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On the Uniqueness of Diffusions*

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§ 1. Introduction

Diffusion Operator. A diffusion operator $\mathbf{A}$ on a manifold $\mathbf{E}$ is a second-order differential operator, expressed in local coordinates (x) as

$$\mathbf{A} f(x)=\sum a_i(x)\,\partial_i f(x)+\tfrac{1}{2}\sum \mathbf{S}_{ij}(x)\,\partial_i\,\partial_j f(x) \tag{1}$$

with all coefficients continuous, and the symmetric matrix $(\mathbf{S}_{ij}(x))$ everywhere non-negative.

In this paper $(\mathbf{A}, a, \mathbf{S})$ has this basic meaning as above. We say $(a, \mathbf{S})$ represents $\mathbf{A}$ in the given coordinates. Properties of $(a, \mathbf{S})$ which are independent of coordinates chosen will be referred to as properties of $\mathbf{A}$. Smoothness conditions are examples of this. In case the manifold $\mathbf{E}$ is Euclidean and (x) any affine coordinate system, we say $\mathbf{A}$ is bounded when $(a, \mathbf{S})$ is bounded. Finally $\mathbf{A}$ (or $\mathbf{S}$) is called nondegenerate if $\det(\mathbf{S}_{ij}(x))$ is strictly positive everywhere; otherwise $\mathbf{A}$ is called degenerate, at the points where the inequality fails.

A Fellerian semigroup $\mathbf{P}=(\mathbf{P}_t)_{t\geqq 0}$ on $\mathbf{E}$, a semicompact, (i.e. a locally compact Hausdorff space with a countable base) is a strongly continuous semigroup of positive contractions on $\mathscr{C}_0(\mathbf{E})$, the Banach space of all continuous functions vanishing at infinity, with $\mathbf{P}_0=$ the identity. A Fellerian semigroup $\mathbf{P}$ is said to be a Fellerian A-diffusion on a manifold $\mathbf{E}$ if the infinitesimal generator $\mathbf{G}(\mathbf{P})$ of $\mathbf{P}$ coincides with $\mathbf{A}$ on a suitable class of functions, which in the following will always be taken to be $\mathscr{C}_k^2(\mathbf{E})$, the set of all twice continuously differentiable functions with compact support.

More generally, a standard semigroup P is called an A-diffusion if:

(1) P is realizable on the space of paths continuous before death; and

(2) the characteristic operator of $\mathbf{P}$ is defined for all $\mathscr{C}^2$-functions, and coincides there with $\mathbf{A}|_{\mathscr{C}^2}$. (Compare [4: § 5.18].)

Note 1. We are not interested in the axiomatics. However, because of the local character of $\mathbf{A}$, the sample paths of a Fellerian $\mathbf{A}$-diffusion $\mathbf{P}$ are surely continuous. Also a term $-b(x)f(x)$ may be added to $\mathbf{A} f(x)$ with $b(x)\geqq 0$ interpreted as the killing rate, but the modification will not be essential.

In this paper the state space $\mathbf{E}$ will always be Polish, i.e., a topologically complete, separable, metrizable space, with the Borel tribe as its measurable class. But when the Fellerian property or diffusion is referred to $\mathbf{E}$ will always be assumed, respectively, a semicompact or a manifold.

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Note 2. A manifold will always be separable, Hausdorff, in this paper. Boundaries will not be allowed, as diffusions with boundary conditions are much more difficult and are not discussed here.

Note 3. By a standard semigroup we mean the transition semigroup of a "standard process." See [12b, Chapter XIV, § 2]. By a process is meant a stochastic process as defined in [3, p. 46], so that a "Markov evolution" is really a collection of Markov processes $(xP; \Omega; X)$, one for each x in $\mathbf{E}$. In the following the underlying set Ω and the function $\mathbf{X}$ will be omitted when there is no possible confusion. In particular, for a good transition-semigroup $\mathbf{P}$ on a Polish space E, Ω will be a good subset on $\mathbf{E}^{[0,\infty)}$ and $\mathbf{X}(t;\cdot)$ the projections. In this case, for each μ in $\mathbf{M}_1(\mathbf{E})$, the set of all probability measures on $\mathbf{E}$, the Markov process determined by $\mathbf{P}$ and μ will be identified with the corresponding probability measure on Ω and denoted by $\mu\mathbf{P}$ or simply $x\mathbf{P}$ if μ is concentrated at the point x.

The Problem. A fundamental problem in diffusion theory is, for given $(\mathbf{A}, \mathbf{E})$, the existence and uniqueness of an $\mathbf{A}$-diffusion.

This problem has been exhaustively studied, we mention only the references [19, 13], and also [9, 6, 15, 7], for their relevance to this paper.

The general existence of a Fellerian $\mathbf{A}$-diffusion on a Euclidean space has been proved by Tanaka ([16]) and also by Krylov ([10]) by explicit contructions, assuming the boundedness and continuity of the coefficients, as well as the non-degeneracy.

Under the same conditions assumed by Tanaka and Krylov, the uniqueness is finally established by Stroock and Varadhan ([16]). The present paper is devoted to the removal of some of the restrictions in the last-mentioned paper. The state space may be a manifold and degeneracy will also be allowed under some compensating restrictions of smoothness.

§ 2. The Statements of the Main Theorems

Theorem 1. *If A is non-degenerate, then the Fellerian A-diffusion is unique.*

In view of the uniqueness theorem of Stroock-Varadhan in Euclidean space, our approach is naturally a localization. This will be carried out in § 7, after the necessary preparations in §§ 3–5. The degenerate case is much more complicated, as some additional requirements are obviously necessary for the uniqueness. With the aid of §§ 3–5, and a result (§ 6) on the Ito-Doob integral, we will arrive at the following theorem. (The pertinent definition of "outward non-degeneracy" will be given in § 3, (6), ad that of quasi-square-root in § 6, (35).)

Theorem 2. *(The uniqueness of the A-diffusion when A is degenerate.)*

There is only one Fellerian A-diffusion provided a countable open covering $\{\mathbf{W}, \mathbf{U}_1, \mathbf{U}_2, \ldots\}$ of $\mathbf{E}$ can be found so that:

(a) *$\mathbf{A}$ is non-degenerate on $\mathbf{W}$.*

(b) *$\mathbf{A}$ is $\mathbf{U}_n$-outwardly non-degenerate.*

(c) *The closure $\overline{\mathbf{U}}_n$ of $\mathbf{U}_n$ is representable as the closed unit ball in some local coordinates, relative to which a is Lipschitz, and there exists a Lipschitz quasi-square-root $\mathbf{R}$ of $\mathbf{S}$.*

Note 1. The Fellerian property can be weakened to:

(d) $m_{\mathbf{U}}\,\mathbf{P}$ is Fellerian on $\mathbf{U}$, if $\mathbf{U}=\mathbf{U}_n$ or if $\mathbf{U}$ is a region relatively compact in $\mathbf{W}$, with smooth boundary. (For the notation $\mathbf{M}_{\mathbf{U}}$, see below.)

Note 2. The patching-up theorem of Courrège-Priouret, to be explained presently, can also be applied to the existence (construction) problem. We have then the following existence theorem:

If $\mathbf{A}$ satisfies the condition (a), (b), (c) above with respect to an open covering $\{\mathbf{W}, \mathbf{U}_1, \mathbf{U}_2, \ldots\}$, then there exists one (and only one) $\mathbf{A}$-diffusion with property (d).

Theorems 3–10, which are of some independent interest, are scattered in §§3–6. The rest of this section is devoted to the description of the patching-up technique. It played a very basic role in several works, e.g. [13, 6, 16] cited above. It is thoroughly studied in [2] and will be utilized in this article.

