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Transition Probabilities and Contractions of L_∞

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1. Definitions and Notations

A Markov process is defined to be a quadruple (X, Σ, m, P) where (X, Σ, m) is a measure space with positive measure m, $m(X)=1$, and where P is an operator on $L_1(m)$ satisfying:

(i) P is a contraction: $\|P\| \leqq 1$,

(ii) P is positive: if $0 \leqq f \in L_1(m)$, then $fP \geqq 0$.

The operator adjoint to P is defined on $L_\infty(m)$. It will also be denoted by P but will be written to the left of the variable. Thus $\langle fP, g\rangle = \langle f, Pg\rangle$ for $f \in L_1(m)$, $g \in L_\infty(m)$.

The usual probabilistic definition of a Markov process is by a function $P(x, A)$ on $X \times \Sigma$ such that for each $x \in X$, $P(x, \cdot)$ is a probability measure and for each $A \in \Sigma$, $P(\cdot, A)$ is a measurable function. Assume that if $m(A)=0$, then $P(x, A)=0$ a.e. m; then $P(x, A)$ induces an operator on $L_\infty(m)$ and $L_1(m)$ as follows:

$$Pf(x) = \int P(x, dy)\, f(y), \qquad f \in L_\infty(m), \tag{1.1}$$

$$\mu P(A) = \int P(x, A)\, \mu(dx), \qquad \mu \prec m. \tag{1.2}$$

A general Markov process need *not* be induced by a transition probability $P(x, A)$. The process is said to be *conservative* if

$$\sum_{n=0}^{\infty} P^n f = \begin{cases} \infty \\ 0 \end{cases} \quad \text{for every } 0 \leqq f \in L_\infty(m). \tag{1.3}$$

The process is said to be *ergodic* if

$$Pf \leqq f \Rightarrow f = \text{const}. \tag{1.4}$$

Let $P^n = Q_n + R_n$ where Q_n is an integral operator with the kernel $f_n(x, y)$, and if K is any integral operator such that $0 \leqq K \leqq R_n$ then $K=0$. (See [6], Chapter V.) (X, Σ, m, P) is said to be a Harris process if $Q_n > 0$ for some integer n, and the process is ergodic and conservative.

If the process is conservative and ergodic and the operator P is induced by a transition probability, then the process is *not* a Harris process if and only if the measures $P^n(x, \cdot)$ are orthogonal to m for almost every x. If P is no longer induced by a transition probability, then the characterization of a Harris process is more complicated.

We shall also define the operator I_A for $A \in \Sigma$ by:

$$I_A f(x) = 1_A(x) \cdot f(x), \qquad \mu I_A(B) = \mu(A \cap B), \tag{1.5}$$

and the operator:

$$P_A = I_A \sum_{n=0}^{\infty} (P I_{A^c})^n P I_A. \tag{1.6}$$

It is well known that $(A, \Sigma_A, m I_A, P_A)$ is a Markov process ([6], Chapter VI).

2. On Spaces Isometric to $L_\infty(X, \Sigma, m)$

Given the space $L_\infty(X, \Sigma, m)$ let $B = \{v \in L_\infty^* \mid v \geqq 0, \|v\| = 1, v(1) = 1\}$. The set B is convex and compact in the W^*-topology, hence by the Krein-Milman Theorem it is the closed convex hull of its extremal points. The following statements are equivalent:

(i) $\tilde{x} \in B$ is an extremal point of B.

(ii) $\forall f, g \in \Sigma$, $\tilde{x}(f \cdot g) = \tilde{x}(f) \cdot \tilde{x}(g)$.

(iii) $\forall A \in \Sigma$, $\tilde{x}(1_A) = 0$ or 1.

Let:

$$\tilde{X} = \{\tilde{x} \in B \mid \tilde{x}(1_A) = 0 \text{ or } 1, \forall A \in \Sigma\}. \tag{2.1}$$

$\tilde{X}$ is a compact Hausdorff space in the W^*-topology. Let us define the mapping: $\tau: L_\infty(X, \Sigma, m) \to C(\tilde{X})$ by

$$(\tau f)(\tilde{x}) = \tilde{x}(f). \tag{2.2}$$

τ is an isometric isomorphism and it maps positive functions into positive functions and $\tau(fg) = (\tau f) \cdot (\tau g)$ ([4], Chapter V.8). It is clear that $\tau 1_A = 1_{\tilde{A}}$ where $\tilde{A} \subset \tilde{X}$. The set $\tilde{A}$ is closed and open.

Lemma 2.1. *The adjoint mapping of τ: τ^*: $C^*(\tilde{X}) \to L_\infty^*(X, \Sigma, m)$ is an isometric isomorphism. Let us denote $\tilde{m} = \tau^{*-1} m$, then for every measure $\mu \in L_\infty^*(X, \Sigma, m)$ the measure $\tau^{*-1}\mu$ is absolutely continuous to $\tilde{m}$, and for every pure charge (a functional on L_∞ such that the only measure dominated by it is the zero measure) $v \in L_\infty^*(X, \Sigma, m)$ the measure $\tau^{*-1} v$ is orthogonal to $\tilde{m}$. In particular, for each $\tilde{x} \in \tilde{X}$, $\tau^{*-1}\tilde{x}$ is the Dirac measure $\delta_{\tilde{x}}$.*

Lemma 2.2. *If $\tilde{A} \in \tilde{\Sigma}$, then there exist $\{E_k\}$, $\{F_k\} \subset \Sigma$ such that* (i) *$E_k \searrow$ and $F_k \nearrow$,* (ii) *$\tau 1_{E_k} \leqq 1_{\tilde{A}}$, $\tau 1_{\tilde{F}_k} \leqq 1_{\tilde{A}}$,* (iii) *$m(E_k) \searrow \tilde{m}(\tilde{A})$ and $m(F_k) \nearrow \tilde{m}(\tilde{A})$.*

In particular, if $m(\tilde{A}) = 0$, then $\tilde{A}$ is nowhere dense.

