential or improperly essential.

Of course, if $X$ itself is improperly essential, the condition is vacuous. The partition of $X$ into $I$ and the $C_n$'s is called a *Doeblin decomposition*. The significance of the decomposition (when $X$ is absolutely essential) is a consequence of the surprisingly extensive knowledge of the ergodic properties of $P(x, B)$ in the special case where $X$ is indecomposable and absolutely essential. For instance, if $X$ is indecomposable and absolutely essential, there is a $\sigma$-finite invariant measure on $(X, \mathscr{B})$ which is unique (up to constant multiples). If $X$ is absolutely essential, if $I + \sum C_k$ is a Doeblin decomposition of $X$, and if for each $x \in X$ there

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is an $n$ and a $k$ for which $P^n(x, C_k)>0$, then there is at least one $\sigma$-finite invariant measure, and every such measure is of the form $\sum a_k \mu_k$, where $a_k \geqq 0$ and $\mu_k$ is the essentially unique $\sigma$-finite invariant measure on $C_k$. In [3] (p. 74–76) Doeblin proves that $(\mathscr{D})$ holds under the following condition:

$(\mathscr{M})$ *There is a finite measure $\varphi$ on $(X, \mathscr{B})$ which assigns positive measure to every stochastically closed set.*

Condition $(\mathscr{M})$ clearly implies the following condition.

$(\mathscr{C})$ *There exists no uncountable disjoint class of stochastically closed subsets of $X$.*

It is natural to conjecture that $(\mathscr{C})$ implies $(\mathscr{D})$. The first section is devoted to a proof that, under the assumption that Suslin's conjecture holds, $(\mathscr{C})$ does indeed imply $(\mathscr{D})$. In the second section, results which can be considered partial converses to the main result of [5] are established.

**1.** Let $(S, <)$ be a partially ordered set ([7], p. 13). Members $x$ and $y$ of $S$ are called *comparable* if either $x<y$ or $y<x$. A subset $Q$ of $S$ is called a *chain* if any two elements of $Q$ are comparable, and it is called an *antichain* if no two elements of $Q$ are comparable. A partially ordered set is called a *tree* if the set of all elements which precede any given element forms a chain. The following statement is a formulation of *Suslin's conjecture* (see [6] for the original formulation and a proof of its equivalence to this one).

$(\mathscr{S})$ *Every tree of cardinality $\aleph_1$ contains either a chain of cardinality $\aleph_1$ or an antichain of cardinality $\aleph_1$.*

This is our main theorem.

**Theorem 1.**

$$\mathscr{S} \Rightarrow (\mathscr{C} \Rightarrow \mathscr{D}).$$

The theorem follows from the following lemma.

**Lemma 1.** *If $(\mathscr{S})$ holds, then $(\mathscr{C})$ implies that $X$ contains a closed set which is either indecomposable or improperly essential.*

To see that the lemma implies the theorem, we reason as follows. Call a collection $\mathscr{A} \subset \mathscr{B}$ admissible if it is a non-void, countable, disjoint collection each of whose members is an indecomposable or improperly essential closed set. Assume $(\mathscr{S})$ and $(\mathscr{C})$ hold. The lemma states that there is at least one admissible collection. Order the class of all such collections by inclusion. It is easy to see that the union of a chain of admissible collections is itself an admissible collection, so Zorn's lemma applies to yield a maximal admissible collection $\mathscr{L}$. Let $L$ be the union of all members of $\mathscr{L}$. Then $L$ is stochastically closed. Also $L^c = X - L$ contains no closed sets, for if it did, $(\mathscr{C})$ and the lemma would combine to yield the existence of a closed $C \subset L^c$ with $C$ either indecomposable or absolutely essential. But then $\mathscr{L} \cup \{C\}$ would be an admissible collection, which would contradict the maximality of $\mathscr{L}$. Since $L^c$ contains no closed sets, it is either inessential or improperly essential by virtue of Proposition 14.1 of [2]. Let $C_1, C_2, \ldots$ be a possibly void or finite enumeration of the members of $\mathscr{L}$ which are indecomposable and absolutely essential. Then $L-\sum C_i$, hence $L^c \cup (L-\sum C_i) = X-\sum C_i$ is inessential or improperly essential. Let $I=X-\sum C_i$. Then $X=I+\sum C_i$ is a Doeblin decomposition of $X$. Having established that the theorem does indeed follow from the lemma, we proceed with a proof of the lemma.

