THE CONSTRUCTION OF A THIRD ORDER SECULAR ANALYTICAL J-S-U-N THEORY BY HORI–LIE TECHNIQUE

Part IV: Derivation of secular perturbation equations and its solution

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Abstract. In this part we find out the 24 equations of secular perturbation equations for the subsystem J-S-U-N. The solution of these equations by the Lagrange–Laplace procedure and the Eigen value Eigen vector is analysed. Also we refer to Hurwitz theorem to test stability.

1. The Lagrange–Laplace Procedure

We have the following set of canonical third order secular perturbation equations for the J-S-U-N subsystem

$$
\begin{aligned}
\frac{\mathrm{d}L'_s}{\mathrm{d}t} &= 0,\\
\frac{\mathrm{d}\lambda'_s}{\mathrm{d}t} &= -\frac{\partial}{\partial L'_s}(F'_0 + F'_1 + F'_2 + F'_3),\\
\frac{\mathrm{d}H'_s}{\mathrm{d}t} &= \frac{\partial}{\partial K'_s}(F'_1 + F'_2 + F'_3),\\
\frac{\mathrm{d}K'_s}{\mathrm{d}t} &= -\frac{\partial}{\partial H'_s}(F'_1 + F'_2 + F'_3),\\
\frac{\mathrm{d}P'_s}{\mathrm{d}t} &= \frac{\partial}{\partial Q'_s}(F'_1 + F'_2 + F'_3),\\
\frac{\mathrm{d}Q'_s}{\mathrm{d}t} &= -\frac{\partial}{\partial P'_s}(F'_1 + F'_2 + F'_3),
\end{aligned}
\tag{1}
$$

where

$$F'_0 = \sum_{s=1}^{4} \frac{k^4 m_0^2 m_{0s} \beta_s^3}{2L_s'^2}; \quad s = 1, 2, 3, 4,$$

and F_1', F_2', F_3' are given in Part III.

We see now that the secular perturbation equations for the case of the four major planets J-S-U-N could be written for H_s', K_s', P_s', Q_s'; $s = 1, 2, 3, 4$ in the following form after dropping the prime of H_s', K_s', P_s', Q_s', for simplicity of writing, and after performing the partial differentiations of F' with respect to the new Poincare' canonical variables.

$$\frac{dH_1}{dt} = aK_1 + bK_2 + cK_3 + dK_4,$$

$$\frac{dH_2}{dt} = a'K_1 + b'K_2 + c'K_3 + d'K_4,$$

$$\frac{dH_3}{dt} = a''K_1 + b''K_2 + c''K_3 + d''K_4,$$

$$\frac{dH_4}{dt} = a'''K_1 + b'''K_2 + c'''K_3 + d'''K_4; \tag{2}$$

$$\frac{dK_1}{dt} = -aH_1 - bH_2 - cH_3 - dH_4,$$

$$\frac{dK_2}{dt} = -a'H_1 - b'H_2 - c'H_3 - d'H_4,$$

$$\frac{dK_3}{dt} = -a''H_1 - b''H_2 - c''H_3 - d''H_4,$$

$$\frac{dK_4}{dt} = -a'''H_1 - b'''H_2 - c'''H_3 - d'''H_4; \tag{3}$$

and

$$\frac{dP_1}{dt} = AQ_1 + BQ_2 + CQ_3 + DQ_4,$$

$$\frac{dP_2}{dt} = A'Q_1 + B'Q_2 + C'Q_3 + D'Q_4,$$

$$\frac{dP_3}{dt} = A''Q_1 + B''Q_2 + C''Q_3 + D''Q_4,$$

$$\frac{dP_4}{dt} = A'''Q_1 + B'''Q_2 + C'''Q_3 + D'''Q_4, \tag{4}$$

$$\frac{dQ_1}{dt} = -AP_1 - BP_2 - CP_3 - DP_4,$$

$$\frac{dQ_2}{dt} = -A'P_1 - B'P_2 - C'P_3 - D'P_4,$$

$$\frac{dQ_3}{dt} = -A''P_1 - B''P_2 - C''P_3 - D''P_4,$$

$$\frac{dQ_4}{dt} = -A'''P_1 - B'''P_2 - C'''P_3 - D'''P_4, \tag{5}$$

where $a, b, c, \ldots, d'''$ and $A, B, C, \ldots, D'''$ are constants and are given by

$$a = 2\left(\frac{1}{2}g_2 - \frac{1}{2}g_{37} + g_{130}\right), \tag{6}$$

$$b = \frac{1}{2}g_{10} - \frac{1}{2}g_{45} + g_{134} + \frac{1}{2}(2g_{30}g_{99} - g_{32}g_{100}) - \frac{1}{2}V_2g_{70}g_{32}, \tag{7}$$