Sketch of the Proof of Lemmas 2.1 and 2.2

Let $\tilde{\Sigma}' = \{\tilde{A} \subset \tilde{X} \mid \tau 1_A = 1_{\tilde{A}}, A \in \Sigma\}$. $\tilde{\Sigma}'$ is a field, and the σ-field, $\tilde{\Sigma}$, generated by $\tilde{\Sigma}'$ is the Baire σ-field of subsets of $\tilde{X}$. If $\mu \in L_\infty^*(X, \Sigma, m)$ is a measure and $\{\tilde{A}_n\} \subset \tilde{\Sigma}'$ and $A_n \nearrow A$, where $1_{\tilde{A}_n} = \tau 1_{A_n}$, then $\tilde{\mu}(\tilde{A}_n) \nearrow \tilde{\mu}(\tilde{A})$, where $1_{\tilde{A}} = \tau 1_A$, $\tau^* \tilde{\mu} = \mu$, and by the Carathéodory extension theorem, we get Lemma 2.2 and the first part of Lemma 2.1.

Let $v \in L_\infty^*(X, \Sigma, m)$ be a positive pure charge with $v(X) = 1$. Then by [8] there exists $A_n \searrow \Phi$ such that $v(A_n) = 1$. Let $\tau 1_{A_n} = 1_{\tilde{A}_n}$, $\tilde{A} = \bigcap \tilde{A}_n$. Then $\tilde{m}(\tilde{A}) = 0$ and $\tau^{*-1} v(\tilde{A}) = 1$, so the second part of Lemma 2.1 is proved.

Lemma 2.3. *There exists an isometric isomorphism Λ between $L_\infty(X, \Sigma, m)$ and $L_\infty(\tilde{X}, \tilde{\Sigma}, \tilde{m})$ which maps positive functions into positive functions and $\Lambda(fg) = (\Lambda f)(\Lambda g)$. Moreover, if $f_n \to f$* a.e. *m, then $\Lambda f_n \to \Lambda f$* a.e. *$\tilde{m}$.*

Proof. It is sufficient to show that every coset in $L_\infty(\tilde{X}, \tilde{\Sigma}, \tilde{m})$ can be represented by a function of $C(\tilde{X})$. Let $\tilde{f} \in L_\infty(\tilde{X}, \tilde{\Sigma}, \tilde{m})$, denote $\mathfrak{A} = \{g \mid \tau g \leqq \tilde{f}$ a.e. $\tilde{m}\}$. By Proposition II.4.1 of [12] there exist $\{g_n\} \subset \mathfrak{A}$ such that $g_n \nearrow h$ and $h \geqq g$ for each $g \in \mathfrak{A}$. By Egorov's theorem there exist $A_k \nearrow X$ such that

$$\|1_{A_k} g_n - 1_{A_k} h\|_\infty \xrightarrow[n \to \infty]{} 0.$$

Hence $\tau(1_{A_k} g_n) \nearrow \tau(1_{A_k} h)$. Let $1_{\tilde{A}_k} = \tau 1_{A_k}$, $\tilde{A} = \bigcup_{k=1}^{\infty} \tilde{A}_k$, then $\tilde{m}(\tilde{A}) = 1$ and $\tau g_n \to \tau h$ on $\tilde{A}$, hence $\tau h \leqq \tilde{f}$ a.e. $\tilde{m}$.

Let $\tilde{E} = \{\tilde{f} - \tau h \geqq \varepsilon\}$; assume that $\tilde{m}(\tilde{E}) > 0$. By Lemma 2.2 there exists a set $\tilde{F} \subset \tilde{E}$ such that $\tau^{-1} 1_{\tilde{F}} = 1_F$. Hence $h \leqq h + \varepsilon 1_F \in \mathfrak{A}$ which contradicts the maximality of h. So, $\tilde{f} = \tau h$ a.e. $\tilde{m}$ and the coset of $L_\infty(\tilde{X}, \tilde{\Sigma}, \tilde{m})$ which is represented by $\tilde{f}$ can be represented by $\tau h \in C(\tilde{X})$.

If $f_n \to f$ a.e. m, then by Egorov's theorem there exist $A_k \nearrow X$ such that $\|1_{A_k}(f_n - f)\|_\infty \to 0$, hence $\tau(1_{A_k} f_n) \to \tau(1_{A_k} f)$. Let $1_{\tilde{A}_k} = \tau 1_{A_k}$, $\tilde{A} = \bigcup_{k=1}^{\infty} \tilde{A}_k$. Then $\tau f_n \to \tau f$ on $\tilde{A}$ and $\tilde{m}(\tilde{A}) = 1$. So, Lemma 2.4 is proved.

Remark. Many results of this section were already noted in [3]. We give them here for completeness in a slightly different approach.

3. The Induced Transition Probability

Let P be a Markov operator on $L_\infty(\tilde{X}, \tilde{\Sigma}, \tilde{m})$. Then $\tau P \tau^{-1} = \tilde{P}$ is a Markov operator on $L_\infty(\tilde{X}, \tilde{\Sigma}, \tilde{m})$. $\tilde{P}$ is a positive contraction on $C(\tilde{X})$.

The adjoint operator of $\tilde{P}$ acts on the regular measures on $\tilde{X}$. It will also be denoted by $\tilde{P}$, but will be written to the right of the variable.

Lemma 3.1. *The operator $\tilde{P}$ is induced by a transition probability $\tilde{P}(\tilde{x}, \tilde{A})$.*

Proof. Let us define the transition probability

$$\tilde{P}(\tilde{x}, \tilde{A}) = \delta_{\tilde{x}} \tilde{P}(\tilde{A}) \tag{3.1}$$

where $\delta_{\tilde{x}}$ is the Dirac measure at $\tilde{x} \in \tilde{X}$ and $\tilde{A} \in \tilde{\Sigma}$.

It is clear that $\tilde{P}(\tilde{x}, \cdot)$ is a measure for all $\tilde{x}$.

On the other hand, if $\tilde{f}$ is a continuous function, then $\langle \delta_{\tilde{x}} \tilde{P}, \tilde{f} \rangle = \tilde{P} \tilde{f}(x)$ is also continuous, and the collection $\mathfrak{A} = \{\tilde{f} \mid \tilde{f} \in B(\tilde{X}, \tilde{\Sigma}); \langle \delta_{\tilde{x}} \tilde{P}, \tilde{f} \rangle \in B(\tilde{X}, \tilde{\Sigma})\}$ is equal to $B(\tilde{X}, \tilde{\Sigma})$, the space of the bounded and $\tilde{\Sigma}$-measurable functions, because $\mathfrak{A}$ contains all the continuous function and is closed under monotonic limits. Hence, if $\tilde{f}$ is measurable, then $\tilde{P} \tilde{f}(x) = \langle \delta_{\tilde{x}} \tilde{P}, \tilde{f} \rangle$ is also measurable.

