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Limiting Behavior of One-sample Rank-order Statistics for Absolutely Regular Processes

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Summary. Let $\{\xi_i\}$ be a strictly stationary, absolutely regular process defined on a probability space $(\Omega, \mathscr{A}, P)$, i.e., ξ_i's satisfy the condition

$$\beta(n)=E\{\sup_{A\in\mathscr{M}_n^\infty} |P(A|M_{-\infty}^0)-P(A)|\}\downarrow 0$$

where $\mathscr{M}_a^b$ $(a\leqq b)$ denote the σ-algebra of events generated by $\xi_a, \ldots, \xi_b$. (It is known that $\{\xi_i\}$ is absolutely regular if $\{\xi_i\}$ is ϕ-mixing, i.e.

$$\phi(n)=\sup_{B\in\mathscr{M}_{-\infty}^0,\, A\in\mathscr{M}_n^\infty} |P(A\cap B)-P(A)P(B)|/P(B)\downarrow 0.$$

For some class of one-sample rank-order statistics, generated by absolutely regular processes, we shall prove theorems concerning the following problems under the assumption that ξ_1 has a continuous (not necessarily symmetric) distribution function.

(a) weak convergence to a process $\{U(t)\colon 0\leqq t\leqq 1\}$ defined by $U(t)=\int_0^t h(s)\,dW(s)$,

(b) functional laws of the iterated logarithm, and

(c) almost sure invariance principles and integral tests.

Some of them are extensions of Sen's results [(Ann. Statist. **2**, 49–62. (1974; Zbl. 273, 60005) ibid. **2**, 1358 (1974; Zbl. **292**, 60012)] and Stigler's ones [ibid. **2**, 676–693 (1974; Zbl. **286**, 62028)].

1. Introduction

Let $\{\xi_i, -\infty<i<\infty\}$ be a strictly stationary sequence of random variables which are defined on a probability space $(\Omega,\mathscr{A},P)$ and have a continuous distribution function (df) $F(x)$, $x\in R$, the real line $(-\infty,\infty)$. For $a\leqq b$, let $\mathscr{M}_a^b$ denote the σ-algebra of events generated by $\xi_a, \ldots, \xi_b$. As in [18], we shall say

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that the sequence is absolutely regular if

$$\beta(n)=E\{\sup_{A\in\mathcal{M}_n^\infty}|P(A|\mathcal{M}_{-\infty}^0)-P(A)|\}\downarrow 0 \tag{1.1}$$

as $n\to\infty$. Further, we shall say that $\{\xi_i\}$ satisfies the ϕ-mixing condition if

$$\phi(n)=\sup_{B\in\mathcal{M}_{-\infty}^0,\, A\in\mathcal{M}_n^\infty}|P(A\cap B)-P(A)P(B)|/P(B)\downarrow 0 \tag{1.2}$$

as $n\to\infty$. Since $\beta(n)\leqq\phi(n)$, so if $\{\xi_i\}$ is ϕ-mixing, then it is absolutely regular (cf. [7]).

Next, let $u(x)$ be equal to 1 or 0 according x is $\geqq 0$ or <0, and for every $n\geqq 1$, let

$$R_{n,i}=\sum_{j=1}^{m}u(|\xi_i|-\xi_j|), \quad 1\leqq i\leqq n. \tag{1.3}$$

Consider the one-sample rank-order statistic

$$T_{n,m}=\sum_{i=1}^{m}c_{n,i}\,s(\xi_i)\,J\left(\frac{R_{m,i}}{m+1}\right), \quad n\geqq m\geqq 1 \tag{1.4}$$

where J is a score-function, $s(x)=\mathrm{sgn}(x)$ and the $c_{n,i}$ are defined by a continuous function $h(x)$ on $I=[0,1]$ as

$$c_{n,i}=h\left(\frac{i}{n+1}\right) \quad 1\leqq i\leqq n,\ n\geqq 1. \tag{1.5}$$

If $h(x)=1$ for all $x\in I$, then we write T_m instead of $T_{n,m}$, i.e.

$$T_m=\sum_{i=1}^{m}s(\xi_i)\,J\left(\frac{R_i}{m+1}\right). \tag{1.6}$$

Assuming that $\{\xi_i\}$ are independent and $h(x)=1$ on I, many authors studied asymptotic normality of $n^{-\frac{1}{2}}(T_n-n\mu)$ and, recently, Sen [15] tried to prove the classical invariance principle or weak convergence to Brownian motion processes for $\{T_n\}$ under some weak conditions.

Sen [15] remarked that, in particular, the Wilconxon signed rank statistic can be expressed as a von Mises' differentiable statistical functional, but, in general, this characterization is not possible for $\{T_n\}$. For U-statistics and von Mises' functionals generated by a strictly stationary, absolutely regular processes, the invariance principle was established by the author [18]. But, for $\{T_n\}$ generated by such processes, different proofs are needed.

On the other hand, Stigler [16] proved asymptotic normality of linear functions of order statistics of the form $n^{-1}\sum h(i/n+1)X_{(i)}$.

In this paper, extending the previous results, we shall prove the following under suitable conditions on J and $\beta(n)$;

(i) weak convergence for $\{T_{n,m}\}$ (Sections 3 and 4),

(ii) a functional law of the iterated logarithm for $\{T_n\}$ (Section 5)

and

(iii) almost sure invariance principles and integral tests for $\{T_n\}$ (with $h(x)=1$ on I) generated by ϕ-mixing sequences.

Some extensions and modifications of the above results are stated in Section 7.

2. Preliminary Lemmas

In this section, we assume that $\{\xi_i\}$ is a strictly stationary, absolutely regular sequence of random variables with $df\ F(x)$. In what follows, we shall agree to denote by the letter M, with suffix or not, some quantity bounded in absolute value.

The following lemma is proved in [18] (cf. Lemma 1 in [18]).

Lemma 2.1. *Let δ be some positive number. Let $g(x_1, x_2, \ldots, x_k)$ be a Borel function such that*

$$\int_{R^k}\cdots\int |g(x_1,x_2,\ldots,x_k)|^{1+\delta}\, dF^{(1)}(x_1,\ldots,x_j)\, dF^{(2)}(x_{j+1},\ldots,x_k) \leqq M_1 \tag{2.1}$$

where $F^{(1)}$ and $F^{(2)}$ are distribution functions of random vectors $(\xi_{i_1},\ldots,\xi_{i_j})$ and $(\xi_{i_{j+1}},\ldots,\xi_{i_k})$, respectively, and $i_1<i_2<\cdots<i_k$. If $E|g(\xi_{i_1},\xi_{i_2},\ldots,\xi_{i_k})|^{1+\delta}\leqq M_1$, then

$$\begin{aligned}|Eg(\xi_{i_1},\xi_{i_2},\ldots,\xi_{i_k}) - \int_{R^k}\cdots\int g(x_1,\ldots,x_j,x_{j+1},\ldots,x_k)\\ \cdot dF^{(1)}(x_1,\ldots,x_j)\, dF^{(2)}(x_{j+1},\ldots,x_k)|\\ \leqq 4M_1^{1/1+\delta}\{\beta(i_{j+1}-i_j)\}^{\delta/1+\delta}.\end{aligned} \tag{2.2}$$

As a special case, if $g(x_1,x_2,\ldots,x_k)$ is bounded, say, $|g(x_1,x_2,\ldots,x_k)|\leqq M_2$, then we can replace the right-hand side of (2.2) by $2M_2\,\beta(i_{j+1}-i_j)$.

For any real number x, define $H(|x|)$ and $S_n(x)$, respectively, as

$$H(|x|) = F(|x|) - F(-|x|) \tag{2.3}$$

and

$$S_n(x) = \sum_{j=1}^{n} Y_j(x)$$

where

$$Y_j(x) = u(|x| - |\xi_j|) - H(|x|) \qquad (j=0,\ \pm 1,\ \pm 2, \ldots). \tag{2.4}$$

Then, the process $\{Y_j(x)\}$ is a strictly stationary, absolutely regular sequence of random variables with the same function $\beta(n)$ as that of $\{\xi_j\}$ and for any x and j, $|Y_j(x)|\leqq 1$, $EY_j(x)=0$ and

$$EY_j^2(x) = H(|x|)\{1-H(|x|)\}. \tag{2.5}$$

Lemma 2.2. *Let $\alpha (0<\alpha<1)$ be fixed. Choose a number ρ $(0<\rho<\alpha)$. Assume that $\{\xi_j\}$ are absolutely regular random variables with $\beta(n)$. If x is a number such that $H(|x|)\geqq 1-n^{-\alpha}$, then*

$$P(n^{-1}|S_n(x)|\geqq t[\exp_n\{-(1-\rho)(1+\alpha)/2\}])$$
$$\leqq M_1 n^{\rho}\{e^{-M_2 t}+n^{1-\rho}\beta(n^{\rho})\} \tag{2.6}$$

where $\exp_n(x)=n^x$.

Proof. Choose an integer $k=k_n=[n^{\rho}]+1$, and write

$$S_n(x)=U_1(x)+\cdots+U_k(x) \tag{2.7}$$

where

$$U_j(x)=Y_j(x)+Y_{j+k}(x)+\cdots+Y_{j+m_jk}(x), \quad 1\leqq j\leqq k \tag{2.8}$$

and $m_j=m_{n,j}$ is the largest integer for which $j+m_jk\leqq n$. We note that

$$m_j\leqq m_1\leqq n^{1-\rho}-1 \quad \text{for } j=1,\dots,k \text{ and } k<n. \tag{2.9}$$

Thus, from (2.8)

$$P(n^{-1}|S_n(x)|\geqq t[\exp_n\{-(1-\rho)(1+\alpha)/2\}])$$
$$\leqq P\left(n^{-1}\sum_{j=1}^{k}|U_j(x)|\geqq t[\exp_n\{-(1-\rho)(1+\alpha)/2\}]\right)$$
$$\leqq \sum_{j=1}^{k} P(n^{-(1-\rho)}|U_j(x)|\geqq t[\exp_n\{-(1-\rho)(1+\alpha)/2\}]). \tag{2.10}$$

Let $j(1\leqq j\leqq k)$ be fixed. Define A_j by

$$A_j=\{(y_1,\dots,y_{m_j}):\ |y_1+\cdots+y_{m_j}|\geqq t[\exp_n\{-(1-\rho)(1-\alpha)/2\}]\}$$

and put

$$g(y_1,\dots,y_{m_j})=\begin{cases}1 & \text{if } (y_1,\dots,y_{m_j})\in A_j\\ 0 & \text{otherwise.}\end{cases}$$

Since $\{y_{j+ik}(x)\}$ are absolutely regular with $\beta(ik)$, so using Lemma 2.1 (with $M_2=1$) repeatedly, we have

$$P(|U_j(x)|\geqq t[\exp_n\{(1-\rho)(1-\alpha)/2\}])$$
$$=Eg(Y_j(x),\dots,Y_{j+m_jk}(x))$$
$$\leqq \int\cdots\int_{R^{m_j}} g(y_1,\dots,y_{m_j})dF(y_1)\dots dF(y_{m_j})+2m_j\beta(k)$$
$$=P(|Z_1(x)+\cdots+Z_{m_j}(x)|\geqq t[\exp_n\{(1-\rho)(1-\alpha)/2\}])+2m_j\beta(k) \tag{2.11}$$

where $\{Z_i(x)\}$ are independently and identically distributed random variables with the same *df* $F(x)$ as that of the random variable $Y_1(x)$. So, from Theorem 1 in [10]

$$P(|Z_1(x)+\cdots+Z_{m_j}(x)|\geqq t[\exp_n\{(1-\rho)(1-\alpha)/2\}])$$
$$\leqq M_1 e^{-M_2 t}. \tag{2.12}$$

Thus, from (2.10)–(2.12), we have (2.6), which completes the proof.