First, for a given subset $\mathbf{U}$ of the state space $\mathbf{E}$, and a given path $\mathbf{X}$ (i.e., a measurable function from $[0, \infty]$ to $\mathbf{E}$), the (first) exit time of $\mathbf{X}$ from $\mathbf{U}$ is defined by

$$\tau_{\mathbf{U}}(\mathbf{X})=\inf\{s\geqq 0: \mathbf{X}(s)\notin \mathbf{U}\}. \tag{2}$$

(The infimum of the empty set is ∞ by convention.) The stopping on leaving $\mathbf{U}$ is the operation

$$a_{\mathbf{U}}\colon \mathbf{X}\mapsto \mathbf{Y}, \tag{3}$$

defined by

$$\mathbf{Y}(t)=\mathbf{X}\big(\text{minimum of } t \text{ and } \tau_{\mathbf{U}}(\mathbf{X})\big). \tag{3'}$$

If a "cemetery" Δ is singled out, so that $\mathbf{E}$ is a "cemetery-equipped space," the killing on leaving $\mathbf{U}$ is the operation

$$m_{\mathbf{U}}\colon \mathbf{X}\mapsto \mathbf{Z}, \tag{4}$$

defined by

$$\mathbf{Z}(t)=\begin{cases}\mathbf{X}(t), & \text{if } t<\tau_{\mathbf{U}}(\mathbf{X}):\\ \Delta, & \text{if } t\geqq\tau_{\mathbf{U}}(\mathbf{X}).\end{cases} \tag{4'}$$

Let now $\big((x\mathbf{P}); \Omega; (\mathbf{X}(t,\omega))\big)$ be a Markov evolution with transition-semigroup $\mathbf{P}$. (That is to say a "Markov process" as defined in [4, §3.1] or [1, §1.3].) Under some conditions $\big((x\mathbf{P}); \Omega; (\mathbf{Y}(t,\omega))\big)$ and $\big((x\mathbf{P}); \Omega; (\mathbf{Z}(t,\omega))\big)$ will also be Markov evolutions, with transition-semigroup: $\mathbf{Q}'$ and $\mathbf{Q}''$ respectively. In this case we will write $\mathbf{Q}'=a_{\mathbf{U}}\,\mathbf{P}$ and $\mathbf{Q}''=m_{\mathbf{U}}\,\mathbf{P}$.

The Patching-Up Theorem of Courrège-Priouret

Suppose $\mathscr{U}$ is a countable open covering of E, a semicompact, and $(\mathbf{P}_{\mathbf{U}}\colon \mathbf{U}\in\mathscr{U})$ is a system of "standard semigroups" on $\mathbf{E}$. Then there is exactly one standard semigroup $\mathbf{P}$ such that $a_{\mathbf{U}}\,\mathbf{P}=a_{\mathbf{U}}\,\mathbf{P}_{\mathbf{U}}$, if and only if,

$$a_{\mathbf{U}\cap\mathbf{V}}\,\mathbf{P}_{\mathbf{U}}=a_{\mathbf{U}\cap\mathbf{V}}\,\mathbf{P}_{\mathbf{V}} \qquad \text{for } \mathbf{U}, \mathbf{V} \text{ in } \mathscr{U}; \tag{5}$$

that is, the family $(\mathbf{P}_{\mathbf{U}}\colon \mathbf{U}\in\mathscr{U})$ is compatible. In case all $\mathbf{P}_{\mathbf{U}}$ are diffusions, the compatibility condition can be replaced by

$$m_{\mathbf{U}\cap\mathbf{V}}\,\mathbf{P}_{\mathbf{U}}=m_{\mathbf{U}\cap\mathbf{V}}\,\mathbf{P}_{\mathbf{V}}. \tag{5'}$$

These assertions are the main part of [2: Theorem 2.3.1 and Theorem 2.4.2, respectively].

§ 3. The Outward Regularity

Given an **A**-diffusion **P** on **E**, a point θ on the boundary $\partial\mathbf{U}$ of an open set **U** is said to be regular, for leaving $\overline{\mathbf{U}}$, the closure of **U**, if $\theta\mathbf{P}\{\bar{\tau}_{\mathbf{U}}=0\}=0$, where $\bar{\tau}_{\mathbf{U}}$ denotes, for ease of printing, the exit time from $\overline{\mathbf{U}}$. This section is devoted to a criterion of this regularity.

Suppose **U** to be smooth at θ, that is to say, coordinates $(x_1, \ldots, x_d)$ can be chosen on a neighborhood **V** of θ so that $x_i(\theta)=0$ and $\mathbf{U}\cap\mathbf{V}$ is the part of **V** where $x_1<0$. We say that **A** is **U**-outwardly non-degenerate at θ, if, corresponding to such coordinates, either

$$S_{11}(\theta)>0 \qquad \text{(second-order case)}, \tag{6'}$$

or

$$S_{11}(\theta)=0 \quad \text{and} \quad a_1(\theta)>0 \qquad \text{(first-order case)}. \tag{6''}$$

We say **A** is **U**-outwardly nondegenerate if it is so at every $\theta\in\partial\mathbf{U}$.

Theorem 3. *If* **A** *is* **U**-*outwardly non-degenerate at* θ, *then* θ *is regular.*

Proof. For the purpose of the proof, we suppose (as we may) the coordinates are chosen so that for a neighborhood **V** of θ, $\mathbf{U}\cap\mathbf{V}$ becomes the part of **V** where

$$x_2^2+\cdots+(1+x_1)^2<1, \tag{7}$$

and $(\partial\mathbf{U})\cap\mathbf{V}$ is the part where

$$x_2^2+\cdots+x_d^2+(1+x_1)^2=1, \tag{7'}$$

and $x_i(\theta)=0$. Corresponding to such coordinates, the criterion for **U**-outward non-degeneracy is trivially the same as (6).

Choose now a function $f\in\mathscr{C}_k^2$ which is given, near θ, by $f=x_1^2+2b\,x_1$. The positive constant b is chosen so that

$$\mathbf{A}f(\theta)=\mathbf{S}_{11}+2b\,a_1>0. \tag{8}$$

Consider a sequence of domains still to be selected, and shrinking to θ. According to the Zero-One Law, and noting that θ is neither a trap nor an exponentially-holding point, the proof will be completed when we prove that for every **C** of this sequence

$$\theta\mathbf{P}\{\text{exit from } \mathbf{C}=\text{entrance to } (\partial\mathbf{C})\setminus\overline{\mathbf{U}}\}\geqq\tfrac{1}{4}. \tag{9}$$

We will achieve this by the use of Dynkin's formula. Indeed from $\mathbf{A}f(\theta)>0$ we see that $\int f\,d\pi>0$, if **C** is small enough and if π is the exit distribution, on leaving **C**, for the process $\theta\mathbf{P}$. An appropriate choice of **C**, fulfilling (9), is the following:

C is a cylinder, having x_1-axis as axis, θ as center, height $2v$ and as basis a disk of radius $[1-(1-v)^2]^{\frac{1}{2}}$ where v is a small positive number.

On $\partial\mathbf{C}$, the boundary, f has the maximum $v(2b+v)$, on the top, and the minimum $-v(2b-v)$, on the bottom $\mathbf{B}=\{x_1=-v;\ x_2^2+\cdots+x_d^2+(1-v)^2\leqq 1\}$. Because of $\int f\,d\pi=\int_{f>0} f\,d\pi-\int_{f<0}(-f)\,d\pi>0$, we have

$$\pi((\partial\mathbf{C})\setminus\mathbf{B})\,v(2b+v)\geqq\int_{f>0} f\,d\pi>\int_{f<0}(-f)\,d\pi\geqq\pi(\mathbf{B})\,v(2b-v),$$

and therefore, with $(\partial \mathbf{C}) \cap \overline{\mathbf{U}} = \mathbf{B}$ and $\pi(\mathbf{C}) = 1$,

$$\pi\big((\partial \mathbf{C}) \setminus \overline{\mathbf{U}}\big) = \pi\big((\partial \mathbf{C}) \setminus \mathbf{B}\big) > (2b - v)/(4b) \geqq \tfrac{1}{4},$$

provided v is very small and less than b. Q.E.D.