$$\begin{aligned} c = {} & \frac{1}{2}g_{12} + \frac{1}{2}g_{47} + g_{135} + \frac{1}{4}\left\{g_{81}\left(\frac{\partial g_{22}}{\partial L_3} - 2\frac{\partial g_{22}}{\partial L_4}\right) + \right. \\ & + g_{22}\left(\frac{\partial g_{81}}{\partial L_3} - 2\frac{\partial g_{81}}{\partial L_4}\right) + 4g_{24}g_{85} - 2g_{26}g_{86} - \\ & -2g_{32}g_{99} + 4g_{31}g_{100} + g_{57}g_{22}\left(\frac{\partial V_1}{\partial L_3} - 2\frac{\partial V_1}{\partial L_4}\right) + \\ & + V_1g_{22}\left(\frac{\partial g_{57}}{\partial L_3} - 2\frac{\partial g_{57}}{\partial L_4}\right) + V_1g_{57} \times \\ & \left. \times \left(\frac{\partial g_{22}}{\partial L_3} - 2\frac{\partial g_{22}}{\partial L_4}\right) + 4V_2g_{70}g_{31}\right\} \end{aligned} \tag{8}$$

$$\begin{aligned} d = {} & \frac{1}{2}g_{20} - \frac{1}{2}g_{51} + g_{136} + \frac{1}{4}\left\{g_{81}\left(2\frac{\partial g_{23}}{\partial L_4} - \frac{\partial g_{23}}{\partial L_3}\right) - \right. \\ & \left. -g_{23}\left(\frac{\partial g_{81}}{\partial L_3} - 2\frac{\partial g_{81}}{\partial L_4}\right) + 4g_{25}g_{86} - 2g_{26}g_{85}\right\} \\ & -\frac{1}{4}\left\{g_{23}g_{57}\left(\frac{\partial V_1}{\partial L_3} - 2\frac{\partial V_1}{\partial L_4}\right) + V_1g_{23}\left(\frac{\partial g_{57}}{\partial L_3} - 2\frac{\partial g_{57}}{\partial L_4}\right) + \right. \\ & \left. + V_1g_{57}\left(\frac{\partial g_{23}}{\partial L_3} - 2\frac{\partial g_{23}}{\partial L_4}\right)\right\}, \end{aligned} \tag{9}$$

$$a' = b, \tag{10}$$

$$\begin{aligned} b' = {} & 2\left[\frac{1}{2}g_4 + \frac{1}{2}\{V_2(4g_{30}^2 + g_{32}^2) - g_{38}\} + g_{131} + \right. \\ & \left. + \frac{1}{2}(4g_{30}g_{102} - g_{32}g_{103}) - \frac{V_2}{2}(4g_{68}g_{30} + g_{32}g_{71})\right], \end{aligned} \tag{11}$$

$$
\begin{aligned}
c' = {} & \frac{1}{2}g_{14} + \frac{1}{2}(g_{49} - 4V_2 g_{32}(g_{30} + g_{31})) + g_{137} + \\
& + \frac{1}{4}\left\{ g_{82}\left(\frac{\partial g_{22}}{\partial L_3} - 2\frac{\partial g_{22}}{\partial L_4}\right) + g_{22}\left(\frac{\partial g_{82}}{\partial L_3} - 2\frac{\partial g_{82}}{\partial L_4}\right) + \right. \\
& \left. + 4g_{24}g_{87} - 2g_{26}g_{88} + 4g_{103}(g_{30} + g_{31}) - 4g_{32}(g_{104} + g_{102}) \right\} + \\
& + \frac{1}{4}\left\{ g_{22}g_{58}\left(\frac{\partial V_1}{\partial L_3} - 2\frac{\partial V_1}{\partial L_4}\right) + V_1 g_{22}\left(\frac{\partial g_{58}}{\partial L_3} - 2\frac{\partial g_{58}}{\partial L_4}\right) + \right. \\
& \left. + V_1 g_{58}\left(\frac{\partial g_{22}}{\partial L_3} - 2\frac{\partial g_{22}}{\partial L_4}\right) + 4V_2 g_{32}(g_{68} + g_{69}) + 4V_2 g_{71}(g_{30} + g_{31}) \right\}, \qquad (12)
\end{aligned}
$$