For any $n(n\geqq 1)$ and for any i $(1\leqq i\leqq n)$, let

$$\xi^*_{n,i}=\frac{1}{n+1}\{(n-1)H(|\xi_i|)+1\} \tag{2.13}$$

and

$$\zeta_{n,i}=\frac{R_{n,i}}{n+1}-\xi^*_{n,i}=\frac{1}{n+1}\sum_{\substack{1\leqq j\leqq n\\ j\neq i}} Y_j(\xi_i). \tag{2.14}$$

Lemma 2.3. *Under the assumptions of Lemma 2.2, there exists an $n_0=n_0(t)$ such that*

$$P(|\zeta_{n,i}|\geqq 5t[\exp_n\{-(1-\rho)(1+\alpha)/2\}],\ H(|\xi_i|)\geqq 1-n^{-\alpha})$$
$$\leqq M_1 n^{-\alpha+\rho}\{e^{-M_2 t}+n^{1-\rho}\beta(n^\rho)\} \tag{2.15}$$

for all $n\geqq n_0$ and for any i $(1\leqq i\leqq n)$.

Proof. Let n and i be fixed. Let $B=\{v\colon 1-n^{-\alpha}\leqq v\leqq 1\}$,

$$\zeta^{(1)}_{n,i}(x)=\frac{1}{n+1}\sum_{j=1}^{i-[n^\rho]} Y_j(x) \quad\text{and}\quad \zeta^{(2)}_{n,i}=\frac{1}{n+1}\sum_{j=i+[n^\rho]+1}^{n} Y_j(x),$$

putting $\zeta^{(1)}_{n,i}(x)=0$ if $i-[n^\rho]\leqq 0$ and $\zeta^{(2)}_{n,i}(x)=0$ if $i+[n^\rho]>n$. As

$$|\zeta_{n,i}|\leqq\sum_{j=1}^{2}|\zeta^{(j)}_{n,i}(\xi_i)|+2n^{-(1-\rho)},$$

so for all n such that $\frac{1}{2}n^{(1-\rho)(1-\alpha)/2}>t$

$$\text{the left-hand side of (2.15)}\leqq\sum_{j=1}^{2}E\chi^{(j)} \tag{2.16}$$

where, for j $(j=1,2)$, $\chi^{(j)}$ is the indicator of the ω-set

$$\{\omega\colon\ |\zeta^{(j)}_{n,i}(\xi_i)|\geqq 2t[\exp_n\{-(1-\rho)(1+\alpha)/2\}],\ H(|\xi_i|)\in B\}.$$

From Lemma 2.1

$$E\chi^{(j)}\leqq\int_{H(|x|)\in B}P(|\zeta^{(j)}_{n,i}(x)|\geqq 2t[\exp_n\{-(1-\rho)(1+\alpha)/2\}])\,dF(x)+2\beta(n^\rho)$$

and so from Lemma 2.2

$$E\chi^{(j)}\leqq M_1 n^\rho\{e^{-M_2 t}+n^{1-\rho}\beta(n^\rho)\}\int_{H(|x|)\in B}dF(x)+2\beta(n^\rho)$$
$$\leqq M n^{\rho-\alpha}\{e^{-M_2 t}+n^{1-\rho}\beta(n^\rho)\}\qquad (j=1,2). \tag{2.17}$$

Thus, from (2.16) and (2.17), we have (2.14), which completes the proof of Lemma 2.3.

By the method of the proof of Remark 1 in [12], we can easily show the following lemma.

Lemma 2.4. *Let $\{\xi_i\}$ be a not necessarily strictly stationary, absolutely regular sequence of random variables with $\beta(n)$ and $E\xi_i=0$. If for all $m(\leqq n)$ and $x>0$*

$$P(|S_n-S_m|\geqq x)\leqq \tfrac{1}{2}, \tag{2.18}$$

then

$$P(\max_{1\leqq m\leqq n}|S_m|\geqq 2x)\leqq 2P(|S_n|\geqq x)+4\frac{n}{p}P\left(\sum_{j=1}^{p}|\xi_i|\geqq x\right)+4\frac{n}{p}\beta(p) \tag{2.19}$$

where $S_n=\sum_{j=1}^{n}\xi_j$ and $p=p(n)\to\infty$ as $n\to\infty$ and $p\leqq n$.

3. Weak Convergence Theorems for $\{T_{n,m}\}$ when J'' is Bounded Inside I

For a score-function $J(u)$ put

$$\mu=\mu_J(F)=\int s(x)J(H(|x|)\,dF(x)). \tag{3.1}$$

It is obvious that if the score-function J is square integrable, then $|\mu|<\infty$.

Let $h(x)$ be a continuous function on I. Let $T_{n,m}$ be defined by (1.4) and put

$$\mu_{n,m}=\mu\sum_{i=1}^{m}c_{n,i}=\mu\sum_{i=1}^{m}h\left(\frac{i}{n+1}\right). \tag{3.2}$$

For every $n\geqq 1$, let

$$X_n(t)=\begin{cases}0 & \text{for } t=0\\ (T_{n,k}-\mu_{n,k})/(\sigma n^{\frac{1}{2}}) & \text{for } t=k/n\ (k=1,\dots,n)\\ \text{linearly interpolated} & \text{for } t\in[(k-1)/n,\ k/n]\ (k=1,\dots,n),\end{cases} \tag{3.3}$$

where σ is a positive constant. Then, the stochastic process $X_n =\{X_n(t)\colon\ 0\leqq t\leqq 1\}$ belong to the space $C=C(I)$ of all continuous function on I with which we associate the usual uniform topology defined by the metric

$$d(f,g)=\sup_{0\leqq t\leqq 1}\{|f(t)-g(t)|\colon f,\ g\in C\}. \tag{3.4}$$

Now, we shall prove the following theorem, which is an extension Theorem 1 in [15].

Theorem 3.1. *Let $\{\xi_i\}$ be a strictly stationary, absolutely regular process with $\beta(n)=O(n^{-4})$. Let J be a score-function having a bounded second derivative. If σ^2, defined by* (3.7), *(below), is positive and finite, then X_n, defined by* (3.3), *converge in distribution in the uniform topology on C to the stochastic process $U=\{U(t)\colon 0\leqq t\leqq 1\}$, where*

$$U(t)=\int_0^t h(s)\,dW(s), \qquad 0\leqq t\leqq 1 \tag{3.5}$$

and $W=\{W(t)\colon\ 0\leqq t\leqq 1\}$ *is a standard Brownian motion process.*

To prove Theorem 3.1, we need some lemmas.

Firstly, let

$$\eta_i=\int s(x)\,\{u(|x|-|\xi_i|)-H(|x|)\}\,J'(H(|x|))\,dF(x)$$
$$+\{s(\xi_i)\,J(H(|\xi_i|))-E\,s(\xi_i)\,J(H(|\xi_i|))\} \qquad (i=0,\ \pm 1,\ \pm 2,\ldots) \tag{3.6}$$

It is obvious that $E\eta_i=0$. Put

$$\sigma^2=\lim_{n\to\infty}\operatorname{Var}\left(\sum_{i=1}^{n}\eta_i\right)\Big/ n \tag{3.7}$$

if the limit exists. It is easily proved that if $\{\xi_i\}$ is a strictly stationary, absolutely regular sequence with $\beta(n)=O(n^{-4})$ and J'' is bounded, then $\{\eta_i\}$ is a strictly stationary, absolutely regular sequence of bounded random variables satisfying $\beta(n)=O(n^{-4})$ and the limit in (3.7) exists. (cf. [13]).

If the score-function possesses a bounded derivative, then by Taylor expansion

$$J\left(\frac{R_{m,i}}{m+1}\right)=J(\xi^*_{m,i})+\zeta_{m,i}\,J'(\xi^*_{m,i})+\zeta^2_{m,i}\,k_{m,i}(H(|\xi_i|)) \tag{3.8}$$

where $\xi^*_{m,i}$ and $\zeta_{m,i}$ are the ones defined in (2.13) and (2.14), respectively, and $k_{m,i}(x)$ is bounded, say $|k_{m,i}(x)|\leqq M_0$, $-\infty<x<\infty$, $1\leqq i\leqq m$. (cf. [3]).