In particular, for every $\tilde{A} \in \tilde{\Sigma}$, $\tilde{P}(\cdot, \tilde{A})$ is a measurable function. Hence, $\tilde{P}(\tilde{x}, \tilde{A})$ defined in (3.1) is indeed a Markov transition probability.

Let us define the functions:

$$i_A(x) = \sum_{n=0}^{\infty} (I_{A^c} P)^n 1_A(x), \tag{3.2}$$

$$j_A(x) = \lim_{k \to \infty} P^k i_A(x). \tag{3.3}$$

We have ([6], Chapter III) $P i_A \leqq i_A$ and $i_A \geqq 1_A$. In the case that the process is induced by a transition probability, $P i_A(x)$ is the probability that x enters A at least once, $j_A(x)$ is the probability that x enters A infinitely many times.

The probabilistic definition of a Harris process is: There exists a set N with $m(N)=0$ such that for each $x \notin N$ and for every set A with $m(A)>0$ we have $j_A(x)=1$ (see [7] and [10]).

Theorem 3.2. *If the Markov process (X, Σ, m, P) is a Harris process as it is defined in section 1, then the process induced by the transition probability $\tilde{P}(\tilde{x}, \tilde{A})$ given by Lemma 3.1 is a Harris process in the probabilistic definition.*

Proof. Let (X, Σ, m, P) be a Harris process where there is an integer k so that $Q_k>0$. Let $q_k(x, y)$ be the integral kernel of Q_k. Hence, there are two positive numbers ε, δ, so that if we define

$$E_x = \{y \mid q_k(x, y) > \varepsilon\},$$

then we can find a set A with $m(A)>0$ such that $m(E_x)>\delta$ for each $x \in A$.

Let B be a set with $m(B)>1-\delta/2$. Then $x \in A \Rightarrow m(B \cap E_x) > \delta/2$, and therefore:

$$P^k 1_B(x) \geqq \int_{B \cap E_x} q_k(x, y)\, m(dy) \geqq \varepsilon\, m(B \cap E_x) \geqq \frac{\varepsilon\,\delta}{2} 1_A(x) \qquad \text{a.e.}$$

Let F be any set with $m(F)>0$, let $B=\left\{x \,\middle|\, \sum_{j=1}^{n} p^j 1_F(x) \geqq 1\right\}$. If n is sufficiently large, then $m(B) \geqq 1-\delta/2$. Hence:

$$\sum_{j=1}^{n+k} P^j 1_F \geqq P^k \sum_{j=1}^{n} P^j 1_F \geqq P^k 1_B \geqq \frac{\varepsilon\,\delta}{2} 1_A.$$

Let $\tau 1_A = 1_{\tilde{A}}$, $\tau 1_F = 1_{\tilde{F}}$. We have

$$\sum_{j=1}^{n+k} \tilde{P}^j(\tilde{x}, \tilde{F}) \geqq \frac{\varepsilon\,\delta}{2} 1_{\tilde{A}}(\tilde{x}),$$

hence for each $\tilde{x} \in \tilde{A}$ there exists an integer $1 \leqq j \leqq n+k$ such that

$$\tilde{P}^j(\tilde{x}, \tilde{F}) \geqq \frac{\varepsilon\,\delta}{2(n+k)}.$$

Hence:

$$\tilde{P} i_{\tilde{F}}(\tilde{x}) \geqq \tilde{P}^j i_{\tilde{F}}(\tilde{x}) \geqq \tilde{P}^j 1_{\tilde{F}}(\tilde{x}) \geqq \frac{\varepsilon\,\delta}{2(n+k)}.$$

By Lemma 2.3 we have $\sum_{k=0}^{\infty} \tilde{P}^k(\tilde{x}, \tilde{A}) = \infty$ a.e. $\tilde{m}$ and hence (see [6], Chapter III) $j_A(\tilde{x})=1$ a.e. $\tilde{m}$ and $\inf_{\tilde{x} \in A} \tilde{P} i_{\tilde{F}}(x) \geqq \frac{\varepsilon\,\delta}{2(n+k)} > 0$. It follows from Proposition 7 of [2] that for all $\tilde{x} \in \tilde{X}$, $j_{\tilde{A}}(\tilde{x}) \leqq j_{\tilde{F}}(x)$. In particular $j_{\tilde{A}}(\tilde{x})=1 \Rightarrow j_{\tilde{F}}(\tilde{x})=1$.

Let $\tilde{G} \in \tilde{\Sigma}$ with $\tilde{m}(\tilde{G})>0$, by Lemma 2.1 there exists $\tilde{F} \subset \tilde{G}$ such that $\tau^{-1} 1_{\tilde{F}} = 1_F$ and $m(F)>0$.

Let $\tilde{N} = \{\tilde{x} \mid j_{\tilde{A}}(\tilde{x}) < 1\}$. We have $m(N)=0$ and for each $x \notin N$:

$$j_{\tilde{G}}(\tilde{x}) \geqq j_{\tilde{F}}(\tilde{x}) = 1.$$

So, the process $(\tilde{X}, \tilde{\Sigma}, \tilde{m}, \tilde{P})$ is a Harris process in the probabilistic definition.

Theorem 3.3. *If the process induced by the transition probability $\tilde{P}(\tilde{x}, \tilde{A})$ is a Harris process, then the process (X, Σ, m, P) is also a Harris process.*

Proof. Let $(\tilde{X}, \tilde{\Sigma}, \tilde{m}, \tilde{P})$ be a Harris process. Then $\tilde{P}^n = \tilde{Q}_n + \tilde{R}_n$ where $\tilde{Q}_n$ is an integral operator on $L_\infty(\tilde{X}, \tilde{\Sigma}, \tilde{m})$ and for some n, $\tilde{Q}_n \not\equiv 0$. Let $Q_n = \Lambda^{-1} \tilde{Q}_n \Lambda$, $R_n = \Lambda^{-1} \tilde{R}_n \Lambda$. Then $P^n = Q_n + R_n$; clearly $Q_n \not\equiv 0$.