$$
\begin{aligned}
d' = {} & \frac{1}{2}g_{16} - \frac{1}{2}g_{53} + g_{138} + \frac{1}{2}\left\{ 4g_{25}g_{88} - \right. \\
& - 2g_{26}g_{87} + 4g_{30}g_{105} - 2g_{32}g_{106} - g_{58}g_{23}\left(\frac{\partial V_1}{\partial L_3} - 2\frac{\partial V_1}{\partial L_4}\right) + \\
& + g_{82}\left(2\frac{\partial g_{23}}{\partial L_4} - \frac{\partial g_{23}}{\partial L_3}\right) - g_{23}\left(\frac{\partial g_{82}}{\partial L_3} - 2\frac{\partial g_{82}}{\partial L_4}\right) - \\
& \left. - V_1 g_{23}\left(\frac{\partial g_{58}}{\partial L_3} - 2\frac{\partial g_{58}}{\partial L_4}\right) - V_1 g_{58}\left(\frac{\partial g_{23}}{\partial L_3} - 2\frac{\partial g_{23}}{\partial L_4}\right) \right\}, \qquad (13)
\end{aligned}
$$

$$a'' = c, \qquad (14)$$

$$b'' = c', \qquad (15)$$

$$
\begin{aligned}
c'' = {} & 2\left[\frac{1}{2}g_6 + \frac{1}{4}\left\{ 2V_1\left(2\frac{\partial g_{22}}{\partial L_4} - \frac{\partial g_{22}}{\partial L_3}\right) - g_{22}^2 \times \right.\right. \\
& \left. \times \left(\frac{\partial V_1}{\partial L_3} - 2\frac{\partial V_1}{\partial L_4}\right) + 4V_1 g_{24}^2 + V_1 g_{26}^2 + 2V_2(4g_{31}^2 + g_{32}^2) - 2g_{39} \right\} + \\
& + g_{132} + \frac{1}{4}\left\{ 8g_{24}g_{89} + g_{83}\left(\frac{\partial g_{22}}{\partial L_3} - 2\frac{\partial g_{22}}{\partial L_4}\right) + g_{22}\left(\frac{\partial g_{83}}{\partial L_3} - 2\frac{\partial g_{83}}{\partial L_4}\right) - \right. \\
& \left. - 2g_{26}g_{90} + 8g_{31}g_{104} - 2g_{32}g_{103} \right\} + \frac{1}{4}\left\{ g_{22}g_{59}\left(\frac{\partial V_1}{\partial L_3} - 2\frac{\partial V_1}{\partial L_4}\right) + \right. \\
& + V_1 g_{22}\left(\frac{\partial g_{59}}{\partial L_3} - 2\frac{\partial g_{59}}{\partial L_4}\right) + V_1 g_{59}\left(\frac{\partial g_{22}}{\partial L_3} - 2\frac{\partial g_{22}}{\partial L_4}\right) + \\
& \left. + V_1(4g_{24}g_{61} - g_{26}g_{63}) - 2V_2(4g_{31}g_{69} + g_{32}g_{71}) \right\} + \frac{1}{48} \times \\
& \times \left\{ -8V_1 g_{22}\left(\frac{1}{2}g_{23}g_{26} + g_{22}g_{24}\right)\left(\frac{\partial V_1}{\partial L_3} - 2\frac{\partial V_1}{\partial L_4}\right) - \right.
\end{aligned}
$$

$$-V_1^2(12g_{22}g_{24}+3g_{23}g_{26})\left(\frac{\partial g_{22}}{\partial L_3}-2\frac{\partial g_{22}}{\partial L_4}\right)-3V_1^2g_{22}g_{26}\times$$
$$\times\left(\frac{\partial g_{23}}{\partial L_3}-2\frac{\partial g_{23}}{\partial L_4}\right)-6V_1^2g_{22}^2\left(\frac{\partial g_{24}}{\partial L_3}-2\frac{\partial g_{24}}{\partial L_4}\right)-$$
$$-3V_1^2g_{22}g_{23}\left(\frac{\partial g_{26}}{\partial L_3}-2\frac{\partial g_{26}}{\partial L_4}\right)\Bigg\}\Bigg], \tag{16}$$