§ 4. Speed-Change without Degeneracy

The theorems in these two sections, §4 and 5, are generalizations of a result implicit in [8], and discussed in some detail in [11]. Though we have a different motivation from that of the last cited paper, our approach is the same, namely, the use of the Skorohod machinery. Some details of the "$\varepsilon - \delta$", trivial though tedious, will be omitted. The generality of the treatment is dictated by its naturality, rather than by its main applications in §7, where only continuous paths are involved.

The Skorohod Space. Let $(\mathbf{E}, \rho)$ be a Polish space. For a compact interval $[c_1, c_2] = \mathbf{I}$, the Skorohod space $\mathscr{J}(\mathbf{I}; \mathbf{E})$ is the set of all functions from $\mathbf{I}$ to $\mathbf{E}$, which are right-continuous, have left-limits, and satisfy $\mathbf{X}(c_2) = \mathbf{X}(c_2 -)$. The topology on $\mathscr{J}(\mathbf{I}; \mathbf{E})$ will be always the Polish topology $\mathbf{J}_1$ of [14a, Definition 2.2] which can be defined by

$$\begin{aligned} &\operatorname{dist}(\mathbf{X}; \mathbf{Y}) = \inf\{\operatorname{dist}(\mathbf{X}; \theta; \mathbf{Y}) \colon \theta \in \operatorname{Aut} \mathbf{I}\}, \\ \text{with} \quad &\operatorname{dist}(\mathbf{X}; \theta; \mathbf{Y}) = \sup\{\rho(\mathbf{X}(t); \mathbf{Y}(\theta t)) + |t - \theta t| \colon t \in \mathbf{I}\} \end{aligned} \tag{10}$$

with θ ranging through all the order-automorphisms of $\mathbf{I}$. We will write $\mathscr{J}$ or $\mathscr{J}\mathbf{I}$ for $\mathscr{J}(\mathbf{I}; \mathbf{E})$ when there is no possible confusion. We also write $\mathscr{J}_K$ for it when $\mathbf{I} = [0; K]$.

There is a closely related space $\bar{\mathscr{J}}$ which is the set of all functions $\mathbf{X}$ from $\mathbf{I}$ to $\mathbf{E}$, right-continuous and with left-limits everywhere, but not necessarily continuous at c_2. Obviously $\mathscr{J}$ is a subset of $\bar{\mathscr{J}}$ and in fact it is a closed subset in the topology of $\bar{\mathscr{J}}$ to be described presently.

The space $\bar{\mathscr{J}}$ is factored as a product $\bar{\mathscr{J}} = \mathscr{J} \times \mathbf{E}$ by $\mathbf{X} \mapsto (\mathbf{X}', \mathbf{X}'')$ where

$$\begin{aligned} \mathbf{X}'(t) &= \begin{cases} \mathbf{X}(t) & \text{for } c_1 \leqq t < c_2 \\ \mathbf{X}(c_2 -) & \text{for } t = c_2 \end{cases} \\ \mathbf{X}'' &= \mathbf{X}(c_2) \end{aligned} \tag{11}$$

We equip $\bar{\mathscr{J}}$ with the product topology, which can be defined by a metric having the same form (10) as above.

Remark 1. $\mathbf{X} \mapsto \mathbf{X}'$ is a retraction of $\bar{\mathscr{J}}$ on $\mathscr{J}$. It will be called the natural retraction.

Remark 2. $\bar{\mathscr{J}}$ plays only a minor role, but its appearance is inevitable if we consider, for example, restricting $\mathbf{X} \in \mathscr{J}(\mathbf{I}; \mathbf{E})$ to a subinterval $\mathbf{I}_1$.

Lemma 1. *If* $\mathbf{X}_n \to \mathbf{X}_0$ *in* $\mathscr{J}(\mathbf{I}; \mathbb{R})$, *then* $\int_{\mathbf{I}} \mathbf{X}_n(t)\, dt \to \int_{\mathbf{I}} \mathbf{X}_0(t)\, dt$.

Lemma 2. *Let $h_n \to h_0$ in the topology of uniform convergence on compacts, in the space $\mathscr{C}(\mathbf{E}, \mathbf{F})$ of continuous functions from* **E** *to* **F**, *both Polish. Then $\mathbf{X}_n \to \mathbf{X}_0$ in $\bar{\mathscr{J}}(\mathbf{I}; \mathbf{E})$ implies $h_n \circ \mathbf{X}_n \to h_n \circ \mathbf{X}_0$ in $\bar{\mathscr{J}}(\mathbf{I}; \mathbf{F})$.*

These lemmas are as trivial as the following observations.

Remark 3. Let $c_3, c_3' \in (c_1, c_2)$, $\mathbf{X}, \mathbf{Y} \in \bar{\mathscr{J}}$ and let θ_1: $[c_1, c_3] \to [c_1, c_3']$, and θ: $[c_3, c_2] \to [c_3', c_2]$ be two order-isomorphisms. We can define $\operatorname{dist}(\mathbf{X}, \theta_i, \mathbf{Y})$ just as in (3) above. If θ is the common extension of θ_1 and θ_2, then we have

$$\operatorname{dist}(\mathbf{X}; \mathbf{Y}) \leqq \operatorname{dist}(\mathbf{X}; \theta; \mathbf{Y}) \leqq \operatorname{dist}(\mathbf{X}; \theta_1; \mathbf{Y}) + \operatorname{dist}(\mathbf{X}; \theta_2; \mathbf{Y}). \tag{12}$$

Remark 4 The above remark is used in the following situation. We have $\mathbf{X}$, $\mathbf{Y}$, c_3, c_3' and θ_1 as above, and moreover,

$$\sup\{\rho(\mathbf{X}(t); \mathbf{Y}(s)) : t \in [c_3, c_2], s \in [c_3', c_2]\} \leqq \varepsilon; \ |c_3 - c_3'| \leqq \delta. \tag{13}$$

In this case, for any order-isomorphism θ_2: $[c_3, c_2] \to [c_3', c_2]$, the last term of (4) is dominated by $(\varepsilon + \delta)$. Therefore

$$\operatorname{dist}(\mathbf{X}, \mathbf{Y}) \leqq \operatorname{dist}(\mathbf{X}; \theta_1; \mathbf{Y}) + (\varepsilon + \delta). \tag{14}$$

Speed-Change. The general notion of change of time associated with an increasing process is defined in [12a, Chapter VIII, D 13]. We are interested in the following special case.

Let v be a bounded and continuous function on the cemetery-equipped state space **E**, valued in $(0, \infty]$, and **X** a path in **E**. We define now

$$\begin{aligned} \varphi(t, \mathbf{X}) &= \int_0^t ds / v(\mathbf{X}(s)), \\ \psi(s, \mathbf{X}) &= \inf\{t : \varphi(t, \mathbf{X}) \geqq s\}, \end{aligned} \tag{15}$$

and also a new path **VX** by

$$\mathbf{VX}(s) = \mathbf{V}(\psi(s, \mathbf{X})) \quad \text{with } \mathbf{X}(\infty) = \Delta. \tag{16}$$

The operation **V** is called the speed-change associated with v.

If $((x\mathscr{P}); \Omega; \mathbf{X})$ is a Markov evolution with transition-semigroup **P**, and $((x\mathscr{P}); \Omega; \mathbf{VX})$ is a Markov evolution with transition-semigroup **Q**, then we will write

$$\mathbf{Q} = \mathbf{VP}.$$

The definitions φ, ψ, and **V** above are still meaningful when we are considering finite-interval paths. In particular when $\mathbf{X} \in \bar{\mathscr{J}}([0, k]; \mathbf{E}) = \bar{\mathscr{J}}_k$ and $k' \leqq \varphi(k, \mathbf{X})$, we can consider **VX** as an element of $\bar{\mathscr{J}}([0, k']; \mathbf{E})$.

