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A Lemma on Regular Variation of a Transient Renewal Function

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Let $F(x)$ be a defective, non lattice distribution with

$$F(0)=0, \quad F(\infty)=\alpha, \qquad 0<\alpha<1, \tag{1}$$

and

$$U(x)=\sum_{n=1}^{\infty} F^{n*}(x), \tag{2}$$

so that

$$U(\infty)=\alpha/(1-\alpha). \tag{3}$$

Lemma. *For $\rho \geqq 0$ and $L(x)$ a slowly varying function at infinity*

$$U(\infty)-U(x)=\frac{\alpha}{1-\alpha}\, x^{-\rho} L(x)$$

iff $F(\infty)-F(x)=\alpha(1-\alpha)\, x^{-\rho} L(x)$.

Proof. From the renewal equation

$$U(x)=F(x)+F(x)*U(x),$$

we have

$$\frac{1-\alpha}{\alpha} U(x)=(1-\alpha)\{F(x)/\alpha\}+\alpha\{F(x)/\alpha\}*\left\{\frac{1-\alpha}{\alpha} U(x)\right\}. \tag{4}$$

Obviously, $F(x)/\alpha$ and $\frac{1-\alpha}{\alpha} U(x)$ are non-defective probability distributions. Let $\underline{x}_1$ and $\underline{x}_2$ be non-negative, independent stochastic variables with distributions

$$\begin{aligned} H_1(x)&=F(x)/\alpha=\Pr\{\underline{x}_1<x\},\\ H_2(x)&=\frac{1-\alpha}{\alpha} U(x)=\Pr\{\underline{x}_2<x\}, \end{aligned} \tag{5}$$

so that

$$\Pr\{\underline{x}_2<x\}=(1-\alpha)\Pr\{\underline{x}_1<x\}+\alpha\Pr\{\underline{x}_1+\underline{x}_2<x\},$$

and hence

$$\Pr\{\underline{x}_1+\underline{x}_2\geqq x\}=\frac{1}{\alpha}\Pr\{\underline{x}_2\geqq x\}-\frac{1-\alpha}{\alpha}\Pr\{\underline{x}_1\geqq x\}. \tag{6}$$

Since $\underline{x}_1$ and $\underline{x}_2$ are nonnegative, we have for $\varepsilon>0$ (cf. Feller, p. 271),

$$\begin{aligned} \Pr\{\underline{x}_1+\underline{x}_2\geqq x\}\geqq{}&\Pr\{\underline{x}_1\geqq x(1+\varepsilon)\}\Pr\{\underline{x}_2<x\varepsilon\}\\ &+\Pr\{\underline{x}_2\geqq x(1+\varepsilon)\}\Pr\{\underline{x}_1<x\varepsilon\}, \end{aligned} \tag{7}$$

$$\begin{aligned}\Pr\{\underline{x}_1+\underline{x}_2\geqq x\}\leqq &\Pr\{\underline{x}_1\geqq x(1-\varepsilon)\}+\Pr\{\underline{x}_2\geqq x(1-\varepsilon)\}\\ &+\Pr\{\underline{x}_1\geqq x\varepsilon)\Pr\{\underline{x}_2\geqq x\varepsilon\}.\end{aligned} \tag{8}$$

Consequently, from (5), (6), (7) and (8),

$$\begin{aligned}&\{1-H_1(x(1+\varepsilon))\}\,H_2(x\varepsilon)+\{1-H_2(x(1+\varepsilon))\}\,H_1(x\varepsilon)\\ &\quad\leqq\frac{1}{\alpha}(1-H_2(x))-\frac{1-\alpha}{\alpha}(1-H_1(x))\\ &\quad\leqq 1-H_1(x(1-\varepsilon))+1-H_2(x(1-\varepsilon))+\{1-H_1(x\varepsilon)\}\{1-H_2(x\varepsilon)\}.\end{aligned} \tag{9}$$

Suppose

$$1-H_1(x)=\Pr\{\underline{x}_1\geqq x\}=x^{-\rho}S(x)\qquad\text{for } x\to\infty, \tag{10}$$

with $S(x)$ slowly varying at infinity. Then from (9),

$$\frac{1}{(1+\varepsilon)^\rho}\left\{1+\liminf_{x\to\infty}\frac{1-H_2(x(1+\varepsilon))}{((1+\varepsilon)x)^{-\rho}S(x(1+\varepsilon))}\right\}\leqq\frac{1}{\alpha}\liminf_{x\to\infty}\frac{1-H_2(x)}{x^{-\rho}S(x)}-\frac{1-\alpha}{\alpha}, \tag{11}$$

$$\frac{1}{\alpha}\limsup_{x\to\infty}\frac{1-H_2(x)}{x^{-\rho}S(x)}-\frac{1-\alpha}{\alpha}\leqq\frac{1}{(1-\varepsilon)^\rho}\left\{1+\limsup_{x\to\infty}\frac{1-H_2(x(1-\varepsilon))}{((1-\varepsilon)x)^{-\rho}S(x(1-\varepsilon))}\right\}, \tag{12}$$

since for $\varepsilon>0$ and $x\to\infty$,

$$\frac{S(x(1+\varepsilon))}{S(x)}\to 1,\qquad \frac{1-H_1(x\varepsilon)}{x^{-\rho}S(x)}\{1-H_2(x\varepsilon)\}\to 0.$$