Let $\mathfrak{B}$ be the field generated by rectangles in $X \times X$, let us define the charge π on $\mathfrak{B}$ by $\pi(A \times B) = \langle Q_n 1_A, 1_B \rangle$. Let $\{E_k\} \subset \mathfrak{B}$ where $E_k = \bigcup_i A_i^k \times B_i^k$ is a finite union of disjoint rectangles. Let $1_{\tilde{A}_i} = \Lambda 1_{A_i}$, $1_{\tilde{B}_i} = \Lambda 1_{B_i}$, $\tilde{E}_k = \bigcup_i \tilde{A}_i^k \times \tilde{B}_i^k$, $\tilde{\pi}(\tilde{A} \times \tilde{B}) = \langle \tilde{Q}_n 1_{\tilde{A}}, 1_{\tilde{B}} \rangle$. Then it is easy to see that $\tilde{\pi}$ can be extended to a measure on $\tilde{\Sigma} \times \tilde{\Sigma}$, and $\dfrac{d\tilde{\pi}}{d\tilde{m} \times \tilde{m}} = \tilde{q}_n(\tilde{x}, \tilde{y})$ where $\tilde{q}_n(\tilde{x}, \tilde{y})$ is the integral kernel of the operator $\tilde{Q}_n$ (for details see [6], Chapter V). But

$$\pi(E_k) = \sum_i \langle Q_n 1_{A_i^k}, 1_{B_i^k} \rangle = \sum_i \langle \tilde{Q}_n 1_{\tilde{A}_i^k}, 1_{\tilde{B}_i^k} \rangle = \tilde{\pi}(\tilde{E}_k).$$

Let $E_k \searrow \Phi$, then

$$\tilde{m} \times \tilde{m}(\tilde{E}_k) = \sum_i m(\tilde{A}_i^k)\, \tilde{m}(\tilde{B}_i^k) = \sum_i m(A_i^k)\, m(B_i^k) = m \times m(E_k) \searrow 0$$

and therefore $\pi(E_k) = \tilde{\pi}(\tilde{E}_k) \searrow 0$, because $\tilde{\pi} \prec \tilde{m} \times \tilde{m}$.

By the Extension Theorem for measures, π can be extended as a measure on $\Sigma \times \Sigma$. Let $F \in \Sigma \times \Sigma$ with $m \times m(F) = 0$, then for each $\delta > 0$ there exists a set $E = \bigcup_i A_i \times B_i$, a countable union of rectangles with $E \supset F$ and $m \times m(E) < \delta$. Let $\tilde{E} = \bigcup_i \tilde{A}_i \times \tilde{B}_i$, then $\tilde{m} \times \tilde{m}(\tilde{E}) < \delta$. But $\tilde{\pi} \prec \tilde{m} \times \tilde{m}$ and for each $\varepsilon > 0$, if δ is small enough, we have $\tilde{\pi}(\tilde{E}) < \varepsilon$. Hence:

$$\pi(F) \leqq \pi(E) = \sum_i \langle Q_n 1_{A_i}, 1_{B_i} \rangle = \sum_i \langle \tilde{Q}_n 1_{\tilde{A}_i}, 1_{\tilde{B}_i} \rangle = \tilde{\pi}(\tilde{E}) < \varepsilon,$$

but ε is arbitrary, hence $\pi(F) = 0$ and $\pi \prec m \times m$. Let $q_n = \dfrac{d\pi}{dm \times m}$; it is easy to see that $Q_n f(x) = \int q_n(x, y) f(y)\, m(dy)$, hence Q_n is an integral operator and $P^n = Q_n + R_n$. Thus (X, Σ, m, P) is a Harris process.

Theorem 3.4. *Let (X, Σ, m, P) be an ergodic and conservative Markov process, then the following are equivalent:*

(a) *The process is a Harris process.*

(b) *There exists a set A and an integer n, and $\varepsilon > 0$, $\alpha > 0$ such that for each set B with $m(B) > 1 - \varepsilon$ we have $P^n 1_B \geqq \alpha 1_A$ a.e.*

(c) *There exists a set A such that for each set E with $m(E) > 0$ there exists an integer n and $\alpha > 0$ (n and α may depend on E) such that $\sum_{k=1}^{n} P^k 1_E \geqq \alpha 1_A$ a.e.*

(d) *The same as* (c) *but only if each $E \subset A$ with $m(E) > 0$.*

Proof. (a) $\Rightarrow$ (b) and (b) $\Rightarrow$ (c): See the proof of Theorem 3.2. (c) $\Rightarrow$ (d) trivial. (d) $\Rightarrow$ (a): for each $E \subset A$ we have $\sum_{k=1}^{n} P^k 1_E \geqq \alpha 1_A$ a.e. for some n and α, hence we have

$$\sum_{k=1}^{n} \tilde{P}\, 1_{\tilde{E}}(\tilde{x}) \geqq \alpha\, 1_{\tilde{A}}(\tilde{x})$$

for *all* $\tilde{x} \in \tilde{X}$ where $1_{\tilde{E}} = \tau\, 1_E$, $1_{\tilde{A}} = \tau\, 1_A$. If the process induced by the transition probability $\tilde{P}(\tilde{x}, \tilde{A})$ is not a Harris process, then by [6], Chapter V, Theorem A, for almost every $\tilde{x}$ there exists a set $\tilde{A}_{\tilde{x}}$ with $\tilde{m}(\tilde{A}_{\tilde{x}}) = 1$ and $\sum_{n=1}^{\infty} \tilde{P}^n(\tilde{x}, \tilde{A}_{\tilde{x}}) = 0$. In particular, we can find a $\tilde{x} \in \tilde{A}$ and a set $\tilde{E} \subset \tilde{A} \cap \tilde{A}_{\tilde{x}}$ with $\tilde{E} \in \tilde{\Sigma}^1$ and $\tilde{m}(\tilde{E}) > 0$ (by Lemma 2.2), hence $\sum_{n=1}^{\infty} \tilde{P}^n(\tilde{x}, \tilde{E}) = 0$, a contradiction. So, $(\tilde{X}, \tilde{\Sigma}, \tilde{m}, \tilde{P})$ is a Harris process and by Theorem 3.3 (X, Σ, m, P) is a Harris process.

Remark. The condition (d) of this theorem is weaker then the condition given in [11]. Hence, the condition given there implies that the process is a Harris process.

4. On Quasi-Compact Operators on $L_\infty(m)$

In [1] some conditions are given on the operator ${}^A T$, in the notation of that paper. We are going to prove that these conditions are equivalent to quasi-compactness.