On p. 70 of [3], Doeblin shows that, if a closed set $E$ is not indecomposable, then there are two disjoint closed subsets $A$ and $B$ of $E$ such that $E-(A\cup B)$ has no closed subset. We say that $(A, B)$ is a *maximal pair* of closed subsets of $E$. It follows from Proposition 14.1 of [2] that $E-(A\cup B)$ is either inessential or improperly essential. Now assume that $(\mathscr{C})$ holds, but that every closed subset of $X$ is decomposable and absolutely essential. We proceed by transfinite induction, in effect pursuing Doeblin's line of reasoning on p. 71 of [3]. We denote the first uncountable ordinal by $\Omega$, and ordinals strictly less than $\Omega$ by lower case Greek letters. Our set theory is that of [7], in which every ordinal is the set of all ordinals strictly less than it. A function $s$ into $\{0, 1\}$ is called a *binary sequence* if its domain is equal to an ordinal $\alpha<\Omega$; $s$ is then said to be of *order* $\alpha$. The binary sequences $s\cup\{(\alpha, 0)\}$ and $s\cup\{(\alpha, 1)\}$ are denoted by $s0$ and $s1$ respectively; they are of order $\alpha+1$. We now use transfinite induction to define on the class of all binary sequences a function $C$ such that

(i) $C(s)$ is either $\emptyset$ or a closed set,

(ii) if $s\subset t$ then $C(t)\subset C(s)$,

(iii) if neither $s\subset t$ or $t\subset s$ then $C(s)\cap C(t)=\emptyset$ (the latter holding, in particular, if $s$ and $t$ are distinct and of the same order).

The unique binary sequence of order 0 is $\emptyset$. We define $C(\emptyset)=X$. Suppose $C(s)$ has been defined for all binary sequences of order less than $\alpha$. Suppose $\alpha$ is not a limit ordinal, that is, $\alpha=\beta+1$ for some $\beta$. Any sequence of order $\alpha$ is equal to $s0$ or $s1$ where $s$ is a sequence of order $\beta$. If $C(s)=\emptyset$, we define $C(s0)=C(s1)=\emptyset$. Otherwise $C(s)$ is closed, and, by assumption, decomposable. Let $(A, B)$ be a maximal pair of closed subsets of $C(s)$, and define $C(s0)=A$, $C(s1)=B$. Suppose, on the other hand, that $\alpha$ is a limit ordinal. Let $s$ be of order $\alpha$. For any $\beta<\alpha$ let $s_\beta$ be the restriction of $s$ to $\beta$. We define $C(s)$ to be $\bigcap_{\beta<\alpha} C(s_\beta)$. Since the intersection is countable, $C(s)$ is either closed or empty. The definition of $C(s)$ for all binary sequences is now complete by virtue of the principle of transfinite induction. It is clear that (i), (ii), and (iii) hold.

Let $\mathscr{R}=\{C(s)\colon s \text{ binary sequence}\}-\{\emptyset\}$, that is, $\mathscr{R}$ is the range of the function $C$ just defined but with $\emptyset$ thrown out. It follows from (i), (ii) and (iii) that the members of $\mathscr{R}$ are closed and that

(iv) $E\in\mathscr{R}$ and $F\in\mathscr{R} \Rightarrow E\subset F$ or $F\subset E$ or $E\cap F=\emptyset$.

We deal separately with the cases where (a) $\mathscr{R}$ is finite, (b) $\mathscr{R}$ is denumerably infinite, and (c) $\mathscr{R}$ is uncountable.

*Case (a). $\mathscr{R}$ finite.*

If $\mathscr{R}$ is finite, there has to be a binary sequence $s$ of finite order such that $C(s)$ is closed, but $C(s0)=C(s1)=\emptyset$. But if $C(s)$ is closed, so are $C(s0)$ and $C(s1)$, for, by definition, they form a maximal pair in $C(s)$.

*Case (b). $\mathscr{R}$ denumerably infinite.*

I claim that in this case $X$ itself is improperly essential. Let $C(s^{(1)})$, $C(s^{(2)}), \ldots$ be an enumeration of $\mathscr{R}$. Let $\alpha_1, \alpha_2, \ldots$ be the orders of $s^{(1)}, s^{(2)}, \ldots$ respectively. Let $\alpha = \sup_n \alpha_n$. Then $\alpha$ is a limit ordinal, and if $s$ is any sequence of order $\alpha$, then $C(s) = \emptyset$. For each $\beta < \alpha$ let

$$C^*(\beta) = \{C(s) \colon s \text{ is of order } \beta\}.$$

The members of $C^*(\beta)$ are pairwise disjoint by (iii). We are assuming that ($\mathscr{C}$) holds, so $C^*(\beta)$ is countable. Let $K(\beta)$ be the union of all the members of $C^*(\beta)$. Then $K(\beta)$ is either empty or closed. To proceed, we require the following