Lemma 3. *Convergence $\mathbf{X}^n \to \mathbf{X}^0$ in $\bar{\mathscr{J}}_1$ implies the uniform convergence of $\varphi(; \mathbf{X}^n) \to \varphi(; \mathbf{X}^0)$ on $[0, 1]$, and also that of $\psi(; \mathbf{X}^n) \to \psi(; \mathbf{X}^0)$ on $[0, \mathbf{M}]$, provided $1/\mathbf{M} \geqq \sup v$.*

Proof. The pointwise convergence of $\varphi(; \mathbf{X}^n) \to \varphi(; \mathbf{X}^0)$ is a consequence of Lemma 1 and 2 above. Observing that the closure of $\{\mathbf{X}^n(t) : 0 \leqq t \leqq 1, n = 0, 1, \ldots\}$ is compact, v may be assumed bounded below by a strictly positive number, say $1/m$. Therefore the one-sided derivatives $1/v(\mathbf{X}^n((t \pm)))$ of $\varphi(; \mathbf{X}^n)$ are bounded by m, and the Ascoli-Arzela theorem implies the uniform convergence of $\varphi(;\mathbf{X}^n)\to\varphi(;\mathbf{X}^0)$. The second assertion is as easy to prove; in fact the one-sided derivatives of $\psi(;\mathbf{X})$ are always bounded by $\mathbf{M}$, for every $\mathbf{X}\in\bar{\mathscr{J}}_1$, while $\varphi(;\mathbf{X})$ and $\psi(;\mathbf{X})$ are inverse to each other.

Lemma 4. *If b is a continuity time of $\mathbf{X}^0$, and if $\mathbf{X}^n\to\mathbf{X}^0$ in $\bar{\mathscr{J}}_1$ then for any $\varepsilon>0$, there exist $\delta_1>0$ and $n_1\in\mathbb{N}$ such that $\rho(\mathbf{X}^n(t);\mathbf{X}^0(b))<\varepsilon$, whenever $n\geqq n_1$ and $|t-b|\leqq\delta_1$.*

This is an elementary fact in the Skorohod theory and the proof is omitted.

Theorem 4. *(Continuity of speed change on sample paths.)*

Let v be continuous and strictly positive on the Polish space $\mathbf{E}$, bounded above by $1/\mathbf{M}$. If $\mathbf{X}^0\in\bar{\mathscr{J}}([0,1],\mathbf{E})$ is continuous at $\psi(\mathbf{M};\mathbf{X}^0)$ then $\mathbf{X}^0$ is a continuity point of the speed-change $\mathbf{V}$ considered as a mapping from $\bar{\mathscr{J}}_1$ to $\bar{\mathscr{J}}_{\mathbf{M}}$.

Remark 5. It is easy to see that the continuity of $\mathbf{X}^0$ at $\psi(\mathbf{M};\mathbf{X}^0)$ is necessary. This is a peculiar feature of the Skorohod theory because here the time interval is, so to speak, not rigid enough.

Proof. Assuming $\mathbf{X}^n\to\mathbf{X}^0$ in $\bar{\mathscr{J}}_1$, a sequence $(\theta_n)\subset\mathrm{Aut}[0,1]$ can be so chosen that

$$\lim \operatorname{dist}(\mathbf{X}^0;\theta_n;\mathbf{X}^n)=0. \tag{17}$$

Let

$$\theta'_n=\varphi(;\mathbf{X}^n)\circ\theta_n\circ\psi(;\mathbf{X}^0),$$

so that

$$\sup_{0\leqq s\leqq\mathbf{M}}\rho\big(\mathbf{V}\mathbf{X}^0(s);\mathbf{V}\mathbf{X}^n(\theta'_n s)\big)=\sup_{0\leqq s\leqq\mathbf{M}}\rho\big(\mathbf{X}^0(\psi(s,\mathbf{X}^0));\mathbf{X}^n(\theta_n\psi(s,\mathbf{X}^0))\big)\to 0 \tag{18}$$

and that, by Lemmas 2 and 3,

$$\lim_n \sup_{0\leqq s\leqq\mathbf{M}}|\theta'_n s-s|=0. \tag{19}$$

We have yet to tailor (θ'_n) into automorphisms of $[0,\mathbf{M}]$.

Given $\varepsilon>0$, we choose (n_1,δ_1) as in Lemma 4, with $b=\psi(\mathbf{M};\mathbf{X}^0)$. Also, applying Lemma 4 again to the uniform convergence: $\psi(;\mathbf{X}^n)\to\psi(;\mathbf{X}^0)$, we can find (n_2,δ_2) such that

$$|\psi(s,\mathbf{X}^n)-b|\leqq\delta_1,\quad \text{whenever } n\geqq n_2 \text{ and } |s-\mathbf{M}|\leqq 2\delta_2. \tag{20}$$

Here $n_2\in\mathbb{N}$ and $0<\delta_1<\varepsilon$. By (19) we now choose n_3 so that

$$\mathbf{M}-2\delta_2<\theta'_n(\mathbf{M}-\delta_2)<\mathbf{M}\quad \text{for } n\geqq n_3. \tag{21}$$

Let us apply Remark 4 to this case, with θ_1 there the restriction of θ'_n to $[0,\mathbf{M}-\delta_2]$; thus (18), (20), (21), together with (14), yield

$$\overline{\lim_n}\operatorname{dist}(\mathbf{V}\mathbf{X}^0;\mathbf{V}\mathbf{X}^n)\leqq 0+(2\varepsilon+\delta_2)<3\varepsilon.\quad \text{Q.E.D.}$$

Remark 6. The "natural" lifetime $\varphi(1,\mathbf{X}^n)$ of $\mathbf{V}\mathbf{X}^n$ depends on n, so that $\mathbf{V}\mathbf{X}^n\to\mathbf{V}\mathbf{X}^0$ is nonsense unless we restrict the consideration to a fixed interval (note that $\varphi(1,\mathbf{X})\geqq\mathbf{M}$; therefore $\varphi(\mathbf{M};\mathbf{X})\leqq 1$). However, by a normalization of time scale, we can interpret the convergence in the following way.

Let $k=\varphi(1;\mathbf{X}^n)/\varphi(1;\mathbf{X}^0)$. (Note that $k_n\to 1$.) We have then $\mathbf{V}\mathbf{X}^n(k_n\cdot)\to\mathbf{V}\mathbf{X}^0(\cdot)$ in the space $\mathscr{J}([0,\varphi(1;\mathbf{X}^0)];\mathbf{E})$.

Remark 7. Let $0\leqq c<\varphi(1;\mathbf{X}^0)$, and let $\psi(c;\mathbf{X}^0)$ be a time of continuity of $\mathbf{X}^0$, then $\mathbf{V}\mathbf{X}^n\to\mathbf{V}\mathbf{X}^0$ in the space $\bar{\mathscr{J}}_c$. This is proved by the same kind of tailoring.

Preservation of Fellerian Property. A transition-semigroup $\mathbf{P}$ on a Polish space $\mathbf{E}$ will be called a Skorohod semigroup if it satisfies the following conditions:

(a) $\mathbf{P}$ is conservative: $\mathbf{P}_t 1=1$ for $t\geqq 0$.

(b) $\mathbf{P}$ can be canonically realized on the space $\mathscr{J}(\mathbb{R}_+;\mathbf{E})$ of all paths which are right-continuous and have left-limits everywhere on $\mathbb{R}_+$.

(c) The canonical realization is strongly Markov and quasi-left-continuous on $\mathbb{R}_+$.

(d) For any $\mathbf{T}>0$, $x\mapsto x\mathbf{P}\in\mathbb{M}_1(\bar{\mathscr{J}}_\mathbf{T})$ is weakly continuous on $\mathbf{E}$; here $x\mathbf{P}$ is considered as a probability measure on $\bar{\mathscr{J}}_\mathbf{T}$.

Remark 8. In particular, $\mathbf{P}$ is a Hunt semigroup. The axiomatics will not be studied here. We observe that, because of (c), one also has:

(c′) For any initial distribution $\mu\in\mathbb{M}_1(\mathbf{E})$ the process $\mu\mathbf{P}$ does not have any fixed time of continuity. Therefore $\mu\mathbf{P}$ can be considered as a probability measure on $\mathscr{J}_\mathbf{T}$, $\mathbf{T}>0$.