Theorem 4.1. *Let (X, Σ, m, P) be an ergodic and conservative Markov process. Then the following are equivalent:*

(a) *There exists* no *invariant pure charge*[1].

(b) *Let R be a contraction on $L_\infty(m)$ with $0 \leqq R \leqq P$ and $R\,1 \neq 1$ (for example $R = I_A P$ where $m(A) > 0$). Then $\|R^n\|_\infty \searrow 0$.*

(c) *Let R be as in* (b). *Then $\sum_{n=1}^{\infty} R^n\, 1 \in L_\infty(m)$.*

(d) *There exists a unique functional $\mu \in L_\infty^*(m)$ such that $\mu P = \mu$ and μ is a measure equivalent to m.*

(e) *Let μ be an invariant measure and denote*

$$L_\infty^0(m) = \{ f \in L_\infty(m) \mid \textstyle\int f\, d\mu = 0 \}. \tag{4.1}$$

Then $\overline{(I-P)\, L_\infty(m)} = L_\infty^0(m)$.

(f) *For each $f \in L_\infty(m)$ we have $\left\| \frac{1}{n} \sum_{k=1}^{n} P^k f - \int f\, d\mu \right\|_\infty \xrightarrow[n \to \infty]{} 0$ where μ is an invariant measure.*

(g) *The process is a Harris process and there exists* no *invariant pure charge.*

(h) *P is a quasi-compact operator on $L_\infty(m)$.*

(i) *There exists invariant measure μ and $(I-P)\, L_\infty(m) = L_\infty^0(m)$ where $L_\infty^0(m)$ is defined in* (4.1).

(j) *$\frac{1}{n} \sum_{k=1}^{n} P^k$ converges in the operator norm to a projection of $L_\infty(m)$ on the one dimensional space of the constants.*

[1] It is sufficient to consider only positive charges, because if ν is any invariant charge, observe the positive and negative parts of it, $\nu = \nu^+ - \nu^-$. But $\nu P = \nu$, $\nu^+ P \geqq \nu^+$ and $\nu^- P \geqq \nu^-$ and $P\,1 = 1$ implies $\nu^+ P = \nu^+$ and $\nu^- P = \nu^-$, hence if there exists any invariant charge (or pure charge) there exists also a positive invariant charge.

Proof. (a) $\Rightarrow$ (b): Let $\nu \in L_\infty^*(m)$ with $\nu R = \nu$, then $\nu P \geqq \nu R = \nu \Rightarrow \nu P = \nu$. Let $\nu = \nu_1 + \nu_2$ where ν_1 is a measure and ν_2 is a pure charge. Then $\nu_1 P \geqq \nu_1 \Rightarrow \nu_1 P = \nu_1 \Rightarrow \nu_2 P = \nu_2 \Rightarrow \nu_2 = 0$ and ν is a measure, equivalent to m, because of the ergodicity of the process. But $m(\{R1<1\})>0$, hence $\langle \nu R, 1\rangle = \langle \nu_1 R1\rangle < \langle \nu_1 1\rangle$ a contradiction, hence $\nu R = \nu \Rightarrow \nu = 0$. Consider the orthogonal complement of the closure of the range of the operator $I-R$, i.e.

$$\overline{(I-R)\, L_\infty(m)}^\perp = \{\nu \in L_\infty^*(m) | \nu R = \nu\} = \{0\}.$$

By the Hahn-Banach theorem $\overline{(I-R)\, L_\infty(m)} = L_\infty(m)$, in particular for each $\varepsilon>0$, there exists a function $g \in L_\infty(m)$ such that $\|g - Rg - 1\|_\infty \leqq \varepsilon$. Therefore,

$$\left\|\frac{1}{n}\sum_{k=1}^{n} R^k 1\right\|_\infty \leqq \left\|\frac{1}{n}\sum_{k=1}^{n} R^k(1-g+Rg)\right\|_\infty + \left\|\frac{1}{n}\sum_{k=1}^{n} R^k(g-Rg)\right\|_\infty \leqq \varepsilon + \frac{2\|g\|_\infty}{n},$$

but $\frac{2\|g\|_\infty}{n}$ tends to zero and ε is arbitrary, hence $\left\|\frac{1}{n}\sum_{k=1}^{n} R^k 1\right\|_\infty \xrightarrow[n\to\infty]{} 0$. But $R^n 1$ is a decreasing sequence and R is a positive operator, hence

$$\|R^n\|_\infty = \|R^n 1\|_\infty \leqq \left\|\frac{1}{n}\sum_{k=1}^{n} R^k 1\right\|_\infty \xrightarrow[n\to\infty]{} 0.$$

(b) $\Rightarrow$ (c): $\|R^n\|_\infty \searrow 0$, hence the operator R has *no* spectrum points on the unit circle. In particular 1 is not a spectrum point and $(I-R)^{-1}$ is a bounded operator on $L_\infty(m)$. Hence, there exists a function $g \in L_\infty(m)$ such that $(I-R)\,g = 1$, thus:

$$\left\|\sum_{n=1}^{N} R^n 1\right\|_\infty = \left\|\sum_{n=1}^{N} R^n(I-R)\,g\right\|_\infty = \|Rg - R^{N+1} g\|_\infty \leqq 2\|g\|_\infty.$$

(c) $\Rightarrow$ (d): Assume that there exists an invariant pure charge ν, so there exists a set A with $m(A)<1$ such that $\nu I_A = I_A$.

Let $R = I_A P$, hence $\nu R = \nu$ and $0 \leqq R \leqq P$, $R1 \neq 0$. Thus $\nu\left(\sum_{n=1}^{N} R^n 1\right) = N \cdot \nu(1)$, but $\sum_{n=1}^{\infty} R^n 1$ is bounded, therefore $\nu = 0$. The set $\{\nu \in L_\infty^*(m) | \nu(1) = 1, \nu > 0\}$ is convex and compact in the weak* topology and P maps this set into itself. Hence by the Fixed Point Theorem there exists a functional $\mu \in L_\infty^*(m)$ such that $\mu P = \mu$. Let $\mu = \mu_1 + \mu_2$ where μ_1 is a measure and μ_2 is a pure charge, then:

$$\mu_1 P \geqq \mu_1 \Rightarrow \mu_1 P = \mu_1 \Rightarrow \mu_2 P = \mu_2 \Rightarrow \mu_2 = 0 \Rightarrow \mu = \mu_1.$$

By the ergodicity of the process, μ is unique and equivalent to m.