**Proposition.** *For each $\beta < \Omega$, $X - K(\beta)$ is either inessential or improperly essential.*

We prove this by induction on $\beta$. For $\beta = 0$, $X - K(0) = X - X = \emptyset$. Suppose $\beta = \gamma + 1$, and that $X - K(\gamma)$ is inessential or improperly essential. For each $s$ of order $\gamma$ for which $C(s) \neq \emptyset$, $C(s0)$ and $C(s1)$ are a maximal pair in $C(s)$, so $C(s) - (C(s0) \cup C(s1))$ is either inessential or improperly essential. Suppose, on the other hand, that $\beta$ is a limit ordinal, and that $X - K(\gamma)$ is inessential or improperly essential for each $\gamma < \beta$. To show that $X - K(\beta)$ is inessential or improperly essential it suffices to show that

$$K(\beta) = \bigcap_{\gamma < \beta} K(\gamma). \tag{1}$$

Suppose $s$ is of order $\beta$. If $t \subset s$ but $t \neq s$ we write $t < s$. Then $C(s)$, is, by definition, the intersection of all the sets $C(t)$ for which $t < s$. Hence

$$K(\beta) = \bigcup_{s} \bigcap_{t < s} C(t). \tag{2}$$

It is easy to verify, using (iii), that

$$\bigcup_{s} \bigcap_{t < s} C(t) = \bigcap_{\gamma < \beta} \bigcup_{t} C(t), \tag{3}$$

where, on the right, $t$ ranges over all binary sequences of order $\gamma$. Since the right hand side of (3) is equal to $\bigcap_{\gamma < \beta} K(\gamma)$, (2) and (3) combine to yield (1).

This completes the proof of the proposition. In particular, $X - K(\alpha)$ is either inessential or improperly essential. But $K(\alpha) = \emptyset$, so $X$ is indeed inessential or improperly essential in case (b).

*Case (c). $\mathscr{R}$ uncountable.*

Order $\mathscr{R}$ by inclusion: that is; $A < B$ iff $A \supset B$. Properties (i), (ii) and (iii) of the function $C$ imply that $\mathscr{R}$ is a tree. Since we are assuming the well-ordering principle, we may, without assuming the continuum hypothesis, select from $\mathscr{R}$ a subcollection $\mathscr{R}^*$ of cardinality $\aleph_1$. Of course, $\mathscr{R}^*$ is also a tree relative to $<$. Since we are assuming that ($\mathscr{S}$) holds, $\mathscr{R}^*$ contains either a chain or an antichain of cardinality $\aleph_1$. The existence of such an antichain is a direct contradiction to ($\mathscr{C}$), however, so $\mathscr{R}^*$, hence $\mathscr{R}$, contains a chain $\{C(s^{(\gamma)}) \colon \gamma \in \Gamma\}$, where the index set $\Gamma$ has cardinality

$\aleph_1$, and where $s^{(\gamma)}$ and $s^{(\gamma')}$ are distinct binary sequences if $\gamma$ and $\gamma'$ are distinct members of $\Gamma$. By (ii) and (iii), the set $\{s^{(\gamma)}: \gamma\in\Gamma\}$ is a chain of binary sequences relative to the inclusion relation $\subset$. Letting $\sigma=\bigcup\{s^{(\gamma)}: \gamma\in\Gamma\}$, we see that $\sigma$ is a function from $\Omega$ into $\{0,1\}$ such that, for each $\gamma\in\Gamma$, the restriction of $\sigma$ to the ordinal which is the order of $s^{(\gamma)}$ is $s^{(\gamma)}$. For each $\alpha<\Omega$ let $\sigma_\alpha$ be the restriction of $\sigma$ to $\alpha$. Then each set $C(\sigma_\alpha)$ is closed and $C(\sigma_{\alpha+1})$ is one member of a maximal pair of closed subsets of $C(\sigma_\alpha)$. Denote the other member of this maximal pair by $D(\alpha+1)$. If $\beta>\alpha$, $C(\sigma_\beta)\subset C(\sigma_{\alpha+1})$, hence $C(\delta_\beta)\cap D(\alpha+1)=\emptyset$. Thus $\{D(\alpha+1): \alpha<\Omega\}$ is a collection of pairwise disjoint closed sets, which contradicts (c). This finishes case (c), so the lemma is proved.