We note that

$$|\xi^*_{m,i}-\mathrm{H}(|\xi_i|)|=\left|\frac{1-2H(|\xi_i|)}{m+1}\right|\leqq\frac{1}{m+1}.$$

So, from (1.4), (3.2), (3.6) and (3.8) and Taylor expansion, it follows that

$$\left|T_{n,m}-\mu_{n,m}-\sum_{i=1}^{m}c_{n,i}\,\eta_i\right|$$
$$=\Bigg|\sum_{i=1}^{m}c_{n,i}[s(\xi_i)\{J(\xi^*_{m,i})+\zeta_{m,i}\,J'(\xi^*_{m,i})+\zeta^2_{m,i}\,k_{m,i}(H(|\xi_i|))\}$$
$$-\{\int s(x)\,(u(|x|-|\xi_i|)-H(|x|))\,J'(H(|x|))\,dF(x)-s(\xi_i)\,J(H(|\xi_i|))\}]\Bigg|$$
$$\leqq\left|\sum_{i=1}^{m}c_{n,i}\,s(\xi_i)\{J(\xi^*_{m,i})-J(H(|\xi_i|))\}\right|$$
$$+\Bigg|\sum_{i=1}^{m}c_{n,i}[s(\xi_i)\,\zeta_{m,i}\,J'(\xi^*_{m,i})$$
$$-\int s(x)\{u(|x|-|\xi_i|)-H(|x|)\}\,J'(H(|x|))\,dF(x)\Bigg|$$
$$+M_0\sum_{i=1}^{m}|c_{n,i}|\,\zeta^2_{m,i}$$

$$\leqq \sum_{i=1}^{m} |c_{n,i}| \left\{ |\xi^*_{m,i} - H(|\xi_i|)| \cdot |J'(H(|\xi_i|))| + \frac{M_0}{2} |\xi^*_{m,i} - H(|\xi_i|)|^2 \right\}$$

$$+ \left[\sum_{i=1}^{m} |c_{n,i}| \cdot |s(\xi_i) \zeta_{m,i} J'(H(|\xi_i|)|\right.$$

$$- \int s(x) \{u(|x| - |\xi_i|) - H(|x|)\} J'(H(|x|)) \, dF(x)$$

$$+ M_0 \sum_{i=1}^{m} |c_{n,i}| \cdot |\xi^*_{m,i} - H(|\xi_i|)| \cdot |\zeta_{m,i}|$$

$$+ M_0 \sum_{i=1}^{m} |c_{n,i}| \zeta^2_{m,i}$$

$$\leqq \frac{1}{m+1} \sum_{i=1}^{m} |c_{n,i}| \left\{ |J'(H(|\xi_i|))| + \frac{M_0}{m+1} \right\}$$

$$+ \left[|V_m| + \frac{2}{m+1} \sum_{i=1}^{m} |c_{n,i}| \cdot \left| \int s(x) \{u(|x| - |\xi_i|) - H(|x|)\} J'(H(|x|)) \, dF(x) \right| \right.$$

$$\left. + \frac{M_0}{m+1} \sum_{i=1}^{m} |c_{n,i}| \cdot |\zeta_{m,i}| \right]$$

$$+ M_0 \sum_{i=1}^{m} |c_{n,i}| \zeta^2_{m,i} \tag{3.9}$$

where

$$V_m = \frac{1}{m+1} \sum_{1 \leqq i \leqq m} \sum_{\substack{1 \leqq j \leqq m \\ j \neq i}} c_{n,i} [s(\xi_i) \{u(|\xi_i| - |\xi_j|) - H(|\xi_i|)\} J'(H(|\xi_i|))$$

$$- \int s(x) \{u(|x| - |\xi_i|) - H(|x|)\} J'(H(|x|)) \, dF(x)]. \tag{3.10}$$

As $|c_{m,i}| \leqq \max\{h(x): 0 \leqq x \leqq 1\} = \|h\|$ and $u(x)$, $s(x)$ and $J'(u)$ are bounded, so from (3.9) we have that for all $m (\leqq n)$

$$\left| T_{n,m} - \mu_{n,m} - \sum_{i=1}^{m} c_{n,i} \eta_i \right| \leqq |V_m| + M_1 \sum_{i=1}^{m} \zeta^2_{m,i} + M_2. \tag{3.11}$$

Lemma 3.1. *Under the conditions of Theorem* 3.1, *the following relations hold:*

$$E|V_m|^2 = O(1), \qquad E|V_m|^4 = O(1). \tag{3.12}$$

Proof. Let

$$g(y, z) = s(y) \{u(|y| - |z|) - H(|y|)\} J'(H(|y|))$$
$$- \int s(x) \{u(|x| - |z|) - H(|x|)\} J'(H(|x|)) \, dF(x).$$

Then

$$\int g(y, z) \, dF(y) = \int g(y, z) \, dF(z) = \iint g(y, z) \, dF(y) \, dF(z) = 0$$

and so, using the same technique as in the proof of Lemma 3 in [18], we have the desired relations.

Lemma 3.2. *Under the conditions of Theorem* 3.1

$$E\left|\sum_{i=1}^{m} \zeta_{m,i}^2\right|^2 = O(m^{-1}).$$

$$E\left(\sum_{i=1}^{m} \zeta_{m,i}^2\right) \leqq \left\{E\left|\sum_{i=1}^{m} \zeta_{m,i}^2\right|^2\right\}^{\frac{1}{2}} = O(m^{-\frac{1}{2}}). \tag{3.13}$$

Proof. (3.13) follows easily from Lemma 2.1.

Lemma 3.3. *Under the conditions of Theorem* 3.1

$$\lim_{m\to\infty} E\left|\sum_{i=1}^{m} s(\xi_i) J\left(\frac{R_{m,i}}{m+1}\right) - m\mu\right|^2 \Big/ (m\sigma^2) = 1 \tag{3.14}$$

if σ^2, *defined by* (3.7), *is positive and finite.*

Proof. The proof is easily obtained from (3.2), (3.7), (3.11)–(3.13) and (1.4) putting $c_{m,i}=1$ $(i=1,\dots,m)$.

Now, we consider stochastic processes $Y_n=\{Y_n(t)\colon 0\leqq t\leqq 1\}$ defined by

$$Y_n(t)=\begin{cases} 0 & \text{for } t=0 \\ \left(\sum_{i=1}^{k} c_{n,i}\eta_i\right) \Big/ (\sigma n^{\frac{1}{2}}) & \text{for } t=k/n \ (k=1,2,\dots,n) \\ \text{linearly interpolated} & \text{for } t\in[(k-1)/n, k/n] \ (k=1,\dots,n). \end{cases} \tag{3.15}$$

Lemma 3.4. *Let* $Y_n=\{Y_n(t)\colon 0\leqq t\leqq 1\}$ $(n=1,2,\dots)$ *be random elements in* C *defined by* (3.15). *Then, under the conditions of Theorem* 3.1, $Y_n \xrightarrow{\mathscr{D}} U$.

Proof. For any $n(n\geqq 1)$, let

$$Z_n(t)=\begin{cases} 0 & \text{for } t=0 \\ \left(\sum_{i=1}^{k} \eta_i\right) \Big/ (\sigma n^{\frac{1}{2}}) & \text{for } t=k/n \ (k=1,2,\dots,n) \\ \text{linearly interpolated} & \text{for } t\in[(k-1)/n, k/n] \ (k=1,2,\dots,n). \end{cases} \tag{3.16}$$

Since $\{\eta_i\}$ is a strictly stationary, absolutely regular sequence of bounded random variables with $\beta(n)=O(n^{-4})$, so from Theorem 2 in [13], we have that $Z_n \xrightarrow{\mathscr{D}} W$. Thus, from Theorem 1 in [11] we have the lemma.

Lemma 3.5. *If the conditions of Theorem* 3.1 *are satisfied, then for any* $\varepsilon>0$

$$P(d(X_n, Y_n)>\varepsilon) \to 0 \qquad (n\to\infty). \tag{3.17}$$

Proof. To prove (3.17), it is enough to prove

$$P\left(\max_{1\leqq m\leqq n}\left|T_{n,m}-\mu_{n,m}-\sum_{i=1}^{m} c_{n,i}\eta_i\right| \geqq 3\varepsilon n^{\frac{1}{2}}\right) \to 0 \qquad (n\to\infty) \tag{3.18}$$

for every $\varepsilon>0$. In order to prove (3.18), from (3.11), it suffices to show that for every $\varepsilon>0$

$$P(\max_{1\leqq m\leqq n}|V_m|>\varepsilon\sigma n^{\frac{1}{2}})\to 0 \quad (n\to\infty) \tag{3.19}$$

and

$$P\left(\max_{1\leqq m\leqq n}\sum_{i=1}^{m}k_{m,i}(H(|\xi_i|))\{\zeta_{m,i}\}^2>\varepsilon\sigma n^{\frac{1}{2}}\right)\to 0 \quad (n\to\infty). \tag{3.20}$$

From (3.12)

$$\begin{aligned}P(\max_{1\leqq m\leqq n}|V_m|>\varepsilon\sigma n^{\frac{1}{2}})&\leqq\sum_{m=1}^{n}P(|V_m|>\varepsilon\sigma n^{\frac{1}{2}})\\&\leqq(\varepsilon^4\sigma^4n^2)^{-1}\sum_{m=1}^{n}E|V_m|^4=O(n^{-1})\end{aligned} \tag{3.21}$$

and from (3.13) (using the boundedness of $k_{m,i}(x)$)

$$\begin{aligned}&P\left(\max_{1\leqq m\leqq n}\left|\sum_{i=1}^{m}k_{m,i}(H(|\xi_i|))\zeta_{m,i}^2\right|>\varepsilon\sigma n^{\frac{1}{2}}\right)\\&\leqq\sum_{m=1}^{n}P\left(M_1\sum_{i=1}^{m}\zeta_{m,i}^2>\varepsilon\sigma n^{\frac{1}{2}}\right)\\&\leqq M_1^2(\varepsilon^2\sigma^2n)^{-1})\sum_{m=1}^{n}E\left\{\sum_{i=1}^{m}\zeta_{m,i}^2\right\}^2\\&=O(n^{-1}\log n).\end{aligned} \tag{3.22}$$

So, (3.19) and (3.20) hold. Thus, we have the lemma.

Proof of Theorem 3.1. Theorem 3.1 follows from Theorem 4.1 in [1] and Lemmas 3.4 and 3.5.

Define the score function $J_n(u)$ by

$$J_n(u)=\begin{cases}J(u) & \text{if } 0\leqq u\leqq n/(n+1),\\ J(n/(n+1)) & \text{if } n/(n+1)<u\leqq 1.\end{cases}$$

If $J(u)$ is twice differentiable, then $J''(u)=J_n''(u)$ $(0\leqq u<n/(n+1))$. If

$$|J_n''(u)|\leqq Mn^{\frac{1}{2}-\delta} \quad (0\leqq u\leqq 1) \tag{3.23}$$

for some $\delta(0<\delta<\frac{1}{2})$, then we can prove the relation

$$P\left(\max_{1\leqq m\leqq n}\left|\sum_{i=1}^{m}k_{m,i}^{(n)}(H(|\xi_i|))\zeta_{m,i}^2\right|>\varepsilon\sigma n^{\frac{1}{2}}\right)\to 0 \quad (n\to\infty) \tag{3.24}$$

by the same method used in (3.22). Thus, we can slightly extend Theorem 3.1 to the following form.