From (11) and (12) it follows for $\varepsilon>0$ that

$$\liminf_{x\to\infty}\frac{1-H_2(x)}{x^{-\rho}S(x)}\geqq\frac{\alpha+(1-\alpha)(1+\varepsilon)^\rho}{(1+\varepsilon)^\rho-\alpha},$$

$$\limsup_{x\to\infty}\frac{1-H_2(x)}{x^{-\rho}S(x)}\leqq\frac{\alpha+(1-\alpha)(1-\varepsilon)^\rho}{(1-\varepsilon)^\rho-\alpha}\qquad\text{with } (1-\varepsilon)^\rho>\alpha;$$

so that by letting $\varepsilon\downarrow 0$,

$$\frac{1}{1-\alpha}\leqq\liminf_{x\to\infty}\frac{1-H_2(x)}{x^{-\rho}S(x)}\leqq\limsup_{x\to\infty}\frac{1-H_2(x)}{x^{-\rho}S(x)}\leqq\frac{1}{1-\alpha}.$$

Hence

$$1-H_2(x)=\frac{1}{1-\alpha}x^{-\rho}S(x)\qquad\text{for } x\to\infty,$$

or by applying (3) and (5),

$$U(\infty)-U(x)=\frac{\alpha}{1-\alpha}x^{-\rho}L(x)\qquad\text{for } x\to\infty,$$

with

$$L(x)=\frac{1}{1-\alpha}S(x),$$

so that the "if" part of the lemma has been proved.

With

$$E\{x\}=\begin{cases}0, & x\leqq 0,\\ 1, & x>0,\end{cases}$$

it follows from (5) and (6),

$$E(x)-H_1(x)=(1-\alpha)\big(E(x)-H_2(x)\big)+\alpha\big(E(x)-H_1(x)\big)*\big(E(x)-H_2(x)\big).$$

From this relation it is seen by iteration that

$$\begin{aligned}E(x)-H_1(x)=(1-\alpha)\sum_{n=1}^{N}\alpha^{n-1}\{E(x)-H_2(x)\}^{n*}\\ +\alpha^N\big(E(x)-H_1(x)\big)*\{E(x)-H_2(x)\}^{N*},\end{aligned}\tag{13}$$

for every integer $N=1,2,\ldots$.

Since the last term in (13) is nonnegative for N even, and nonpositive for N odd it follows from (13),

$$\sum_{n=1}^{N}\alpha^{n-1}\{E(x)-H_2(x)\}^{n*}\leqq\frac{E(x)-H_1(x)}{1-\alpha}\leqq\sum_{n=1}^{N+1}\alpha^{n-1}\{E(x)-H_2(x)\}^{n*},\tag{14}$$

for every $N=2,4,6\ldots$.

It is readily verified that for $m=1,2,\ldots$,

$$\{E(x)-H_2(x)\}^{m*}=\sum_{k=1}^{m}(-1)^{k+1}\binom{m}{k}\big(E(x)-H_2^{k*}(x)\big).\tag{15}$$

Suppose

$$1-H_2(x)=x^{-\rho}L(x)\qquad\text{for } x\to\infty,$$

so that (cf. Feller, p. 272),

$$1-H_2^{k*}(x)=k\,x^{-\rho}L(x)\qquad\text{for } x\to\infty.\tag{16}$$

Hence from (15) and (16),

$$\lim_{x\to\infty}\frac{\{E(x)-H_2(x)\}^{m*}}{x^{-\rho}L(x)}=\sum_{k=1}^{m}(-1)^{k+1}k\binom{m}{k}=\begin{cases}1 & \text{for } m=1,\\ 0 & \text{for } m=2,3,\ldots.\end{cases}\tag{17}$$

Dividing the terms in (14) by $x^{-\rho}L(x)$ and letting $x\to\infty$ we obtain by using (17),

$$\liminf_{x\to\infty}\frac{1-H_1(x)}{x^{-\rho}L(x)}\geqq 1-\alpha,$$

$$\limsup_{x\to\infty}\frac{1-H_1(x)}{x^{-\rho}L(x)}\leqq 1-\alpha,$$

i.e.

$$1-H_1(x)=(1-\alpha)\,x^{-\rho}L(x)\qquad\text{for } x\to\infty,$$

or from (5)

$$F(\infty)-F(x)=\alpha(1-\alpha)\,x^{-\rho}L(x)\qquad\text{for } x\to\infty.$$

The proof is complete.

Reference

Feller, W.: An Introduction to Probability Theory and its Applications, II. 1st ed., New York: Wiley 1966.

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