Recent work of Jech, Solovay and Tennenbaum (see [6] for proofs and references) has established the independence of Suslin's conjecture from the other axioms of set theory. This being so, we do not expect anybody to turn up with a transition probability function $P(x,B)$ for which $(\mathscr{C})$ but not $(\mathscr{D})$ holds. Nevertheless, it would be desirable either to give a proof that $(\mathscr{C})\Rightarrow(\mathscr{D})$ in which $(\mathscr{S})$ is not used, or else to show that if $(\mathscr{C})\Rightarrow(\mathscr{D})$, then $(\mathscr{S})$ holds.

We shall require in the next section the following result, the simple proof of which we omit.

**Lemma 2.** *If any disjoint collection of closed sets is finite, then $(\mathscr{D})$ holds (in this case, $\{C_n\}$ is finite).*

**2.** Let $\{X_n, n\geqq 0\}$ be a Markov process having $P(x,B)$ as its transition probability function. Each $X_n$ is an $X$-valued $\mathscr{B}$-measurable function on a probability space $(\Lambda,\mathscr{F},P)$. We denote by $X^\infty$ the product space $\prod_{i=0}^{\infty} X^{(i)}$, where $X^{(i)}=X$ for each $i=0,.,\dots$, and by $\mathscr{B}^\infty$ the corresponding product $\sigma$-field $\prod_{i=0}^{\infty}\mathscr{B}^{(i)}$, where $\mathscr{B}^{(i)}=\mathscr{B}$ for each $i=0,1,\dots$. For each $n$ let $\mathscr{C}_n$ be the $P$-completion of the smallest $\sigma$-field over which $X_m$ is measurable for each $m\geqq n$. Then $\mathscr{T}=\bigcap_{n=1}^{\infty}\mathscr{C}_n$ is called the *tail $\sigma$-field* of the process $\{X_n, n\geqq 1\}$. It is easy to see that $A\in\mathscr{T}$ if and only if there is a sequence $f_0, f_1, \dots$ of bounded $\mathscr{B}^\infty$-measurable functions on $X^\infty$ such that

$$I_A=f_n(X_n, X_{n+1},\dots) \qquad P-\text{a.s.} \tag{4}$$

for each $n=0,1,\dots$, where $I_A$ is the indicator of $A$. If there is a bounded $\mathscr{B}^\infty$-measurable function in $X^\infty$ with

$$I_A=f(X_{n+1},\dots) \qquad P-\text{a.s.} \tag{5}$$

we say that $A$ is *invariant*, and such $A$'s constitute a sub $\sigma$-field $\mathscr{I}$ of $\mathscr{T}$. If $\mathscr{I}$ (or $\mathscr{T}$) consists of a single $P$-atom, that is, if $P(A)=0$ or 1 for each $A$ in $\mathscr{I}$ (or $\mathscr{T}$), we say that $\mathscr{I}$ (or $\mathscr{T}$) is *trivial.*

We say that $X$ is *recurrent in the sense of Harris* if there is a $\sigma$-finite measure $\varphi$ on $\mathscr{B}$ such that $Q(x,B)=1$ for all $x\in X$ whenever $\varphi(B)>0$ (see [5] and [8]).

**Theorem.** *Suppose $P(x,B)$ is a transition function on $(X,\mathscr{B})$, and that $X$ is absolutely essential. Then $\mathscr{I}$ is trivial for each process $\{X_n, n\geqq 0\}$ having $P(x,B)$ as transition function if and only if $X$ is recurrent in the sense of Harris. If $(X,\mathscr{I})$*

*consists of a finite number of P-atoms for each such process, then* $X=I+\sum_{i=1}^{k} H_i$, *where I is either inessential or improperly essential and where* $H_1, \ldots, H_k$ *are closed and recurrent in the sense of Harris.*