Theorem 3.2. *If, among conditions in Theorem* 3.1, *the score-function J is replaced by a score-function J^* for which* (3.23) *holds, then the conclusion of Theorem* 3.1 *remains true.*

4. Functional Central Limit Theorem for $\{T_{n,m}\}$ when J is a General Score-function

In this section, we shall consider a more general score-function.

First, we note that $\int_0^1 |J(u)|^{2+\gamma} du \leqq M$ for some $\gamma > 0$, then $E|\eta_i|^{2+\gamma} \leqq M$ where η_i is the random variable defined by (3.6).

Lemma 4.1. *Let $\{\eta_i\}$ be a strictly stationary, absolutely regular sequence of random variables with $\beta(n)$. If $E\eta_i = 0$ and $E|\eta_i|^{2+\gamma} = \lambda < \infty$ and $\sum \{\beta(n)\}^{\gamma/(2+\gamma)} < \infty$ for some $\gamma > 0$, then*

$$E\left(\sum_{i=1}^{n} \eta_i\right)^2 \leqq M\lambda^\tau$$

where $\tau = 2/(2+\gamma)$.

Proof. The proof is easily obtained since from Lemma 2.1

$$|E\eta_i \eta_j| \leqq M\{E|\eta_0|^{2+\gamma}\}^{2/(2+\gamma)} \{\beta(i-j)\}^{\gamma/(2+\gamma)}.$$

Let δ be any number such that $0 \leqq \delta < \frac{1}{2}$. Let the class $\mathscr{L}_\delta = \{L\}$ consisting of functions L on I possessing the following properties:

(i) L is twice differentiable inside $[0, 1)$
(ii) L is nondecreasing, and
(iii) as $u \nearrow 1$

$$(d^{(i)}/du^{(i)})\, L(u) = O((1-u)^{-\frac{1}{2}-i+\delta}) \qquad (i = 0, 1, 2). \tag{4.1}$$

It is obvious that for $L \in \mathscr{L}_\delta \int_0^1 |L(u)|^{2+\gamma} du \leqq M$ where $0 < \gamma < 3\delta/(1-2\delta)$.

Theorem 4.1. *Let $\{\xi_i\}$ be a strictly stationary, absolutely regular process with function $\beta(n)$. Let $J \in \mathscr{L}_\delta (0 < \delta < \frac{1}{2})$ be given. If $\int_0^1 |J(u)|^{2+\gamma} du < \infty$ where $\gamma = 4\delta/(1-2\delta)$ and $\lambda > 0$ is sufficiently small and if*

$$\beta(n) = O(n^{-60(2-\delta)/\delta}), \tag{4.2}$$

then for any $\varepsilon > 0$

$$P\left(\max_{1 \leqq m \leqq n} \left| T_{n,m} - \mu_{n,m} - \sum_{i=1}^{m} c_{n,i}\eta_i \right| \geqq \varepsilon n^{\frac{1}{2}}\right) \leqq M_0 \lambda^\tau \qquad (n \to \infty) \tag{4.3}$$

where M_0 depends only on ε and σ, and $\tau = 2/(2+\gamma)$.

Proof. Let $\alpha = 1 - \delta/3$ and define d as the smallest integer j for which $\alpha^j \leqq 1/5$ holds. For any $n \geqq 1$, let

$$A_1 = A_{n,1} = \{u:\ 1 - n^{-\alpha} < u \leqq 1\}$$
$$A_j = A_{n,j} = \{u:\ 1 - n^{-\alpha^j} < u \leqq 1 - n^{-\alpha^{j-1}}\} \qquad (j = 2, \ldots, d-1)$$
$$A_d = A_{n,d} = \{u:\ 0 \leqq u \leqq 1 - n^{-\alpha^d}\}. \tag{4.4}$$

Let $\chi(u\colon A)$ be the indicator of the set A. For every $n \geqq 1$, we define functions $K_j(u) = K_{n,j}(u)$ $(j=1,\dots,d)$ on I by

$$K_1(u) = K_{n,1}(u) = \begin{cases} 0 & \text{if } 0 \leqq u \leqq 1 - n^{-\alpha} \\ J(u) & \text{if } 1 - n^{-\alpha} < u \leqq n/(n+1) \\ J(n/(n+1)) & \text{if } n/(n+1) \leqq u \leqq 1. \end{cases} \tag{4.5}$$

$$K_j(u) = K_{n,j}(u) = J(u)\,\chi(u\colon A_j) \qquad (j=2,\dots,d).$$

Then, for almost all $u \in [0, n/(n+1))$ $(n \geqq 1)$

$$J(u) = \sum_{j=1}^{d} K_j(u), \qquad J'(u) = \sum_{j=1}^{d} K_j'(u), \qquad J''(u) = \sum_{j=1}^{d} K_j''(u) \tag{4.6}$$

and

$$\begin{aligned} &M_1\,\chi(u\colon A_j)\,[\exp_n\{(\tfrac{1}{2}+i-\delta)\,\alpha^j\}] \\ &\quad \leqq (d^{(i)}/du^{(i)})\,K_j(u) \leqq M_2\,\chi(u\colon A_j)\,[\exp_n\{(\tfrac{1}{2}+i-\delta)\,\alpha^{j-1}\}] \\ &\qquad (i=0,1,2;\ j=1,\dots,d). \end{aligned} \tag{4.7}$$

Since it is obvious that

$$E\left(T_{n,m} - \mu_{n,m} - \sum_{i=1}^{m} c_{n,i}\,\eta_i\right)^2 \leqq M m^2,$$

so, putting $n_0 = [\exp_n \alpha^d]$, we have that for any $\varepsilon > 0$

$$\begin{aligned} &P\left(\max_{1 \leqq m \leqq n_0} \left| T_{n,m} - \mu_{n,m} - \sum_{i=1}^{m} c_{n,i}\,\eta_i \right| \geqq \varepsilon n^{\frac{1}{2}}\right) \\ &\quad \leqq \sum_{m=1}^{n_0} P\left(\left| T_{n,m} - \mu_{n,m} - \sum_{i=1}^{m} c_{n,i}\,\eta_i \right| \geqq \varepsilon n^{\frac{1}{2}}\right) \\ &\quad \leqq M n_0^3/(\varepsilon n^{\frac{1}{2}})^2 = O(n^{-r}) \end{aligned} \tag{4.8}$$

for some $r > 0$. Thus, to prove (4.3), it is enough to show that for any $\varepsilon > 0$ there exists an integer $N = N(\varepsilon)$ such that for all $n \geqq N$ and for some τ $(0 < \tau < 1)$

$$P\left(\max_{n_0 \leqq m \leqq n} \left| T_{n,m} - \mu_{n,m} - \sum_{i=1}^{m} c_{n,i}\,\eta_i \right| \geqq \varepsilon n^{\frac{1}{2}}\right) \leqq M_1\,\lambda^{\tau}. \tag{4.9}$$

From (3.9) we have that

$$\begin{aligned} &\left| T_{n,m} - \mu_{n,m} - \sum_{i=1}^{m} c_{n,i}\,\eta_i \right| \\ &\quad = \left| \sum_{i=1}^{m} \{c_{n,i}\,s(\xi_i)(J(\xi_{m,i}^*) + \zeta_{m,i}\,J'(\xi_{m,i}^*) + \zeta_{m,i}^2\,k_{m,i}(H(|\xi_i|)))\} \right. \\ &\qquad \left. - \mu_{n,m} - \sum_{i=1}^{m} c_{n,i}\,\eta_i \right| \end{aligned}$$

$$\leqq M_2 \sum_{i=1}^{m} |J_n(\xi^*_{m,i}) - J_n(H(|\xi_i|))|$$

$$+ \sum_{j=1}^{d} \left[V^{(j)}_{n,m} + \frac{2}{m+1} \left| \sum_{j=1}^{m} \int s(x) \{u(|x| - |\xi_j|) \right.\right.$$

$$\left.\left. - H(|x|)\} K'_j(H(|x|))\, dF(x) \right| \right]$$

$$+ \sum_{j=1}^{d} \sum_{i=1}^{m} |c_{n,i}| \cdot k^{(j)}_{n,m,i}(H(|\xi_i|))\, \zeta^2_{m,i} \tag{4.10}$$

where $V^{(j)}_{n,m}$ and $k^{(j)}_{n,m,i}(H(|\xi_i|))$ are the ones obtained in (3.10) and (3.8) on replacing J by K_j, respectively. We note here that if $H(|\xi_i|) \in A_j$ $(j=1, \ldots, d-1)$ then $H(|\xi_i|) > \frac{1}{2}$ and so

$$\xi^*_{m,i} = \frac{m-1}{m+1} H(|\xi_i|) + \frac{1}{m+1} \leqq \frac{m}{m+1} H(|\xi_i|) \leqq H(|\xi_i|).$$

Now,

$$\sum_{i=1}^{m} |J_n(\xi^*_{m,i}) - J_n(H(|\xi_i|))|$$

$$= \sum_{i=1}^{m} (H(|\xi_i|) - \xi^*_{m,i}) J'_n(\vartheta_i) \leqq 3 \sum_{j=1}^{d} \sum_{i=1}^{m} (m+1)^{-1} K'_j(\vartheta_i)$$

where for each i $(i=1, \ldots, m)$ $\vartheta_i = H(|\xi_i|) - \kappa_i(H(|\xi_i|) - \xi^*_{m,i})$ and κ_i is a random variable such that $|\kappa_i| \leqq 1$. Since for each j $(j=1, \ldots, d-1)$ and i $(i=1, \ldots, m)$

$$E\chi(\vartheta_i\colon A_j) \leqq P(\vartheta_i \geqq 1 - n^{-\alpha^j}) \leqq P(H(|\xi_i|) \geqq 1 - n^{-\alpha^j}) \leqq M n^{-\alpha^j}$$

so for each j $(j=1, \ldots, d-1)$ and i $(i=1, \ldots, m)$

$$EK'_j(\vartheta_i) \leqq [\exp_n\{(\tfrac{3}{2} - \delta)\alpha^{j-1}\}]\, E\chi(\vartheta_i\colon A_j) \leqq \exp_n\{(\tfrac{1}{2} - \delta)\alpha^{j-1}\}.$$