We also observe that, as a trivial consequence of (c) and (d),

(d′) $\mathbf{P}_t\mathscr{C}_b(\mathbf{E})\subset\mathscr{C}_b(\mathbf{E})$, $t\geqq 0$.

Theorem 5. *(Preservation of the Skorohod character of semigroup under speed-change.)*

Under the same assumption as in Theorem 4, $\mathbf{VP}$ is a Skorohod semigroup *if* $\mathbf{P}$ *is one.*

Proof. If $\mathbf{P}$ is canonically realized as the Markov evolution $((x\mathbf{P});\mathscr{J}(\mathbb{R}_+;\mathbf{E});\mathbf{X})$, then $\mathbf{VP}$ is the transition semigroup of $((x\mathbf{P}),\mathscr{J}(\mathbb{R}_+;\mathbf{E});\mathbf{VX})$. Therefore (a) and (b) are trivial as $\mathbf{VX}\in\mathscr{J}(\mathbb{R}_+;\mathbf{E})$ whenever $\mathbf{X}\in\mathscr{J}(\mathbb{R}_+;\mathbf{E})$. By the transformation rule of stopping times, (c) is valid. It remains to check (d). For any $x\in\mathbf{E}$ and $\mathbf{T}>0$, we know $\psi(\mathbf{TM};\mathbf{X})$ is almost $(x\mathbf{P}-)$ surely a continuity time of $\mathbf{X}$, because of the quasi-left-continuity. Therefore by Theorem 4, $\mathbf{V}$: $\bar{\mathscr{J}}[0,\mathbf{T}]\to\bar{\mathscr{J}}[0,\mathbf{TM}]$ is almost $(x\mathbf{P}-)$ surely continuous. By (d), we see finally that the mapping $x\in\mathbf{E}\mapsto x\mathbf{VP}\in\mathbb{M}_1(\bar{\mathscr{J}}[0,\mathbf{TM}])$ is weakly continuous. Q.E.D.

Let $\mathbf{P}$ be a Fellerian semigroup on $\mathbf{E}$. It is canonically extended to be a conservative Fellerian semigroup on $\mathbf{E}_\Delta$ (the one-point compactification). In this way $\mathbf{P}$ is always considered as a Skorohod semigroup on $\mathbf{E}_\Delta$. As a corollary of the above, we have the following theorem of Hunt-Lamperti. See [11, §2].

Theorem 6. *(Preservation of Fellerian property under speed-change.)*

If v is strictly positive and continuous on $\mathbf{E}_\Delta$, *and* $\mathbf{P}$ *is Fellerian on* $\mathbf{E}$, *then* $\mathbf{VP}$ *is also Fellerian on* $\mathbf{E}$, *and* $\mathbf{G}(\mathbf{VP})=v\,\mathbf{G}(\mathbf{P})$.

The second assertion holds because of the definition of Dynkin's characteristic operator and Dynkin's theorem. See [4, Theorem 5.5].

§ 5. Speed-Change with Degeneracy

A degenerate speed-field v is a function on $\mathbf{E}$, nonnegative and null somewhere. For example, v may be vanishing at infinity or v may be the indicator function of a certain proper subset $\mathbf{U}$ of $\mathbf{E}$. The associated speed-change in the last case will be the stopping on leaving $\mathbf{U}$, as we will define

$$\varphi(t;\mathbf{X})=\infty \quad \text{if} \quad t \geqq \tau_{\mathbf{U}}, \qquad \text{where } \mathbf{U}=\{x; v(x)>0\}, \tag{22}$$

while ψ and $\mathbf{V}$ are defined as before. See (15) and (16) of §4.

Remark 1. Some sort of limitation is necessary in order to guarantee the continuity of $\mathbf{V}$. This is seen from the following example.

Example. Let $\mathbf{E}=\mathbb{R}$, $v=$ the indicator function of $(-1, +1)$, and, for $n \in \mathbb{N}_0$,

$$\mathbf{X}^n(t)=\begin{cases}\alpha_n+t, & t<\frac{1}{2};\\ \alpha_n+1-t, & t\geqq\frac{1}{2};\end{cases}$$

where

$$\alpha_1<\alpha_2<\cdots\to\alpha_0=\tfrac{1}{2}.$$

The Continuity of the Exit Time and the Detachment Time from a Set

Lemma 1. *If* $\mathbf{F}$ *is closed* $\tau=\tau_{\mathbf{F}}$ *is upper-semi-continuous on* $\bar{\mathscr{J}}_1$.

Proof. In proving the semi-continuity of $\mathbf{X}^0$ we may assume $\tau(\mathbf{X}^0)<1$, since otherwise the proof is trivial. Given $\varepsilon>0$, there is a $t_0<1$ such that

$$\tau(\mathbf{X}^0)\leqq t_0<\tau(\mathbf{X}^0)+\varepsilon \quad \text{and} \quad \mathbf{X}^0(t_0)\in\mathbf{E}\setminus\mathbf{F}.$$

If now $\mathbf{X}^n\to\mathbf{X}^0$, i.e., $\operatorname{dist}(\mathbf{X}^0,\theta_n,\mathbf{X}^n)\downarrow 0$ for some sequence $(\theta_n)\subset\operatorname{Aut}[0,1]$, we see that, for large n, $\mathbf{X}^n(\theta_n t_0)\in\mathbf{E}\setminus\mathbf{F}$. Therefore

$$\theta_n t_0\geqq\tau(\mathbf{X}^n) \quad \text{and} \quad \overline{\lim_n}\,\tau(\mathbf{X}^n)\leqq t_0=\tau(\mathbf{X}^0). \quad \text{Q.E.D.}$$

If $\mathbf{X}\in\mathscr{J}(\mathbf{I},\mathbf{E})$ and $b\in\mathbf{I}$, the trace of $\mathbf{X}$ up to time b is defined as

$$\Gamma_b(\mathbf{X})=\text{the closure of } \{\mathbf{X}(t)\colon t\leqq b\}. \tag{23}$$

The first detachment time of $\mathbf{X}$ from $\mathbf{U}$ is

$$\hat{\tau}_{\mathbf{U}}(\mathbf{X})=\inf\{s;\Gamma_s(\mathbf{X})\setminus\mathbf{U}\neq\emptyset\}. \tag{24}$$

Lemma 2. *If* $\mathbf{U}$ *is open,* $\hat{\tau}=\hat{\tau}_{\mathbf{U}}$ *is lower-semi-continuous on* $\bar{\mathscr{J}}_1$.

Proof. Let $\mathbf{X}^n\to\mathbf{X}^0$. If $b=\hat{\tau}(\mathbf{X}^0)\leqq 1$, then for any $\varepsilon>0$ the compact set $\Gamma_{b-\varepsilon}(\mathbf{X}^0)\subset\mathbf{U}$, therefore, when n is large, $\Gamma_{b-\varepsilon}(\mathbf{X}^0)\subset\mathbf{U}$. And we have $\hat{\tau}(\mathbf{X}^n)\geqq b-\varepsilon$, so that $\underline{\lim}_n\,\hat{\tau}(\mathbf{X}^n)\geqq\hat{\tau}(\mathbf{X}^0)$ follows. If $\hat{\tau}(\mathbf{X}^0)=\infty$ then $\hat{\tau}(\mathbf{X}^n)=\infty$ for lage n. Q.E.D.

Lemma 3. *If* $\mathbf{U}$ *is open and* $\hat{\tau}_{\mathbf{U}}(\mathbf{Z})=\bar{\tau}_{\mathbf{U}}(\mathbf{X})$ *for an* $\mathbf{X}\in\bar{\mathscr{J}}_1$, *then* $\mathbf{X}$ *is a continuity point of* $\tau_{\mathbf{U}}$.

Proof. The inequality $\tilde{\tau}_{\mathbf{U}}\leqq\tau_{\mathbf{U}}\leqq\bar{\tau}_{\mathbf{U}}$ is generally valid. Now apply the above two lemmas.