*Proof.* It is well known and easy to show if $X$ is recurrent in the sense of Harris then $\mathscr{I}$ is trivial. Suppose, then, that $X$ is absolutely essential, and that $\mathscr{I}$ is trivial for every Markov process with transition function $P(x, B)$. First, $X$ is indecomposable. For suppose $E$ and $F$ are disjoint closed sets. Let $\{X_n\}$ be the process with $P(x, B)$ as transition function for which $P(X_0 \in E)=P(X_0 \in F)=\frac{1}{2}$. Clearly $\{X_0 \in E\} \in \mathscr{I}$, so $\mathscr{I}$ is non-trivial. Second, $X$ has no improperly essential subsets. For suppose $E$ is one. By Definition 4 and Proposition 23 of [2], there is an improperly essential $F \supset E$ and an $y \in F$ with $L(y, F^c)<1$. Also, $Q(y, F)<1$ by Proposition 19 of [2]. Let $\{X_n\}$ be the Markov process with $P(x, B)$ as transition function and initial distribution concentrated at $y$. On the one hand, $A=\bigcap_{n=0}^{\infty} \{X_n \in F\}$ belongs to $\mathscr{I}$, on the other, $0<1-L(y, F^c)=P(A) \leqq Q(y, F)<1$. This contradicts the triviality of $\mathscr{I}$. Thus $X$ is an indecomposable absolutely essential set containing no improperly essential sets. (In the terminology of Doeblin [3], $X$ is a "final set".) Recurrence in the sense of Harris now follows from Theorem 3 of [4].

Now suppose that $\mathscr{I}$ consists of only a finite number of $P$-atoms. It is then clear that any disjoint collection of closed sets if finite. By Lemma 2,

$$X=J+\sum_{i=1}^{k} C_i,$$

where $C_1, \ldots, C_k$ is an indecomposable and absolutely essential closed set. Such a set is called *normal* if it contains a final set, or, what is the same (see [5]), a closed set recurrent in the sense of Harris. I claim that each $C_i$ is normal. For suppose $C_i$ is not. Then there is an uncountable disjoint collection $\{E_\alpha\colon \alpha \in A\}$ of pairwise disjoint $\mathscr{B}$-measurable subsets of $C_i$ such that, for each $\alpha \in A$, there is an $x_\alpha \in E_\alpha$ with $Q(x_\alpha, E_\alpha) \geqq \frac{1}{2}$. (See p. 83 of [3].) Pick a sequence $x_{\alpha_1}, x_{\alpha_2}, \ldots$ of the $x_\alpha$'s, and let $\{S_n, n \geqq 0\}$ be a process with $P(x, B)$ as transition function for which $P\{X_0=x_{\alpha_m}\}=p_m$, $m=1, 2, \ldots$, where $p_m>0$ and $\sum_m p_m=1$. Let $A_m=\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \{X_k \in E_{\alpha_m}\}$. Then $A_m \in \mathscr{I}$ and $P(A_m)=p_m\, Q(x_{\alpha_m}, E_m)>0$ for each $m=1, 2, \ldots$. This is, of course, impossible if $\mathscr{I}$ consists of a finite number of $P$-atoms. Thus each $C_i$ is indeed normal, and therefore $K_i=H_i+I_i$, where $H_i$ is recurrent in the sense of Harris and $I_i$ is either inessential or improperly essential (see [4], Theorem 3). Define $I=J+\sum_{i=1}^{k} I_i$. Then $I$ is inessential or improperly essential, and $X=I+\sum_{i=1}^{k} H_i$, where each $H_i$ is recurrent in the sense of Harris. This completes the proof of the theorem.

If one adds to the hypothesis of the theorem that $\mathscr{B}$ is generated by a countable subclass of $\mathscr{B}$, the above proof can be shortened by making use of the fact ([4], Theorem 3) that then every closed, indecomposable and absolutely essential set is the disjoint union of a set which is recurrent in the sense of Harris and a set which is either inessential or improperly essential.

If $X$ is recurrent in the sense of Harris, then $\mathscr{T}$ consists of a finite number of atoms each describable in terms of the cyclic structure of $X$ ([6], Theorem 1). Thus Theorem 2 tells us that, unless $X$ is improperly essential, if $\mathscr{I}$ is trivial for each process $\{X_n\}$ then $\mathscr{T}$ consists only of all unions of a finite number of $P$-atoms for such processes. This result fails spectacularly if $X$ is improperly essential, for Blackwell and Freedman have given (in [1]) an example of Markov chain consisting of a countable number of transient states such that, for every Markov process $\{X_n\}$ with $P(x,E)$ as transition function, $\mathscr{I}$ is trivial but $\mathscr{T}$ is purely non-atomic (in fact $\mathscr{T}$ is the $\sigma$-field generated by $X_0, X_1, \ldots$).

Theorem 2 seems incomplete, for the following question naturally arises. Suppose that $X$ is absolutely essential, and that, for each Markov process $\{X_n, n \geqq 0\}$ with $P(x, E)$ as transition probability function, $\mathscr{I}$ consists of a countable number of $P$-atoms. Does ($\mathscr{D}$) then hold? We have been unable to settle this question.

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