Let k_0 and k_1 be the integers such that $2^{k_0} \leqq n_0 < 2^{k_0+1}$ and $2^{k_1-1} < n \leqq 2^{k_1}$. Then for each j $(j=1, \ldots, d-1)$ and $\varepsilon > 0$

$$P\left(\max_{n_0 \leqq m \leqq n} \sum_{i=1}^{m} (m+1)^{-1} K'_j(\vartheta_i) \geqq 4\varepsilon n^{\frac{1}{2}} \right)$$

$$\leqq \sum_{k=k_0}^{k_1-1} P\left(\max_{2^k \leqq m \leqq 2^{k+1}} \sum_{j=1}^{m} K'_j(\vartheta_i) \geqq 2^{k+2} \varepsilon n^{\frac{1}{2}} \right)$$

$$\leqq (k_1 - k_0) \max_{k_0 \leqq k \leqq k_1} P\left(\max_{1 \leqq m \leqq 2^{k+1}} \sum_{i=1}^{m} K'_j(\vartheta_i) \geqq 2^{k+2} \varepsilon n^{\frac{1}{2}} \right).$$

Let j $(j=1, \ldots, d-1)$ and k $(k_0 \leqq k \leqq k_1)$ be fixed. We note that $K'_j(\vartheta_i)$ is an absolutely regular sequence of random variables with the same $\beta(n)$ as that of $\{\xi_i\}$, and for any $v (\leqq 2^{k+1})$

$$P\left(\sum_{i=\nu}^{2^{k+1}} K_j'(\vartheta_i) \geqq 2^{k+1}\varepsilon n^{\frac{1}{2}}\right)$$
$$\leqq P\left(\sum_{i=\nu}^{2^{k+1}} |K_j'(\vartheta_i)-EK_j'(\vartheta_i)| \geqq 2^{k}\varepsilon n^{\frac{1}{2}}\right)$$
$$\leqq (2^k\varepsilon n^{\frac{1}{2}})^{-1}\sum_{i=\nu}^{2^{k+1}} E|K_j'(\vartheta_i)-EK_j'(\vartheta_i)| \leqq M_2 n^{-\gamma}$$

where γ and M_2 are positive numbers which are independent on j, k and ν. Hence, putting $p=[2^{k/4}]$, we have from Lemma 2.4 that

$$P\left(\max_{1\leqq m\leqq 2^{k+1}}\sum_{i=1}^{m} K_j'(\vartheta_i) \geqq 2^{k+2}\varepsilon n^{\frac{1}{2}}\right)$$
$$\leqq 2P\left(\left|\sum_{i=1}^{2^{k+1}} (K_j'(\vartheta_i)-EK_j'(\vartheta_i)\right| \geqq 2^{k}\varepsilon n^{\frac{1}{2}}\right)$$
$$+2^{\frac{3k}{4}}\beta(2^{\frac{k}{4}})+M_2 n^{-\gamma}=O(n^{-\gamma}).$$

As $k_1-k_0=O(\log n)$, so for each j ($j=1,\dots,d-1$)

$$\mathrm{P}\left(\max_{n_0\leqq m\leqq n}\sum_{i=1}^{m}(m+1)^{-1}K_j'(\vartheta_i)\geqq 4\varepsilon n^{\frac{1}{2}}\right)=O(n^{-\gamma'}) \tag{4.11}$$

where $0<\gamma'<\gamma$.

On the other hand, for $j=d$, using the properties of $J\in\mathscr{L}_\delta$, we have that for any $\varepsilon>0$

$$\sum_{i=1}^{m}(m+1)^{-1}K_j'(\vartheta_i)\leqq M[\exp_n\{\tfrac{1}{5}(\tfrac{3}{2}-\delta)\}]<\varepsilon n^{\frac{1}{2}}$$

and thus (4.11) also holds for $j=d$. Hence, for any $\varepsilon>0$

$$P\left(\max_{n_0\leqq m\leqq n}\sum_{i=1}^{m}|J_n(\xi_{m,i}^{*})-J_n(H(|\xi_i|))|\geqq\varepsilon n^{\frac{1}{2}}\right)\to 0 \tag{4.12}$$

as $n\to\infty$.

Since the sequence $\theta_{n,i}^{(j)}$, defined by

$$\theta_{n,i}^{(j)}=\int s(x)\{u(|x|-|\xi_i|)-H(|x|)\}K_j'(H(|x|))\,dF(x),$$

is a strictly stationary, absolutely regular sequence of random variables with the function $\beta(n)$ satisfying (4.2), and $E\theta_{n,i}^{(j)}=0$ and $E|\theta_{n,i}^{(j)}|^{2+\gamma'}<\infty$ for some $\gamma'>0$, so

$$E\left(\sum_{i=1}^{m}\theta_{n,i}^{(j)}\right)^2=\mathrm{cm}(1+o(1))$$

for some constant c (cf. [12]). Thus, from the Bonferonni inequality

$$P\left(\max_{n_0 \leqq m \leqq n} 2(m+1)^{-1}\left|\sum_{i=1}^{m} \theta_{n,i}^{(j)}\right| \geqq \varepsilon n^{\frac{1}{2}}\right)$$
$$\leqq \sum_{m=n_0}^{n} P\left((m+1)^{-1}\left|\sum_{i=1}^{m} \theta_{n,i}^{(j)}\right| \geqq \varepsilon n^{\frac{1}{2}}\right)$$
$$\leqq M(\varepsilon^2 n)^{-1} \sum_{m=n_0}^{n} m^{-1} = O(n^{-1}\log n) \qquad (j=1,\dots,d). \tag{4.13}$$

Next, from Lemma 4.1 it follows easily that for all $m \geqq n_0$

$$E|V_{n,m}^{(j)}|^2 \leqq M \operatorname{Var}(\eta_0) \leqq M_0 \lambda^\tau$$

and so

$$P(\max_{n_0 \leqq m \leqq n} |V_{n,m}^{(j)}| \geqq \varepsilon n^{\frac{1}{2}})$$
$$\leqq (\varepsilon^2 \sigma^2 n)^{-1} \sum_{m=1}^{n} E|V_{n,m}^{(j)}|^2 \leqq M \lambda^\tau \qquad (j=1,\dots,d). \tag{4.14}$$

Finally, let

$$n_j = \exp_n\{\alpha^{d-j}\} \qquad (j=1,\dots,d).$$

Let $\{\chi_{1.m,i}^{(j)}: n_0 \leqq m \leqq n_j\}$ $(j=1,\dots,d)$ and $\{\chi_{2,m,i}^{(j)}: n_j \leqq m \leqq n\}$ $(j=1,\dots,d)$ be collections of random variables, defined, respectively, by

$$\chi_{1,m,i}^{(j)} = \begin{cases} 1 & \text{if } |\zeta_{m,i}| \leqq (\log m)^2 [\exp_m\{-(1-\rho)(1+\alpha)\alpha^{j-1}/2\}] \\ 0 & \text{otherwise} \end{cases} \tag{4.15}$$

and

$$\chi_{2,m,i}^{(j)} = \begin{cases} 1 & \text{if } |\zeta_{m,i}| \leqq (\log m)^2 [\exp_m\{-(1-\rho)(1+\alpha^j)/2\}] \\ 0 & \text{otherwise} \end{cases} \tag{4.16}$$

where $\rho = \delta/6$.

Noting $|\zeta_{m,i}| \leqq 1$ for all m and i, from (4.10) we have that for any j $(1 \leqq j \leqq d-1)$ and for any m $(n_0 \leqq m \leqq n_{d-j-1})$

$$\sum_{i=1}^{m} |c_{n,i}|\,|k_{n,m,i}^{(j)}(H(|\xi_i|))|\,\zeta_{m,i}^2$$
$$\leqq \|h\| \sum_{i=1}^{m} \chi(H(|\xi_i|): A_{n,j})$$
$$\cdot \{(\log n)^4 [\exp_n\{(\tfrac{5}{2}-\delta)\alpha^{j-1} - (1-\rho)(1+\alpha)\alpha^{j-1}\}]$$
$$+(1-\chi_{1,m.i}^{(j)})[\exp_n\{(\tfrac{5}{2}-\delta)\alpha^{j-1}\}]\}. \tag{4.17}$$

Accordingly, for each j $(1 \leqq j \leqq d-1)$

$$P\left(\max_{n_0 \leqq m \leqq n_{d-j-1}} \sum_{j=1}^{m} |c_{n,i}|\,|k_{n,m,i}^{(j)}(H(|\xi_i|))|\,\zeta_{m,i}^2 > 2\varepsilon n^{\frac{1}{2}}\right)$$
$$\leqq P\left(\|h\| \sum_{i=1}^{n_{d-j-1}} \chi(H(|\xi_i|)\colon A_{n,j})(\log n)^4 \cdot \left[\exp_n\left\{\left(\frac{1}{2}-\frac{\delta}{2}\right)\alpha^{j-1}+\frac{\delta\alpha^j}{6}\right\}\right] \geqq \varepsilon n^{\frac{1}{2}}\right)$$
$$+P\left(\max_{n_0 \leqq m \leqq n} \|h\| \sum_{i=1}^{m} \chi(H(|\xi_i|)\colon A_{n,j})(1-\chi_{1,m,i}^{(j)}) \cdot [\exp_n\{(\tfrac{5}{2}-\delta)\,\alpha^{j-1}\}] \geqq \varepsilon n^{\frac{1}{2}}\right) = I_1 + I_2, \quad \text{(say)}. \tag{4.18}$$

Since for any i and j $(1 \leqq j \leqq d-1)$

$$P(H(|\xi_i|) \in A_{n,j}) = P(H(|\xi_i|) \geqq 1 - n^{-\alpha^j}) \leqq M n^{-\alpha^j},$$

so for any j $(1 \leqq j \leqq d-1)$

$$I_1 \leqq M n^{-\frac{1}{2}} \sum_{i=1}^{n_{d-j-1}} P(H(|\xi_i|) \in A_{n,j})(\log n)^4 \cdot \left[\exp_n\left\{\left(\frac{1}{2}-\frac{\delta}{2}\right)\alpha^{j-1}+\frac{\delta\alpha^j}{6}\right\}\right]$$
$$\leqq M(\log n)^4 \left[\exp_n\left\{-\tfrac{1}{2}+\alpha^{j-1}-\alpha^j+\left(\frac{1}{2}-\frac{\delta}{2}\right)\alpha^{j-1}+\frac{\delta\alpha^j}{6}\right\}\right]$$
$$= O(n^{-r'}) \tag{4.19}$$

where r' is a positive constant.