(d) $\Rightarrow$ (e): By the Hahn-Banach Theorem:

$$\overline{(I-P)\, L_\infty(m)}^\perp = \{\nu | \nu P = \nu\} = \{\alpha\, \mu\}.$$

But $L_\infty^0(m)^\perp = \{\alpha\, \mu\}$, hence $\overline{(I-P)\, L_\infty(m)} = L_\infty^0$.

(e)⇒(f): For any $f \in L_\infty(m)$ we have $(f - \int f\,d\mu) \in L_\infty^0(m)$, hence for each $\varepsilon > 0$ there exists a function g such that $\|f - \int f\,d\mu - g + Pg\| \leqq \varepsilon$. Therefore:

$$\left\|\frac{1}{n}\sum_{k=1}^{n} P^k - \int f\,d\mu\right\|_\infty = \left\|\frac{1}{n}\sum_{k=1}^{n} P^k(f - \int f\,d\mu)\right\|_\infty$$
$$\leqq \left\|\frac{1}{n}\sum_{k=1}^{n} P^k(f - \int f\,d\mu - g + Pg)\right\|_\infty + \left\|\frac{1}{n}\sum_{k=1}^{n} P^k(g - Pg)\right\|_\infty \leqq \varepsilon + \frac{2\|g\|_\infty}{n},$$

but ε is arbitrary and $2\|g\|_\infty \cdot n^{-1}$ tends to zero. Hence

$$\left\|\frac{1}{n}\sum_{k=1}^{n} P^k f - \int f\,d\mu\right\|_\infty \xrightarrow[n\to\infty]{} 0.$$

(f)⇒(g): For every A there exists an integer n such that $\frac{1}{n}\sum_{k=1}^{n} P^k 1_A \geqq \frac{1}{2}\mu(A)$. In particular $\sum_{k=1}^{n} P^k 1_A > \alpha > 0$ and by Theorem 3.4 the process is a Harris process. Assume that ν is an invariant pure charge, then there exists a set A with $\nu(A) = 0$. On the other hand $\nu\left(\sum_{k=1}^{n} P^k 1_A\right) > \alpha > 0$, a contradiction.

(g)⇒(h): Let us first prove the following propositions.

Proposition 1. *Let the process be a Harris process, then there exists an integer k such that P^k can be written as a sum $P^k = Q + R$ where Q is a positive compact operator on $L_\infty(m)$ $(0 < Q < P)$ and P^k is ergodic.*

Proof. There exists an integer n such that P^n can be written as $P^n = Q_1 + R_1$ where Q is an integral operator with the bounded kernel $0 \not\equiv q(x, y) < K$. By Theorem D, Chapter V of [6] there exists a minimal set W and an integer d such that $1_W + P 1_W + \cdots + P^{d-1} 1_W = 1$ and $P^d 1_W = 1_W$. Hence P^{jd+1} is ergodic for each j. Take $jd \geqq 2n$ and $k = jd + 1$. Then P^k is ergodic and

$$P^k = P^{k-2n} P^{2n} = P^{k-2n}(Q_1 + R_1)^2 = P^{k-2n} Q_1^2 + P^{k-2n}(Q_1 R_1 + R_1 Q_1 + R_1^2).$$

Denote

$$Q = P^{k-2n} Q_1^2, \qquad R = P^{k-2n}(Q_1 R_1 + R_1 Q_1 + R_1^2).$$

Q_1^2 and therefore also Q is a compact operator on $L_\infty(m)$.

Proposition 2. *If P has* no *invariant pure charge, then P^k, for any integer k, has the same property.*

Proof. Assume that ν is a pure charge and $\nu = \nu P^k$, then

$$0 = \nu(I - P^k) = \sum_{n=0}^{k-1} \nu P^n (I - P),$$

i.e. $\sum_{n=0}^{k-1} \nu P^n$ is a functional invariant under P. This implies that $\sum_{n=0}^{k-1} \nu P^n$ is a measure and therefore there exists $0 < n < k$ such that νP^n is not a pure charge and hence $\nu P^k = \nu P^n P^{k-n}$ is not a pure charge, a contradiction.

Proof of (g)$\Rightarrow$(h). According to the previous propositions, we can find an integer k such that $P^k = Q + R$ where P^k is ergodic and has no invariant pure charges, Q is compact and $R1 \neq 1$. Define $Q_n = P^{nk} - R^n$. It is easy to see that Q_n is a compact operator and the proof of (a)$\Rightarrow$(b) shows that $\|R^n\|_\infty \searrow 0$. So, if n is sufficiently large, then $P^{nk} = Q_n + R^n$ where Q_n is compact and $\|R^n\| < 1$. Thus P is a quasi-compact operator on $L_\infty(m)$ (see [12], Lemma V.3.1).

(h)$\Rightarrow$(i): P is a quasi-compact operator, hence there exists an integer k such that $P^k = Q + R$ where Q is compact and $\|R\|_\infty < 1$.

Let ν be any charge. Then νQ is a measure because if $A_n \searrow \emptyset$, then the compactness of Q implies that $\|Q 1_{A_n}\|_\infty \to 0$ and therefore $\nu Q(A_n) \searrow 0$.

Let ν be an invariant pure charge. Then we have $\nu Q = 0$ and $\nu R = \nu$, but $\langle \nu, R1 \rangle < 1$, a contradiction. Thus, there is no invariant pure charge and the proof of (c)$\Rightarrow$(d) gives that there exists an invariant measure μ, and the space $L^0_\infty(m)$ is invariant under P. It is clear that P is a quasi-compact operator on $L^0_\infty(m)$ and hence every spectrum points on the unit circle is an eigenvalue, but 1 is not an eigenvalue (because of the ergodicity), hence $(I-P)^{-1}$ is a bounded operator on $L^0_\infty(m)$. So, $(I-P)\, L^0_\infty(m) = L^0_\infty(m)$.