Remark 2. We will be concerned with "Hunt processes" only; thus $\hat{\tau}_{\mathbf{U}}=\tau_{\mathbf{U}}$ is (almost surely) valid when $\mathbf{U}$ is open. (Actually valid when $\mathbf{U}$ is analytic.) Also if $\mathbf{X}(\tau_{\mathbf{U}}(\mathbf{X}))\in\mathbf{E}\setminus\overline{\mathbf{U}}$, then $\bar{\tau}_{\mathbf{U}}(\mathbf{X})=\tau_{\mathbf{U}}(\mathbf{X})$. So the only troublesome situation is when $\mathbf{X}(\tau_{\mathbf{U}}(\mathbf{X}))\in\partial\mathbf{U}$. In this case the identity $\bar{\tau}_{\mathbf{U}}(\mathbf{X})=\tau_{\mathbf{U}}(\mathbf{X})$ means, in the context of Markov processes, that $\mathbf{X}(\tau_{\mathbf{U}}(\mathbf{X}))$ is regular for leaving $\overline{\mathbf{U}}$. This is discussed in §3.

Continuity of Speed-Change with Degeneracy. In the rest of this section v will be positive, bounded above by $1/\mathbf{M}$ and continuous on the set $\mathbf{U}=\{x\colon v(x)>0\}$, which we assume to be open. The exit times from $\mathbf{U}$, $\overline{\mathbf{U}}$, and the detachment time from $\mathbf{U}$ are denoted by τ, $\bar{\tau}$, and $\hat{\tau}$, respectively.

Two numbers $b=\hat{\tau}(\mathbf{X}^0)$ and $c=\varphi(b-;\mathbf{X}^0)$ will be important in our case-by-case study of the continuity of $\mathbf{V}\colon\bar{\mathscr{J}}_1\to\bar{\mathscr{J}}_{\mathbf{M}}$ at the point $\mathbf{X}^0$.

Cace (I). $b>\psi(\mathbf{M};\mathbf{X}^0)$. Just as in Theorem 4, the continuity of $\mathbf{V}$ at $\mathbf{X}^0$ is ensured by the continuity of $\mathbf{X}^0$ at $\psi(\mathbf{M};\mathbf{X}^0)$.

Case (II). $b=\psi(\mathbf{M};\mathbf{X}^0)$, $c=\mathbf{M}$. This is by no means a general situation but it is a prototype for a more complicated situation. We assert that the same criterion as in Case (I) applies, i.e., the continuity of $\mathbf{V}$ at $\mathbf{X}^0$ is ensured by the continuity of $\mathbf{X}^0$ at b.

For a given $\varepsilon>0$, we can find (n_1,δ_1) as in Lemma 4, §4. Choose then $\delta_2>0$ so that

$$\delta_2<\mathbf{M}\delta_1,\qquad \delta_2<\varepsilon. \tag{25}$$

$$\psi(\mathbf{M}-\delta_2,\mathbf{X}^0)\quad\text{lies in}\quad\left(b-\frac{\delta_1}{2},b\right)\quad\text{and is a continuity time of }\mathbf{X}^0.$$

Now by Lemma 3 of §4, and (25), $\psi(\mathbf{M}-\delta_2,\mathbf{X}^n)\in(b-\delta_1,b]$ for n large, say $n\geqq n_2$. Noting that $\psi(;\mathbf{X})$ has one-sided derivatives always dominated by $\mathbf{M}$, we have

$$\psi(s,\mathbf{X}^n)\in(b-\delta_1,b+\delta_1)\quad\text{for } n\geqq n_2\quad\text{and}\quad \mathbf{M}-\delta_2\leqq s\leqq\mathbf{M}. \tag{26}$$

According to Remark 7 of §4, $\mathbf{V}\mathbf{X}^n\to\mathbf{V}\mathbf{X}^0$ in the sense of $\bar{\mathscr{J}}[0,\mathbf{M}-\delta_2]$. On the other hand, we have, by the choice of (n_1,δ_1) above and (26),

$$\rho(\mathbf{V}\mathbf{X}^n(s);\mathbf{V}\mathbf{X}^0(\mathbf{M}))\leqq\varepsilon\quad\text{for } n\geqq\max(n_1,n_2)\quad\text{and}\quad s\in[\mathbf{M}-\delta_2,\mathbf{M}]. \tag{27}$$

These two facts together imply, by the Remark 4 of §4, that

$$\overline{\lim_n}\,\operatorname{dist}(\mathbf{V}\mathbf{X}^n,\mathbf{V}\mathbf{X}^0)\leqq 2\varepsilon+\delta_2<3\varepsilon,\qquad\text{in }\bar{\mathscr{J}}_{\mathbf{M}}.$$

Case (III). $\mathbf{X}^0(b-)\in\mathbf{U}$, $\tau(\mathbf{X}^0)=b$, $c<\mathbf{M}$. In this case b is necessarily a jumping time of $\mathbf{X}^0$ as $\mathbf{X}^0(b)\notin\mathbf{U}$. We will show that $\mathbf{V}$ is continuous at $\mathbf{X}^0$ provided the following condition holds true *(the regular-or-trapping condition)*:

Either

$$\bar{\tau}(\mathbf{X}^0)=b=\hat{\tau}(\mathbf{X}^0), \tag{28R}$$

or

$$\mathbf{X}^0(t)=\mathbf{X}^0(b)\quad\text{for } t\geqq b\qquad(\mathbf{X}^0(b)\text{ is a "trap"}). \tag{28T}$$

Let

$$\mathbf{X}^n\to\mathbf{X}^0,\quad\text{i.e.,}\quad\lim_n\operatorname{dist}(\mathbf{X}^0;\theta_n;\mathbf{X}^n)=0,\qquad\text{for }(\theta_n)\subset\operatorname{Aut}\mathbf{I}.$$

First of all we can find a sequence (θ'_n) of functions on $[0, c]$, continuous, vanishing at 0, strictly increasing, and satisfying

$$\lim_n \operatorname{dist}(\mathbf{VX}^0; \theta'_n; \mathbf{VX}^n)=0 \qquad \psi(\theta'_n c; \mathbf{X}^n)=\theta_n b. \tag{29}$$

To achieve this, let $k'_n=\theta_n b/b$, $\theta_0 b=b$, and $\bar{\mathbf{Y}}^n=\mathbf{X}^n(k'_n \cdot)$ restricted to $[0, b]$. Let us take the natural retraction (Remark 1, §4) $\mathbf{Y}^n$ of $\bar{\mathbf{Y}}^n$. For n sufficiently large, say $n \geqq n_0$, $\mathbf{Y}^n \to \mathbf{Y}^0$ in $\mathscr{J}([0, b]; \mathbf{U})$; therefore the Remark 6 of §4 is applicable, resulting in

$$\mathbf{VY}^n(k_n \cdot) \to \mathbf{VY}^0 \quad \text{in } \bar{\mathscr{J}}_c, \tag{30}$$

with

$$k_n=\phi(b, \mathbf{Y}^n)/\phi(b, \mathbf{Y}^0)=\phi(\theta_n b-; \mathbf{X}^n)/c\, k'_n. \tag{30'}$$

Because $\mathbf{X}^n(k'_n b) \to \mathbf{X}^0(b)$, we have also

$$\mathbf{V}\bar{\mathbf{Y}}^n(k_n \cdot) \to \mathbf{V}\bar{\mathbf{Y}}^0 \quad \text{in } \bar{\mathscr{J}}_c, \quad \text{or} \quad \mathbf{VX}^n(k_n k'_n \cdot) \to \mathbf{VX}^0 \quad \text{in } \bar{\mathscr{J}}_{\mathbf{c}},$$

and (29) is proved if $\theta'_n s=k_n k'_n s$.