On the other hand, from Lemma 2.3 and (4.2)

$$I_2 \leqq \sum_{m=n_0}^{n_{d-j-1}} P\left(\|h\| \sum_{i=1}^{m} \chi(H(|\xi_i|)\colon A_{n,j}) \cdot (1-\chi_{1,m,i}^{(j)})[\exp_n\{(\tfrac{5}{2}-\delta)\,\alpha^{j-1})] \geqq \varepsilon n^{\frac{1}{2}}\right)$$
$$\leqq M n^{-\frac{1}{2}} \sum_{m=n_0}^{n_{d-j-1}} [\exp_n\{(\tfrac{5}{2}-\delta)\,\alpha^{j-1}\}]\,\|h\| \cdot \sum_{i=1}^{m} E\{\chi(H(|\xi_i|)\colon A_{n,j})(1-\chi_{1,m,i}^{(j)})\}$$
$$\leqq M n^{-\frac{1}{2}} \sum_{m=n_0}^{n_{d-j-1}} [\exp_n\{(\tfrac{5}{2}-\delta)\,\alpha^{j-1}\}] \cdot m^{1-\alpha+\rho}\{e^{-M_2(\log m)^2}+m^{1-\rho}\beta(m^\rho)\}$$
$$\leqq M[\exp_n\{-\tfrac{1}{2}+(\tfrac{5}{2}-\delta)\,\alpha^{j-1}\}]\, n_0^{-17-\alpha+10\delta} = O(n^{-r''}) \tag{4.20}$$

for any j $(1 \leqq j \leqq d-1)$ and for some $r''>0$. From (4.19) and (4.20) it follows that

$$P\left(\max_{n_0 \leqq m \leqq n_{d-j-1}} \sum_{i=1}^{m} |c_{n,i}|\,|k_{n,m,i}^{(j)}(H(|\xi_i|))|\,\zeta_{m,i}^2 > 2\varepsilon n^{\frac{1}{2}}\right) = O(n^{-r}) \tag{4.21}$$

for any j $(1 \leqq j \leqq d-1)$ and for some $r>0$.

Similarly, for j $(1 \leqq j \leqq d-1)$ and for some $r>0$

$$\begin{aligned} &P\left(\sum_{n_{d-j-1} \leqq m \leqq n} \sum_{i=1}^{m} |c_{n,i}|\,|k_{n,m,i}^{(j)}(H(|\xi_i|))|\,\zeta_{m,i}^2 \geqq 2\varepsilon n^{\frac{1}{2}}\right) \\ &\leqq P\left(\|h\| \sum_{i=1}^{n} \chi(H(|\xi_i|)\colon A_{n_j})(\log n)^4 \right. \\ &\quad \left. \cdot [\exp_n\{(\tfrac{5}{2}-\delta)\alpha^{j-1} - (1-\rho)(1+\alpha^j)\}] \geqq \varepsilon n^{\frac{1}{2}}\right) \\ &\quad + P\left(\max_{n_{d-j-1} \leqq m \leqq n} \|h\| \sum_{i=1}^{m} \chi(H(|\xi_i|)\colon A_{n,j}) \right. \\ &\quad \left. \cdot (1-\chi_{2,m,i}^{(j)})[\exp_n\{(\tfrac{5}{2}-\delta)\}] \geqq \varepsilon n^{\frac{1}{2}}\right) \\ &= O(n^{-r}). \end{aligned} \tag{4.22}$$

For $j=d$ and for all m $(n_0 \leqq m \leqq n)$ it follows from Lemma 3.2.

$$\begin{aligned} &E\left(\sum_{i=1}^{m} |c_{n,i}|\,|k_{n,m,i}^{(d)}(H(|\xi_i|))|\,\zeta_{m,i}^2\right)^2 \\ &\leqq \|h\|\,[\exp_n\{2(\tfrac{5}{2}-\delta)\alpha^d\}] \cdot E\left(\sum_{i=1}^{m} \zeta_{m,i}^2\right)^2 \\ &\leqq M[\exp_n\{(5-2\delta)\alpha^d\}] \cdot m^{-1}. \end{aligned} \tag{4.23}$$

Since, from the definition of d, $(5-2\delta)\alpha^d - 1 < 0$, so

$$\begin{aligned} &P\left(\max_{n_0 \leqq m \leqq n} \sum_{i=1}^{m} |c_{n,i}|\,|k_{n,m,i}^{(d)}(H(|\xi_i|))|\,\zeta_{m,i}^2 \geqq 2\varepsilon n^{\frac{1}{2}}\right) \\ &\leqq M n^{-1}[\exp_n\{(5-2\delta)\alpha^d\}] \sum_{m=n_0}^{n} m^{-1} = O(n^{-r}) \end{aligned} \tag{4.24}$$

for some $r>0$. Hence, (4.9) follows from (4.12)–(4.14), (4.21), (4.22) and (4.24), and the proof of Theorem 4.1 is completed.

Theorem 4.2. *Assume that the conditions of Theorem* 4.1 *are satisfied. If* $h(t)$ *satisfies the Lipschitz condition*

$$|h(t_2) - h(t_1)| \leqq M_0 |t_2 - t_1| \qquad (t_1, t_2 \in I), \tag{4.25}$$

then

$$P\left(\max_{1 \leqq m \leqq n} \left|\sum_{i=1}^{m} c_{n,i}\eta_i\right| \geqq \varepsilon n^{\frac{1}{2}}\right) \leqq M\lambda^{\tau} \qquad (n \to \infty) \tag{4.26}$$

for any $\varepsilon > 0$ *where* $\tau = 2/(2+\gamma)$.

Proof. Let

$$\sigma_0^2=\lim_{n\to\infty}\frac{1}{n}\operatorname{Var}\left(\sum_{i=1}^{n}\eta_i\right). \tag{4.27}$$

If $\sigma_0>0$, then from Theorem 1 in [11], $Y_n \xrightarrow{\mathscr{D}} U$ where $\{Y_n\}$, U are random elements, defined by (3.15) and (3.5), respectively. Thus, noting $\sigma_0<\lambda^{1/(2+\gamma)}$, from Theorem 1 in [4, Chap. 1, §3]

$$\begin{aligned}
\lim_{n\to\infty} P\left(\max_{1\leqq m\leqq n}\left|\sum_{i=1}^{m} c_{n,i}\eta_i\right|\geqq\varepsilon n^{\frac{1}{2}}\right)
&=\lim_{n\to\infty} P(\sup_{0\leqq t\leqq 1}|U_n(t)|\geqq\varepsilon/\sigma_0)\\
&=P(\sup_{0\leqq t\leqq 1}|U(t)|\geqq\varepsilon/\sigma_0)\\
&\leqq\sigma_0^2/\varepsilon^2\int_0^1 h^2(t)\,dt\leqq M\lambda^{2/(2+\gamma)}.
\end{aligned} \tag{4.28}$$

Now, we consider the case where $\sigma_0=0$, i.e.,

$$\sigma_0^2=E\eta_0^2+2\sum_{j=1}^{\infty}E\eta_0\eta_j=0.$$

As

$$|E\eta_0\eta_j|\leqq M\lambda^{\tau}\{\beta(j)\}^{\gamma/(2+\gamma)}\leqq M_1\lambda^{\tau}j^{-120(2-\delta)}$$

and from (4.25)

$$|c_{n,j+i}-c_{n,i}|\leqq M_0 j/(m+1),$$

so

$$\begin{aligned}
E\left(\sum_{i=1}^{m}c_{n,i}\eta_i\right)^2 &=\sum_{i=1}^{m}c_{n,i}^2E\eta_0^2+2\sum_{1\leqq i<j\leqq m}c_{n,i}c_{n,j}E\eta_0\eta_{j-i}\\
&=\sum_{i=1}^{m}c_{n.i}^2\left(E\eta_0^2+2\sum_{j=1}^{m-i}E\eta_0\eta_j\right)\\
&\quad+2\sum_{i=1}^{m-1}\sum_{j=1}^{m-i}c_{n,i}(c_{n,j+i}-c_{n,i})E\eta_0\eta_j\\
&\leqq\|h\|^2 m\left|E\eta_0^2+2\sum_{j=1}^{\infty}E\eta_0\eta_j\right|+\|h\|^2\left|\sum_{i=1}^{m}\sum_{j=m-i+1}^{\infty}E\eta_0\eta_j\right|\\
&\quad+2\|h\|\sum_{i=1}^{m-1}\sum_{j=1}^{m-i}|c_{n,j+i}-c_{n,i}|\,|E\eta_0\eta_j|\\
&\leqq\|h\|^2M_1\lambda^{\tau}\sum_{i=1}^{m}\sum_{j=m-i+1}^{\infty}j^{-120(2-\delta)}\\
&\quad+2\|h\|M_2\lambda^{\tau}\sum_{j=1}^{\infty}j\{\beta(j)\}^{\gamma/(2+\gamma)}\leqq M\lambda^{\tau}.
\end{aligned}$$

Hence, we have that for all m

$$E\left(\sum_{i=1}^{m} c_{n,i}\eta_i\right)^2 \leqq M\lambda^{\tau}$$

and so

$$P\left(\max_{1\leqq m\leqq n}\left|\sum_{i=1}^{m} c_{n,i}\eta_i\right| \geqq \varepsilon n^{\frac{1}{2}}\right)$$
$$\leqq (\varepsilon n^{\frac{1}{2}})^{-2}\sum_{m=1}^{n} E\left(\sum_{i=1}^{m} c_{n,i}\eta_i\right)^2 \leqq M\varepsilon^{-2}\lambda^{\tau}. \tag{4.29}$$

Thus, we have the theorem.

Theorem 4.3. *Let $\{\xi_i\}$ be a strictly stationary, absolutely regular process with function $\beta(n)$. Let J be a twice differentiable score-function which admits the polynomial approximation using function in $\mathscr{L}_\delta$ $(0<\delta<\frac{1}{2})$ as follows: For every $\lambda>0$ there exists a decomposition*

$$J(u)=L_0(u)+L_1(u)-L_2(u) \tag{4.30}$$

where L_0 is a polynomial and $L_i\in\mathscr{L}_\delta$ $(i=1,2)$ for which

$$\int_0^1 \{|L_1(u)|^{2+\gamma}+|L_2(u)|^{2+\gamma}\}\,du<\lambda \tag{4.31}$$

and $\gamma=4\delta/(1-2\delta)$. Assume that $h(t)$ satisfies the condition (4.25). *If* (4.2) *holds and σ^2, defined by* (3.7), *is positive and finite, then the conclusion in Theorem* 3.1 *remains true.*

Proof. The proof easily follows from Theorems 3.1, 4.1 and 4.2.