(i)$\Rightarrow$(j): Let $f \in L_\infty(m)$. Then $f - \int f\, d\mu \in L^0_\infty(m)$, and there exists $g \in L^0_\infty(m)$ such that $g - Pg = f - \int f\, d\mu$ and $\|g\|_\infty \leqq 2\, \|(I-P)^{-1}\|_{L_\infty} \|f\|_\infty$. Hence:

$$\left\| \frac{1}{n} \sum_{k=1}^{n} P^k f - \int f\, d\mu \right\|_\infty = \left\| \frac{1}{n} \sum_{k=1}^{n} P^k \left(f - \int f\, d\mu\right) \right\|_\infty$$
$$= \left\| \frac{1}{n} \sum_{k=1}^{n} P^k (I-P)\, g \right\| \leqq \frac{2}{n}\, \|g\|_\infty \leqq \frac{4}{n}\, \|(I-P)^{-1}\|_{L^0_\infty}\, \|f\|_\infty .$$

Thus $\frac{1}{n} \sum_{k=1}^{n} P^k f$ converges to $\int f\, d\mu$ uniformly in the unit ball of $L_\infty(m)$.

(j)$\Rightarrow$(a): $\frac{1}{n} \sum_{k=1}^{n} P^k f$ converges in $L_\infty(m)$ and $\frac{1}{n} \sum_{k=1}^{n} P^k 1 = 1$, therefore $\lim \frac{1}{n} \sum_{k=1}^{n} m P^k = \mu$ where μ is an invariant measure and $\left\| \frac{1}{n} \sum_{k=1}^{n} P^k 1_A - \mu(A) \right\|_\infty \to 0$ for every set A. Assume that ν is an invariant pure charge. Then there exists a set A such that $\mu(A) > 0$ and $\nu(A) = 0$. So, $\frac{1}{n} \sum_{k=1}^{n} \nu P^k(A) = 0$. But,

$$\left\| \frac{1}{n} \sum_{k=1}^{n} P^k 1_A - \mu(A) \right\|_\infty \xrightarrow[n\to\infty]{} 0 \Rightarrow \frac{1}{n} \sum_{k=1}^{n} \nu P^k(A) \xrightarrow[n\to\infty]{} \mu(A),$$

a contradiction. Thus, there exist *no* invariant pure charges.

Remark. Some parts of this theorem can be found in [1] and [5]; we give them here for completeness.

5. On Sets A where P_A Is Quasi-Compact

A set A is called "bounded" if P_A is quasi-compact. Such sets are discussed in [1]; it is proved there that if $\operatorname{supp} f \subset A$ and $\int f\, d\mu = 0$ where $\mu P_A = \mu$ then

$\left\| \sum_{n=1}^{N} P^n f \right\|_\infty$ is bounded. We shall prove that such sets exist if and only if the process is a Harris process.

Lemma 5.1. *$\Lambda P_A \Lambda^{-1} = I_{\tilde{A}} \sum_{n=0}^{\infty} \tilde{P}(I_{\tilde{A}^c}\tilde{P}) I_{\tilde{A}}$ where $1_{\tilde{A}} = \Lambda 1_A$.*

Proof. Let $f \geqq 0$ and $\operatorname{supp} f \subset A$, then

$$I_A \sum_{n=0}^{N} P(I_{A^c}P)^n I_A f \nearrow P_A f \quad \text{and} \quad \Lambda I_A \sum_{n=0}^{N} P(I_{A^c}P)^n I_A \Lambda^{-1} = I_{\tilde{A}} \sum_{n=0}^{N} \tilde{P}(I_{\tilde{A}^c}\tilde{P})^n I_{\tilde{A}}$$

and Lemma 2.3 gives $\Lambda P_A \Lambda^{-1} \tilde{f} = I_{\tilde{A}} \sum_{n=0}^{\infty} \tilde{P}(I_{\tilde{A}^c}\tilde{P})^n I_{\tilde{A}} \tilde{f}$ where $\tilde{f} = \Lambda f$.

Denote $\tilde{P}_{\tilde{A}} = I_{\tilde{A}} \sum_{n=0}^{N} \tilde{P}(I_{\tilde{A}^c}\tilde{P})^n I_{\tilde{A}}$. This is an operator on $L_\infty(\tilde{A}, \tilde{\Sigma}_A, \tilde{m}\, I_{\tilde{A}})$. It is easy to see that P_A is quasi-compact if and only if $\tilde{P}_{\tilde{A}}$ is quasi-compact.

Theorem 5.2. *A process has "bounded" sets if and only if it is a Harris process.*

Proof. P_A is quasi-compact and so is $\tilde{P}_{\tilde{A}}$. By Theorem 4.1 $\tilde{P}_{\tilde{A}}$ is a Harris operator. Assume that $\tilde{P}$ is not a Harris operator, then for almost every $\tilde{x} \in X$ there exists a set $\tilde{E}_{\tilde{x}}$ with $\tilde{m}(\tilde{E}_{\tilde{x}}) = 1$ and $\sum_{n=1}^{\infty} \tilde{P}^n(\tilde{x}, \tilde{E}_{\tilde{x}}) = 0$. It is easy to see that for almost every $\tilde{x} \in A$ we have $\sum_{n=1}^{\infty} \tilde{P}_{\tilde{A}}^n(\tilde{x}_1 \tilde{E}_x \cap \tilde{A}) = 0$ where $\tilde{P}_{\tilde{A}}(x, E)$ is the transition probability which induces the operator $\tilde{P}_{\tilde{A}}$. Hence, $\tilde{P}_{\tilde{A}}$ is not a Harris operator, a contradiction. Thus $\tilde{P}$ is a Harris operator, and by Theorem 3.3 P is also a Harris operator.

Conversely, if P is a Harris operator, then by the proof of Theorem 3.2 there exists a set A such that for every set E with $m(E) > 0$ there is an integer n and $\alpha > 0$ such that $\sum_{k=1}^{n} P^k 1_E \geqq \alpha 1_A$ a.e. By Lemma 3 of [5] we have that for each $E \subset A$ with $m(E) > 0$,

$$\sum_{k=1}^{n} P^k{}_A 1_E \geqq I_A \sum_{k=1}^{n} P^k 1_E \geqq \alpha 1_A.$$

Let ν be a pure charge invariant under P_A, then there exists a set $E \subset A$ with $m(E) > 0$ but $\nu(A) = 0$. Hence, $\alpha \nu(A) \leqq \sum_{k=1}^{n} \nu P_A^k(E) = n \nu(E) = 0$, a contradiction. Thus, P_A has *no* invariant pure charges and by Theorem 4.1 P_A is quasi-compact and A is a "bounded" set.