According to Remark 4, §4, it is now only left for us to prove the next assertion: For any $\varepsilon>0$, we can choose $n_1 \geqq n_0$ such that

$$\rho(\mathbf{VX}^n(s); \mathbf{VX}^0(c)) \leqq \varepsilon, \quad \text{for all } s \geqq \theta'_n c \quad \text{and} \quad n \geqq n_1. \tag{31}$$

In Case (28 **T**) of trap, this is trivial. Turning to Case (28 **R**), we first choose (n_2, δ_2), $n_2 \in \mathbb{N}$, $\delta_2>0$, such that $\rho(\mathbf{X}^n(t); \mathbf{X}^0(b))<\varepsilon$, whenever $\theta_n b \leqq t \leqq \theta_n v+\delta_2$ and $n \geqq n_2$. This means, putting $\delta_1=\delta_2/\mathbf{M}$, that

$$\rho(\mathbf{VX}^n(s); \mathbf{VX}^0(c))<\varepsilon, \quad \text{whenever } \theta'_n c \leqq s \leqq \theta'_n c+\delta_1 \text{ and } n \geqq \max(n_0, n_2). \tag{32}$$

The hypothesis $\bar{\tau}(\mathbf{X}^0)=\hat{\tau}(\mathbf{X}^0)$ is now invoked for Lemma 3, this section, resulting in $\tau(\mathbf{X}^n) \to \tau(\mathbf{X}^0)$, which means that

$$\theta_n b \leqq \tau(\mathbf{X}^n) \leqq \theta_n b+\delta_2, \quad \text{for } n \text{ large, say } n \geqq n_3. \tag{33}$$

Letting $n_1=\max(n_0, n_2, n_3)$ we see that the paths $\mathbf{VX}^n$ have no chance to violate (31), for $n \geqq n_1$.

Case (IV). $\tau(\mathbf{X}^0)=b$, $\mathbf{X}^0(b-)=\mathbf{X}^0(b)$, and $c<\mathbf{M}$. We can show that $\mathbf{V}$ is continuous at $\mathbf{X}^0$, provided the regular-or-trapping condition (28) is satisfied. The proof consists of a combination of the arguments in the Cases (II) and (III), and will be omitted.

Preservation of Fellerian Property Under Exit-Stopping. Summarizing the above discussions in the context of Markov processes, we have

Theorem 7. *(Preservation of Skorohod character of semigroups under degenerate speed-change.)*

Let $\mathbf{P}$ *be a Skorohod semigroup on* $\mathbf{E}$, *and* v *a bounded non-negative function on* $\mathbf{E}$, *continuous on* $\mathbf{U}=\{x: v(x)>0\}$, *which is assumed open. If every point of* $\partial \mathbf{U}$ *is either trapping or regular for leaving* $\bar{\mathbf{U}}$, *then* $\mathbf{VP}$ *is also a Skorohod semigroup.*

Proof. Whatever the initial distribution μ may be, almost $(\mu \mathbf{P})$ surely the conditions (III) or (I) are fulfilled and hence almost all sample paths are continuity points for $\mathbf{V}$: $\bar{\mathscr{J}}_1 \to \bar{\mathscr{J}}_{\mathbf{M}}$. The proof is now completed just as in Theorem 5. Q.E.D.

Theorem 8. *(Preservation of Fellerian property under speed-change.)*

Let $\mathbf{P}$ *be a Fellerian semigroup on a semicompact* $\mathbf{E}$ *(without cemetery) and let* v *be a bounded strictly positive continuous function on* $\mathbf{E}$. *Then* $\mathbf{VP}$ *is also Fellerian, and*

$$\mathbf{G}(\mathbf{VP}) \supset v\,\mathbf{G}(\mathbf{P}).$$

That is, for every f *in the domain of definition of* $\mathbf{G}(\mathbf{P})$, $G(\mathbf{P})\,f = v\,\mathbf{G}(\mathbf{P})\,f$.

Proof. In this case the cemetery Δ is a trap for the (extended) Skorohod semigroup $\mathbf{P}$ on $\mathbf{E}_\Delta$ and v is extended to be zero on $\{\Delta\} \supset \partial\mathbf{E}$. The second assertion is proved by Dynkin's theorem, cf. Theorem 6, §4. Q.E.D.

Theorem 9. *(Preservation of Fellerian property of a semigroup under stopping or killing.)*

Let $\mathbf{P}$ *be Fellerian on* $\mathbf{E}$, *and* $\mathbf{U}$ *an open set in* $\mathbf{E}$. *If every point on the boundary* $\partial\mathbf{U}$ *of* $\mathbf{U}$ *is either trapping, or regular for leaving* $\overline{\mathbf{U}}$, *then* $\mathbf{Q}' = a_{\mathbf{U}}\mathbf{P}$ *is Fellerian on* $\mathbf{E}$ *and* $\mathbf{Q}'' = m_{\mathbf{U}}\mathbf{P}$ *is Fellerian on* $\mathbf{U}$. *If, moreover,* $\mathbf{P}$ *is an* $\mathbf{A}$*-diffusion on* $\mathbf{E}$, *then* $\mathbf{Q}''$ *is an* $\mathbf{A}|_{\mathbf{U}}$*-diffusion on* $\mathbf{U}$.

Proof of the Second Assertion. If $f \in \mathscr{C}_0(\mathbf{U})$, then $\mathbf{Q}_t'' f = \mathbf{Q}_t' f$ and vanishes off $\mathbf{U}$, so that $\mathbf{Q}_t'' \mathscr{C}_0(\mathbf{U}) \subset \mathscr{C}_0(\mathbf{U})$. On the other hand $\mathbf{Q}''$ is obviously stochastically continuous on the semicompact $\mathbf{U}$. Now apply [4, Lemma 2.11]. The last assertion is obvious. Q.E.D.

§ 6. Representation of a Martingale as an Ito-Wiener Integral

This section is devoted to the generalization of a theorem of Doob in higher dimension. This almost trivial generalization is essential in Ito's approach to the A-diffusion problem.

Consider Ito's equation

$$d\mathbf{X}(t) = a(\mathbf{X}(t))\,dt + \mathbf{R}(\mathbf{X}(t))\,d\mathbf{W}(t) \tag{34}$$

where $\mathbf{W}$ is a Wiener process. The definition of the stochastic differential or the stochastic integral is given in [9]. (See also [3] and [15].) It is known that when the "Ito fields" $\Psi = (a; \mathbf{R})$ is bounded and (locally) Lipschitzian, the Eq. (34) has a unique solution for any given initial position $\mathbf{X}(0)$. All the solution processes, for varying initial positions, form a conservative Fellerian $\mathbf{A}$-diffusion $\mathbf{P}$ where $\mathbf{A} = \sum a_i\,\partial_i + \frac{1}{2}\sum \mathbf{S}_{ij}\,\partial_i\,\partial_j$ is connected with $\Psi = (a; \mathbf{R})$ by

$$\mathbf{S}(x) = \mathbf{R}(x)\,\widetilde{\mathbf{R}(x)}, \tag{35}$$

$\tilde{\mathbf{B}}$ being the matrix-transpose of $\mathbf{B}$. We call $\mathbf{S}$ *the square* of $\mathbf{R}$ and $\mathbf{R}$ *a quasi-square-root* of $\mathbf{S}$.

Remark 1. A basic problem in It's approach is to find a Lipschitzian $\Psi = (a; \mathbf{R})$ from a given $\mathbf{A} = (a, \mathbf{S})$. This is an ill-posed "section problem". (For some results on this problem see [5].)

We want an affirmative answer to the following question which is in some sense

A Converse Problem to Ito's Construction

Suppose $\Psi=(a,\mathbf{R})$ is a bounded continuous Ito field on $\mathbb{R}^d$ and $\mathbf{P}$ is a Fellerian $\mathbf{A}$-diffusion with $\mathbf{A}=(a,\mathbf{S})$ connected with Ψ by (35). For a given initial distribution μ, can the process $\mathbf{X}$ on $(\mathscr{C}(\mathbb{R}_+;\mathbb{R}^d);\mu\mathbf{P})$ be expressed as a solution of Ito's equation (34) for a certain Wiener process $\mathbf{W}$?

The answer is an easy consequence of Lemma 3 in §7 and the following theorem, due to Doob ([3, p. 449]) in the one-dimensional case.