Remark. If $\{\xi_i\}$ is a sequence of i.i.d. random variables, in Theorem 4.3 we can replace the condition (4.25) by a weaker condition that $h(x)$ is continuous on I. The proof is obtained from the proofs of Theorems 4.2 and 4.3.

5. A Functional Law of the Iterated Logarithm for $\{T_n\}$

In this section we assume that $h(x)=1$ for all $x\in I$.

Let $C_0(\subset C)$ be the space of continuous functions on I vanishing at 0, with the uniform topology and for each $\omega\in\Omega$, define the functions $X_n^*(t,\omega)$ in C_0 as follows:

$$X_n^*(t,\omega)=X_n(t,\omega)(2\log\log n\sigma^2)^{-1} \qquad (n\geqq 3/\sigma^2) \tag{5.1}$$

and $X_n(t,\omega)$ is the one defined by (3.3) with $h(x)=1$ for all $x\in I$. We denote K the set of absolutely continuous functions on I with $f(0)=0$ and

$$\int_0^1 \{f'(t)\}^2\,dt\leqq 1. \tag{5.2}$$

Theorem 5.1. *If the conditions of Theorem 3.1 are satisfied, then for almost all $\omega\in\Omega$, the sequence of functions $\{X_n^*(t,\omega),\ n\geqq 3/\sigma^2\}$ is precompact in C_0 and its derived set coincides with the set K.*

To prove Theorem 5.1, we need the following lemmas.

Lemma 5.1. *If the conditions of Theorem 5.1 are satisfied, then almost all $\omega\in\Omega$, the sequence of functions $\{Y_n^*(t,\omega),\ n\geqq 3/\sigma^2\}$ is precompact in C_0 and its derived set coincides with the set K, where*

$$Y_n^*(t,\omega)=(2\log\log n\sigma^2)^{-\frac{1}{2}}\,Y_n(t,\omega) \tag{5.3}$$

and $Y_n(t,\omega)$ is defined by (3.15) with $h(x)=1$ for all $x\in I$.

Proof. For each $\omega\in\Omega$, let

$$Z_n^*(t,\omega)=(2\log\log n\sigma^2)^{-\frac{1}{2}}\,Z_n(t,\omega) \tag{5.4}$$

where $Z_n(t,\omega)$ is defined by (3.16) with $h(x)=1$ for all $x\in I$. Since $\{\eta_i\}$ are strictly stationary absolutely regular and bounded, and satisfy the conditions of Theorem 1 in [13], so from the theorem we have that for almost all $\omega\in\Omega$, the sequence of functions $\{Z_n^*(t,\omega),\ n\geqq 3/\sigma^2\}$ is precompact in C_0 and its derived set coincides with the set K. Thus, from the definition of $Y_n(t,\omega)$, we have the lemma.

Lemma 5.2. *Under the conditions of Theorem 3.1*

$$P(\lim_{n\to\infty} d(X_n^*, Y_n^*)=0)=1. \tag{5.5}$$

Proof. To prove (5.5), it suffices to show that for every $\varepsilon>0$

$$P\left(\left|T_n-\mu_n-\sum_{i=1}^{n}\eta_i\right|>\varepsilon\,\chi(n)\ \text{i.o.}\right)=0 \tag{5.6}$$

where

$$\chi(n)=(2n\sigma^2\log\log n\sigma^2)^{\frac{1}{2}}. \tag{5.7}$$

From (3.11)–(3.13) and the Bonferonni inequality

$$\begin{aligned}
&\sum_{k=1}^{\infty} P\left(\max_{1\leqq m\leqq k^2}\left|T_m-\mu_m-\sum_{i=1}^{m}\eta_i\right|>3\varepsilon\,\chi(k^2)\right)\\
&\quad\leqq \sum_{k=1}^{\infty} P(\max_{1\leqq m\leqq k^2}|V_m|>\varepsilon\,\chi(k^2))\\
&\qquad+P\left(\max_{1\leqq m\leqq k^2} M_1\sum_{i=1}^{m}\zeta_{m,i}^2>\varepsilon\,\chi(k^2)\right)\\
&\quad\leqq M\sum_{k=1}^{\infty}\{k^{-2}+k^{-\frac{3}{2}}\}<\infty.
\end{aligned} \tag{5.8}$$

So, from the Borel-Cantelli lemma we have

$$\begin{aligned}
&P\left(\left|T_n-\mu_n-\sum_{i=1}^{n}\eta_i\right|>\varepsilon\chi(n)\text{ i.o.}\right)\\
&\leqq P\left(\max_{k^2\leqq n\leqq (k+1)^2}\left|T_n-\mu_n-\sum_{i=1}^{n}\eta_i\right|>\varepsilon\chi(k^2)\text{ i.o.}\right)\\
&\leqq P\left(\max_{1\leqq n\leqq (k+1)^2}\left|T_n-\mu_n-\sum_{i=1}^{n}\eta_i\right|>(\varepsilon/2)\chi((k+1)^2)\text{ i.o.}\right)=0,
\end{aligned}\tag{5.9}$$

which implies (5.6). Thus, the proof is completed.

Proof of Theorem 5.1. The proof is obtained from Lemmas 5.1 and 5.2.

By the same method of the proof of Theorem 5.1, we have the following theorems

Theorem 5.2. *If the conditions of Theorem* 4.3 *are satisfied, then the conclusion of Theorem* 5.1 *remains true.*

6. Almost Sure Invariance Principles and Integral Tests of $\{T_n\}$ for Some ϕ-mixing Processes

In this section, we assume that $\{\xi_j\}$ is a strictly stationary, ϕ-mixing sequence of random variables with function $\phi(n)$, and that $h(x)=1$ for all $x\in I$.

$\{\eta_i\}$, defined by (3.6), is a strictly stationary ϕ-mixing sequence of random variables with the same function $\phi(n)$ as that of $\{\xi_j\}$. Thus, if $E|\eta_i|^{4+\gamma}<\infty$ for some $\gamma>0$, then we can use the martingale approximation method in [8] and [10], from which we have the following:

Let T be an ergodic one to one measure preserving transformation defined on the probability space $(\Omega,\mathscr{A},P)$. Write $L_2(P)$ for the Hilbert space of random variables with finite second moment and define the unitary operator U on $L_2(P)$ by $UX(\omega)=X(T\omega)$ for $X\in L_2(P)$, $\omega\in\Omega$. We define

$$\begin{aligned}
&Y_0=\sum_{j=0}^{\infty}[E\{\eta_j|\mathscr{M}_{-\infty}^{0}\}-E\{\eta_j|\mathscr{M}_{-\infty}^{-1}\}]\in L_2(P)\\
&Y_k=U^kY_0,\qquad k\geqq 1
\end{aligned}\tag{6.1}$$

and

$$Z_0=\sum_{j=0}^{\infty}E\{\eta_j|\mathscr{M}_{-\infty}^{-1}\};\qquad Z_k=U^kZ_0, k\geqq 1.\tag{6.2}$$

Then, for every non-negative integer k

$$EY_k=EZ_k=0,\qquad E|Y_k|^{4+\gamma}<\infty,\qquad E|Z_k|^{4+\gamma}<\infty\tag{6.3}$$

and

$$\eta_k = Y_k - UZ_k + Z_k \tag{6.4}$$

and the sequence $(Y_k, \mathscr{M}^k_{-\infty})$ is a stationary ergodic martingale difference sequence. (cf. Theorem 8.1 in [8]).

Now, we put

$$V_n = \sum_{i=1}^{n} E\{Y_i^2 \mid Y_1, \ldots, Y_{i-1}\}. \tag{6.5}$$

Finally, we define random process $S = \{S(t),\ 0 \leqq t < \infty\}$ by

$$S(t) = \begin{cases} T_k - k\mu & \text{for } t = k \ (k \geqq 0) \\ \text{linearly interpolated} & \text{for } t \in [k, k+1],\ k \geqq 0. \end{cases} \tag{6.6}$$

By the same reason in [8], we use a phrase "if necessary redefining the X_i's on a new probability space" will imply that the joint distributions of the X_i's are kept the same.

Theorem 6.1. *Let $\{\xi_n\}$ be a strictly stationary, ϕ-mixing sequence. Let J be a score-function having a bounded second derivative and assume that $\phi(n) = O(n^{-4})$. For $\alpha \geqq 0$, let*

$$f_\alpha(t) = t(\log\log t)^{-\alpha}, \qquad t > e^e \tag{6.7}$$

and suppose that as $t \to \infty$

$$|V_n - n\sigma^2| = o(f_\alpha(t)) \quad \text{a.s.} \tag{6.8}$$

Then, upon redefining $\{S(t),\ 0 \leqq t < \infty\}$ on a new probability space, if necessary, there exists a Brownian motion $W = \{W(t),\ 0 \leqq t < \infty\}$ such that as $n \to \infty$

$$|S(t) - \sigma W(t)| = o(t^{\frac{1}{2}}(\log\log t)^{(1-\alpha)/2}) \quad \text{a.s.} \tag{6.9}$$

The following is a theorem concerning integral tests for $\{T_n\}$.

Theorem 6.2. *Under the conditions of Theorem 6.1, we have the followings:*

(a) *For every real function φ, $0 < \varphi \nearrow$*

$$P(S(n) > V_n^{\frac{1}{2}} \varphi(V_n) \text{ i.o.}) = 0 \text{ (or } 1)$$

according as $I(\varphi) < \infty$ (or $= \infty$), where

$$I(\varphi) = \int_1^\infty \frac{\varphi(t)}{t} \exp\left(\frac{\varphi^2(t)}{2}\right) dt. \tag{6.11}$$

(b) *Let $M_n = \max_{1 \leqq i \leqq n} |S(i)|$. Then for every real function φ, $0 < \varphi \nearrow$,*

$$P(M_n < V_n^{\frac{1}{2}} \{\varphi(V_n)\}^{-1} \text{ i.o.}) = 0 \text{ (or } 1) \tag{6.12}$$

according as $I_1(\varphi)<\infty$ (or $=\infty$), *where*

$$I_1(\varphi)=\int_1^\infty \frac{\varphi^2(u)}{u}\exp\left(-\frac{8\varphi^2(u)}{\pi^2}\right)du. \tag{6.13}$$

The proofs of Theorems 6.1 and 6.2 need following lemmas.