In [1] is proved that if the function f is supported on a "bounded" set and $\int f\, d\mu = 0$ where μ is the invariant measure then

$$\left\| \sum_{k=1}^{n} P^k f \right\|_\infty \leqq K \|f\|_\infty \tag{5.1}$$

where K is a constant independent on n. Theorem 5.2 shows that this is proved only for Harris process. The next theorem will show that (5.1) can be satisfied only in the case of a Harris process.

Theorem 5.3. *Let μ be a σ-finite invariant measure. Let A be a set with $\mu(A)=1$ and for each $E\subset A$ we have*

$$\left\|\sum_{k=1}^{n} P^k\left(1_A-\frac{1}{\mu(E)}1_E\right)\right\|_\infty \leqq K,$$

where K is a constant independent on n. Then the process is a Harris process.

Proof. By Egorov's Theorem there exists a set $B\subset A$ such that $\sum_{k=1}^{n} P^k 1_A \xrightarrow[n\to\infty]{} \infty$ uniformly on B. Hence, there exists an integer n such that $\sum_{k=1}^{n} P^k 1_A \geqq 2K 1_B$. Therefore $2K1_B\leqq \sum_{k=1}^{n} P^k 1_A \leqq K+\frac{1}{\mu(E)}\sum_{k=1}^{n} P^k 1_E$ or $\sum_{k=1}^{n} P^k 1_E\geqq \mu(E)K1_B$ and by Theorem 3.4 the process is a Harris process.

Remark. In [13] the Property (5.1) is proved for random walks assuming that the measure of the random walk is not orthogonal to the Lebesgue Measure. Theorem 5.3 shows that this assumption is necessary.

6. On the Existence of Finite Invariant Measures

In this section we shall prove Theorem E of [6], Chapter IV by methods developed in this paper.

Theorem 6.1. *The following are equivalent:*

(a) *There exists a finite invariant measure μ equivalent to m.*

(b) *There exists no set A with $m(A)>0$ for which*

$$\lim_{n\to\infty}\left\|\frac{1}{n}\sum_{k=1}^{n} P^k 1_A\right\|_\infty = 0. \tag{6.1}$$

Proof. We need only prove that (b)$\Rightarrow$(a). It is sufficient to prove that if there exists no finite invariant measure then there exists a set A such that $\nu\in L_\infty^*(m)$ and $\nu P=\nu$ then $\nu(A)=0$ and this implies (6.1). (See [9], Theorem 1.)

Assume that for each A with $m(A)>0$ there exists $\nu\in L_\infty^*(m)$ with $\nu(A)>0$ and $\nu P=\nu$.[2] If $\tilde A\in\tilde\Sigma$ and $\tilde m(\tilde A)>0$ then there exist a set $A\in\Sigma$ such that $m(A_1)>0$ and $\tau 1_{A_1}\leqq 1_{\tilde A}$ (by the results of Section 2). Hence, by our assumption there exists $\nu\in L_\infty^*(m)$ such that $\nu(A_1)>0$ and $\nu P=\nu$. This implies $\tilde\nu(\tilde A)>0$ and $\tilde\nu\tilde P=\tilde\nu$ where $\tilde\nu$ is the measure $\tau^{*-1}\nu$. Consider the set

$$\tilde E=\left\{\tilde x\,\middle|\,\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\tilde P^k 1_{\tilde A}(\tilde x) \text{ exists}\right\}.$$

Assume $\tilde m(\tilde E^c)>0$, then by our assumption there exists a measure $\tilde\nu$ with $\nu(E^c)>0$ and $\tilde\nu\tilde P=\tilde\nu$. There are two possibilities:

(i) $\tilde\nu(\tilde A)>0$. Let us observe the Markov process $(\tilde X,\tilde\Sigma,\tilde\nu,\tilde P)$. By Birkhoff's Ergodic Theorem (Theorem A of [6], Chapter VII)

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\tilde P^k 1_{\tilde A}(\tilde x) \text{ exists a.e. } \tilde\nu,$$

in particular there exists $\tilde x\in\tilde E^c$ such that the limit exists, a contradiction.

[2] By the footnote on p. 268 we may assume that ν is positive.

(ii) $\tilde{\nu}(\tilde{A})=0$. $\tilde{\nu}\tilde{P}=\tilde{\nu}$ implies $\sum_{k=1}^{\infty}\tilde{P}^k 1_{\tilde{A}}=0$ a.e. $\tilde{\nu}$, in particular there exists $\tilde{x}\in\tilde{E}^c$ such that $\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\tilde{P}^k 1_{\tilde{A}}(\tilde{x})=0$. A contradiction.

Hence: $\tilde{m}(\tilde{E})=1$. Consider now the set $\tilde{E}\supset\tilde{E}_1=\left\{\tilde{x}\,\middle|\,\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\tilde{P}^k 1_{\tilde{A}}(\tilde{x})>0\right\}$.

Assume $\tilde{m}(\tilde{E}_1^c\cap\tilde{A})>0$, then by our assumption there exists a measure $\tilde{\nu}$ with $\tilde{\nu}(\tilde{E}_1^c\cap\tilde{A})>0$ and $\tilde{\nu}\tilde{P}=\tilde{\nu}$. By Birkhoff's Ergodic Theorem $\lim_{n\to\infty}\frac{1}{n}\sum_{n=1}^{n}\tilde{P}^k 1_{\tilde{A}}(\tilde{x})$ exists and is positive a.e. $\tilde{\nu}$, in particular there exists $\tilde{x}\in\tilde{E}_1^c\cap\tilde{A}$ such that $\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\tilde{P}^k 1_{\tilde{A}}(\tilde{x})>0$, a contradiction. Hence $m(\tilde{E}_1)>0$ $\big($or $\tilde{m}(\tilde{E}_1\cap\tilde{A})=\tilde{m}(\tilde{A})\big)$. By the Dominated Convergence Theorem the limit $\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\tilde{m}\tilde{P}^k(\tilde{A})$ exists and is positive, for each $\tilde{A}\in\tilde{\Sigma}$ with $\tilde{m}(\tilde{A})>0$.

Define $\tilde{\mu}(\tilde{A})=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\tilde{m}\tilde{P}^k(\tilde{A})$. $\tilde{\mu}$ is a finite invariant measure equivalent to $\tilde{m}$, and hence $\mu=\tau^*\tilde{\mu}$ is a finite invariant measure equivalent to m. So Theorem 6.1 is proved.

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