Theorem 10. *Let $(\mathbf{X}(t),\mathscr{F}_t)$ be a continuous d-dimensional vector martingale, and let $(\mathbf{R}(t),\mathscr{F}_t)$ be a process $(d\times d')$ matrix-valued, on the same probability space $(\Omega,\mathscr{P})$. If the conditional expectations*

$$\mathscr{F}_s\left\{\int_s^t \mathbf{S}_{ij}(u)\,du\right\}\cong\mathscr{F}_s\{[\mathbf{X}_i(t)-\mathbf{X}_i(s)]\,[\mathbf{X}_j(t)-\mathbf{X}_j(s)]\}\tag{36}$$

where $\mathbf{S}=\mathbf{R}\tilde{\mathbf{R}}$, $t>s$, then there exists a d'-dimensional Brownian motion $(\mathbf{Y}(t))$ such that

$$\mathbf{X}(t)=\mathbf{X}(0)+\int_0^t \mathbf{R}(\mathbf{X}(u))\,d\mathbf{Y}(u).\tag{37}$$

Proof. Without loss of generality we may assume $d'=d$, by augmenting $\mathbf{R}$ by zeroes if necessary.

Let us consider a polarization $\mathbf{R}=\mathbf{R}'\mathbf{U}$, with $\mathbf{R}'$ non-negative symmetric and $\mathbf{U}$ orthogonal, and then diagonalize $\mathbf{R}'$, i.e., write $\mathbf{R}'=\mathbf{V}\Lambda\tilde{\mathbf{V}}$ with $\mathbf{V}$ orthogonal and Λ diagonal. There may be ambiguity as the polarization and the diagonalization are not unique, but we will choose these measurably. The diagonal matrix is again decomposed into a direct sum of two blocks: $\Lambda=\Lambda_1+0_k$, where Λ_1 is a strictly-positive diagonal matrix, say of dimension $(d-k)$, and 0_k is the k-dimensional zero-matrix, $0\leqq k\leqq d$.

Define now in the same decomposition, $\Lambda'=\Lambda_1^{-1}+0_k$ and $\Lambda''=0_{d-k}+1_k$ where 1_k is the k-dimensional unit matrix. Take finally a d-dimensional Brownian motion $(\mathbf{Z}(t))$, independent of the tribes $(\mathscr{F}_s)$. (See the discussion in [3]: In general it will be necessary to go outside the given probability space to do this.)

We now construct

$$\mathbf{Y}(t)=\int_0^t \tilde{\mathbf{U}}\mathbf{V}\Lambda'\tilde{\mathbf{V}}\,d\mathbf{X}+\int_0^t \tilde{\mathbf{U}}\mathbf{V}\Lambda''\tilde{\mathbf{V}}\,d\mathbf{Z}\tag{38}$$

or

$$d\mathbf{Y}=\tilde{\mathbf{U}}\mathbf{V}\Lambda'\tilde{\mathbf{V}}\,d\mathbf{X}+\tilde{\mathbf{U}}\mathbf{V}\Lambda''\tilde{\mathbf{V}}\,d\mathbf{Z},\quad\text{symbolically}.\tag{38'}$$

It is readily seen that, for $t>s$,

$$\mathscr{F}_s'\{[\mathbf{Y}_i(t)-\mathbf{Y}_i(s)]\,[\mathbf{Y}_j(t)-\mathbf{Y}_j(s)]\}\cong(t-s)\,\delta_{ij},\tag{39}$$

if $\mathscr{F}_s'$ is generated by $\mathscr{F}_s$ and $\{\mathbf{Z}(u)\colon u\leqq s\}$.

By [3, Chapter VII, Theorem 1.9], $\mathbf{Y}(t)$ is a d-dimensional Brownian motion. Also we have $\mathbf{R}\tilde{\mathbf{U}}\mathbf{V}\Lambda''\tilde{\mathbf{V}}=0$, and $\mathbf{V}\Lambda''\tilde{\mathbf{V}}\mathbf{R}=0$; hence, by the composition rule of stochastic integrand, [3, p. 448],

$$\mathbf{R}\,d\mathbf{Y}=\mathbf{V}\Lambda\Lambda'\tilde{\mathbf{V}}\,d\mathbf{X}+\mathrm{O}\,d\,\mathbf{Z}=(1-\mathbf{V}\Lambda''\tilde{\mathbf{V}})\,d\mathbf{X}=d\mathbf{X}.\qquad\text{Q.E.D.}$$

§ 7. The Uniqueness Theorems

Lemma 1. *Let* **A** *be non-degenerate on a Euclidean* **E**, *without any boundedness condition. Then Fellerian* **A**-*diffusion is unique.*

Proof. Let **P** and **P**′ be two Fellerian **A**-diffusions. Let

$$v=\left[\sum (\mathbf{S}_{ij})^2+\sum (a_i)^2+1\right]^{-\frac{1}{2}}$$

and apply the speed-change **V** to **P** and **P**′, resulting in **Q** and **Q**′ respectively. By Theorem 8 (§ 5), $\mathbf{Q}=\mathbf{VP}$ and $\mathbf{Q}'=\mathbf{VP}'$ are both Fellerian ($v\mathbf{A}$)-diffusions, hence identical by the theorem of Stroock-Varadhan.

But now $\mathbf{V}\colon \mathscr{J}(\mathbb{R}_+, \mathbf{E}_\Delta)\to \mathscr{J}(\mathbb{R}_+, \mathbf{E}_\Delta)$ is almost surely injective, relative to the measures $x\mathbf{P}$ and $x\mathbf{P}'$. Therefore $x\mathbf{P}=x\mathbf{P}'$ for every x.

Proof of Theorem 1, § 2. We cover the manifold **E** by smooth balls ($\mathbf{W}_n$). Then on any such ball **W** the murdered evolution $\mathrm{m}_{\mathbf{W}}\mathbf{P}$ of a given Fellerian **A**-diffusion **P** is a Fellerian ($\mathbf{A}|_{\mathbf{W}}$)-diffusion by Theorem 9, § 5, and Theorem 3, § 3, hence uniquely determined by Lemma 1 above, as a ball is diffeomorphic to a Euclidean space. By the patching-up theorem of Courrège-Priouret, **P** is unique.

Proof of Theorem 2, § 2. By patching-up and a speed-change just as in the proof of Theorem 1, it suffices to prove, on a Euclidean space, the following.

Lemma 2. *Let a be bounded and Lipschitz, and let there be a (globally) Lipschitz, bounded, quasi-square root* **R** *of* **S**. *Then the Fellerian* **A**-*diffusion is unique.*

To prove this lemma we invoke the following result [16, Lemma 11.3].

Lemma 3. *Let* **A** *be continuous and bounded on* $\mathbb{R}^d$, *then the following two processes* ξ, η *are continuous martingales valued in* $\mathbb{R}^d$ *and* $\mathbb{R}^{d\times d}$ *respectively, with respect to any Fellerian* **A**-*diffusion process* ($\mathbf{X}(t)$) *with any initial distribution:*

$$\begin{aligned}\xi_i(t)&=\mathbf{X}_i(t)-\mathbf{X}_i(0)-\int_0^t a_i(X(u))\,du,\\ \eta_{ij}(t)&=\xi_i(t)\,\xi_j(t)-\int_0^t \mathbf{S}_{ij}(\mathbf{X}(u))\,du.\end{aligned}\tag{40}$$

Now still following the idea of [16], we apply Theorem 10, § 6, so that there exists a d'-dimensional Brownian motion **Y** such that

$$\xi(t)=\xi(0)+\int_0^t \mathbf{R}(\mathbf{X}(u))\,d\mathbf{Y}(u),$$

or, equivalently,

$$\mathbf{X}(t)=\mathbf{X}(0)+\int_0^t a(\mathbf{X}(u))\,du+\int_0^t \mathbf{R}(\mathbf{X}(u))\,d\mathbf{Y}(u).\tag{41}$$

The solution of this last equation, given ($\mathbf{Y};\mathbf{X}(0)$), is unique, because ($a, \mathbf{R}$) is globally Lipschitz. By the uniqueness (in the sense of identical distributions) of Brownian motion **Y**, the distributions of $\mathbf{X}(t)$ are uniquely determined. Q.E.D.

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