Lemma 6.1. *Under the conditions of Theorem* 6.1 *we have that*

$$S^{(1)}(t)=\sigma W(t)+o(t^{\frac{1}{2}}(\log\log t)^{(1-\alpha)/2}) \quad \text{a.s.} \tag{6.14}$$

as $t\to\infty$, *where* $S^{(1)}=\{S^{(1)}(t),\ 0\leqq t<\infty\}$ *is a random process defined by*

$$S^{(1)}(t)=S_k^{(1)}=(T_k-k\mu) \qquad \text{if } k\leqq t<k+1,\ k\geqq 0. \tag{6.15}$$

Proof. The proof is analogous to the proof of Lemma 6 in [18].

Lemma 6.2. *Under the conditions of Theorem* 6.1 *we have that as* $n\to\infty$

$$\sup_{k>n}\left\{\left|T_k-k\mu-\sum_{i=1}^k \eta_i\right|[k^{\frac{1}{2}}(\log\log k)^{(1-\alpha)/2}]^{-1}\right\}\xrightarrow{P}0. \tag{6.16}$$

Proof. Let

$$c_k=\{k(\log\log k)\}^{-\frac{1}{2}}, \qquad k\geqq e^e.$$

From the proof of Lemma 3.5, we have that there exists a γ $(0<\gamma<1/3)$ such that for any $\varepsilon>0$

$$P\left(\max_{1\leqq m\leqq n}\left|T_m-m\mu-\sum_{i=1}^m \eta_i\right|>\varepsilon\sigma n^{\frac{1}{2}}\right)=O(n^{-1+\gamma}). \tag{6.17}$$

Thus, for any $\varepsilon>0$

$$\begin{aligned}
&P\left(\sup_{k>n} c_k\left|T_k-k\mu-\sum_{i=1}^k \eta_i\right|>2\varepsilon\right)\\
&\quad\leqq \sum_{k=[n^{\frac{1}{2}}]} P\left(\max_{k^2\leqq m\leqq (k+1)^2}\left|T_k-k\mu-\sum_{i=1}^k \eta_i\right|>\varepsilon(k+1)\right)\\
&\quad\leqq \sum_{k=[n^{\frac{1}{2}}]} P\left(\max_{1\leqq m\leqq (k+1)^2}\left|T_k-k\mu-\sum_{i=1}^k \eta_i\right|>\varepsilon(k+1)\right)\\
&\quad\leqq M\sum_{k=[n^{\frac{1}{2}}]}(k+1)^{-\frac{4}{3}}=O(n^{-1/6})\to 0
\end{aligned}$$

as $n\to\infty$. Hence, we have the lemma.

The proof of Theorem 6.1 is obtained from Theorem 4.3 in [8] and Lemmas 6.1 and 6.2, and that of Theorem 6.2 follows from Theorems 5.2 and 6.3 in [8] and Lemmas 6.1 and 6.2.

For a more general score-function the following theorems hold.

Theorem 6.3. *Let $\{\xi_i\}$ be a strictly stationary, ϕ-mixing sequences. Let J be the score-function defined in Theorem 4.3. Then, under the analogous conditions to the ones in Theorem 6.1, the conclusion in Theorem 6.1 remains true.*

Theorem 6.4. *Under the conditions of Theorem 6.3, the conclusion in Theorem 6.2 remains true.*

The proofs of these theorems are similar to those of Theorems 6.1 and 6.2 and so are omitted.

7. Some Concluding Remarks

Remark 7.1. All the results in the preceding sections are concerned with scores $a_n(i)$ defined by

$$a_n(i)=J\left(\frac{i}{n+1}\right), \quad 1\leqq i\leqq n.$$

But, we can prove the analogous results for more general scores $a_n(i)$. Firstly, Theorem 4.3 remains true if T_n is replaced by

$$T_n^{(0)}=\sum_{i=1}^{n} c_{n,i}\, s(\xi_i)\, a_n(i)$$

where the scores $a_n(i)$ satisfying the following condition

(C) For given scores $a_n(i)$ there exists a function J of the type defined in Theorem 3.2 for which the following relations hold:

(i) $\lim_{n\to\infty} a_n(1+[u\,n])=J(u),\ 0\leqq u<1;$ (7.1)

(ii) for any $\varepsilon>0$

$$q_n=q_n(\varepsilon)=P\left(n^{-\frac{1}{2}}\left|\sum_{\substack{i=1\\ R_i<n}}^{n}\left\{a_n(R_i)-J\left(\frac{R_i}{n+1}\right)\right\}\right|>\varepsilon\right)\to 0\ \ (n\to\infty) \tag{7.2}$$

(iii) $r_n=n^{-\frac{1}{2}}\,a_n(n)\to 0\ \ (n\to\infty).$ (7.3)

Secondly, Theorems 5.2, 6.3 and 6.4 remain true, if $J(u)$ is replaced by the scores $a_n(i)$ satisfying the condition (C′): For given scores $a_n(i)$ there exists a function J of the type defined in Theorem 3.2 for which (7.1), (ii′) $q_n=O((\log n)^{-1})$ and (iii′) $r_n=O((\log n)^{-1})$ hold. The proofs of these assertions are easily obtained and so are omitted.

Remark 7.2. In the preceding sections and Remark 7.1, we studied the case where regression constants are defined by a continuous function $h(x)$ (*cf.* (1.5)). If $\{\xi_i\}$ is a sequence of i.i.d. random variables, we can consider the case where regression constants are arbitrary but bounded. More specifically, let $\{\xi_i\}$ be i.i.d. random variables with continuous *df* $F(x)$. Consider the simple linear rank statistics.

$$T_n' = \sum_{i=1}^{n} c_i s(\xi_i) J\left(\frac{R_{n,i}}{n+1}\right), \quad n \geqq 1 \tag{7.4}$$

where $c_1, \ldots, c_n$ are arbitrary bounded constants, i.e., $|c_i| \leqq M_0$ $(i \geqq 1)$ and $\sum_{i=1}^{n} c_i^2 = O(n)$. Then $\eta_j (j \geqq 1)$ (defined by (3.6)) are i.i.d. random variables with $E\eta_j = 0$. If $J \in \mathscr{L}_\delta (0 < \delta < \frac{1}{2})$, and $\int_0^1 |J(u)|^{2+\gamma} < \infty$ for some small $\lambda > 0$ and $\gamma = 4\delta/(1-2\delta)$, then from Kolmogorov's inequality

$$\begin{aligned} &P\left(\max_{1 \leqq m \leqq n} \left|\sum_{j=1}^{m} c_j \eta_j\right| \geqq \varepsilon \left(\sum_{j=1}^{n} c_j^2\right)^{\frac{1}{2}}\right) \\ &\leqq \left(\varepsilon^2 \sum_{j=1}^{n} c_j^2\right)^{-1} E\left(\sum_{j=1}^{n} c_j \eta_j\right)^2 \leqq M \lambda^{2/(2+\gamma)} \end{aligned} \tag{7.5}$$

and from the proof of Theorem 4.1

$$P\left(\max_{1 \leqq m \leqq n} \left| T_m' - \mu\left(\sum_{i=1}^{m} c_j\right) - \sum_{i=1}^{m} c_j \eta_i \right| \geqq \varepsilon \left(\sum_{i=1}^{n} c_i^2\right)^{\frac{1}{2}}\right) \leqq M \lambda^{2/(2+\gamma)} \tag{7.6}$$

where $\varepsilon > 0$ is arbitrary and n is arbitrary large integer. Thus we have the following:

(i) Let J be a twice differentiable score-function which admits the polynomial approximation stated in Theorem 4.3. Let $\sigma_0 = \{\mathrm{Var}(\eta_0)\}^{\frac{1}{2}}$. If Z_n', defined by

$$Z_n'(t) = \begin{cases} 0 & \text{for } t = 0 \\ \left(\sum_{i=1}^{k} c_i \eta_i\right) \Big/ \sigma_0 \left(\sum_{i=1}^{n} c_i^2\right)^{\frac{1}{2}} & \text{for } t = k/n \ (k = 1, \ldots, n) \\ \text{linearly interpolated} & \text{for } t \in [(k-1)/n, k/n] \ (k = 1, \ldots, n). \end{cases} \tag{7.7}$$

converges weakly to W, then X_n', defined by

$$X_n'(t) = \begin{cases} 0 & \text{for } t = 0 \\ \left(T_k' - \mu\left(\sum_{i=1}^{k} c_i\right)\right) \Big/ \sigma_0 \left(\sum_{i=1}^{n} c_i^2\right)^{\frac{1}{2}} & \text{for } t = k/n \ (k = 1, 2, \ldots, n) \\ \text{linearly interpolated} & \text{for } t \in [(k-1)/n, k/n] \\ & (k = 1, 2, \ldots, n) \end{cases} \tag{7.8}$$

converges weakly to W.

(ii) Under the conditions of (i), for almost all $\omega \in \Omega$, the sequence of functions

$$\left\{\left(2 \log\log \sigma_0^2 \sum_{i=1}^{n} c_i^2\right)^{-\frac{1}{2}} X_n'(t, \omega), n \geqq 3/\sigma_0^2\right\}$$

is precompact in C_0 and its derived set coincides with the set K.

(iii) Under the condition of (i), the conclusions of Theorems 6.3 and 6.4 remain true.

The proofs of (i)–(iii) are obtained by the same methods used in the preceeding sections and so are omitted.

Remark 7.3. Let T_n^* be a simple linear rank statistic defined by

$$T_n^* = \sum_{i=1}^{n} c_{n,i} J\left(\frac{R_{n,i}^*}{n+1}\right) \tag{7.9}$$

where $c_{n,i} = h(i/n+1)\,(i=1,\ldots,n)\,(h\in C)$ and

$$R_{n,i}^* = \sum_{j=1}^{n} u(\xi_i - \xi_j). \tag{7.10}$$

Then, all results in the preceding sections (being replaced T_n by T_n^*) are proved by the same methods as the corresponding ones. Some of them are the extensions of Stigler's results in [16].

If $\xi_i (i \geqq 1)$ are i.i.d. random variables, corresponding results to Remarks 7.1 and 7.2 are also obtained.

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