$$d''=\frac{1}{2}g_{18}+\frac{1}{2}\left\{g_{22}g_{23}\left(\frac{\partial V_1}{\partial L_3}-2\frac{\partial V_1}{\partial L_4}\right)+V_1g_{23}\left(\frac{\partial g_{22}}{\partial L_3}-2\frac{\partial g_{22}}{\partial L_4}\right)+\right.$$
$$\left.+V_1g_{22}\left(\frac{\partial g_{23}}{\partial L_3}-2\frac{\partial g_{23}}{\partial L_4}\right)-2V_1g_{26}(g_{24}+g_{25})-g_{55}\right\}+g_{139}+$$
$$+\frac{1}{2}\left\{g_{84}\left(\frac{\partial g_{22}}{\partial L_3}-2\frac{\partial g_{22}}{\partial L_4}\right)-g_{83}\left(\frac{\partial g_{23}}{\partial L_3}-2\frac{\partial g_{23}}{\partial L_4}\right)+\right.$$
$$+g_{22}\left(\frac{\partial g_{84}}{\partial L_3}-2\frac{\partial g_{84}}{\partial L_4}\right)-g_{23}\left(\frac{\partial g_{83}}{\partial L_3}-2\frac{\partial g_{83}}{\partial L_4}\right)+$$
$$+4g_{90}(g_{24}+g_{25})-4g_{26}(g_{91}+g_{89})+$$
$$\left.+4g_{31}g_{106}-2g_{32}g_{105}\right\}+\frac{1}{4}\{(g_{23}g_{59}+g_{22}g_{60})\times$$
$$\times\left(2\frac{\partial V_1}{\partial L_4}-\frac{\partial V_1}{\partial L_3}\right)-V_1g_{23}\left(\frac{\partial g_{59}}{\partial L_3}-2\frac{\partial g_{59}}{\partial L_4}\right)-$$
$$-V_1g_{22}\left(\frac{\partial g_{60}}{\partial L_3}-2\frac{\partial g_{60}}{\partial L_4}\right)-V_1g_{60}\left(\frac{\partial g_{22}}{\partial L_3}-2\frac{\partial g_{22}}{\partial L_4}\right)-$$
$$-V_1g_{59}\left(\frac{\partial g_{23}}{\partial L_3}-2\frac{\partial g_{23}}{\partial L_4}\right)-2V_1g_{26}(g_{61}+g_{62})+$$
$$+2V_1g_{63}(g_{24}+g_{25})\}+\frac{1}{48}\{(8V_1g_{22}g_{23}\times$$
$$\times(g_{24}+g_{25})+4V_1g_{26}(g_{22}^2+g_{23}^2))\left(\frac{\partial V_1}{\partial L_3}-2\frac{\partial V_1}{\partial L_4}\right)+$$
$$+6V_1^2(g_{22}g_{26}+g_{23}(g_{24}+g_{25}))\left(\frac{\partial g_{22}}{\partial L_3}-2\frac{\partial g_{22}}{\partial L_4}\right)+$$
$$+6V_1^2(g_{23}g_{26}+g_{22}(g_{24}+g_{25}))\left(\frac{\partial g_{23}}{\partial L_3}-2\frac{\partial g_{23}}{\partial L_4}\right)+$$
$$+6V_1^2\left\{g_{22}g_{23}\left(\frac{\partial g_{24}}{\partial L_3}-2\frac{\partial g_{24}}{\partial L_4}+\frac{\partial g_{25}}{\partial L_3}-2\frac{\partial g_{25}}{\partial L_4}\right)+\right.$$
$$\left.+\frac{1}{2}(g_{23}^2+g_{22}^2)\left(\frac{\partial g_{26}}{\partial L_3}-2\frac{\partial g_{26}}{\partial L_4}\right)\right\}\Bigg\}, \tag{17}$$

$$a'''=d, \tag{18}$$

$$b'''=d', \tag{19}$$

$$c''' = d'', \tag{20}$$

$$\begin{aligned} d''' = {} & 2\left[\frac{1}{2}g_8 + \frac{1}{4}\left\{g_{23}^2\left(2\frac{\partial V_1}{\partial L_4} - \frac{\partial V_1}{\partial L_3}\right) - 2g_{40} + \right.\right. \\ & +2V_1 g_{23}\left(2\frac{\partial g_{23}}{\partial L_4} - \frac{\partial g_{23}}{\partial L_3}\right) + V_1(4g_{25}^2 + g_{26}^2)\bigg\} + \\ & +g_{133} + \frac{1}{4}\left\{g_{84}\left(2\frac{\partial g_{23}}{\partial L_4} - \frac{\partial g_{23}}{\partial L_3}\right) + 8g_{25}g_{91} - \right. \\ & -g_{23}\left(\frac{\partial g_{84}}{\partial L_3} - 2\frac{\partial g_{84}}{\partial L_4}\right) - 2g_{26}g_{90} + g_{23}g_{60} \times \\ & \times\left(\frac{\partial V_1}{\partial L_3} - 2\frac{\partial V_1}{\partial L_4}\right) + V_1 g_{23}\left(\frac{\partial g_{60}}{\partial L_3} - 2\frac{\partial g_{60}}{\partial L_4}\right) + \\ & +V_1 g_{60}\left(\frac{\partial g_{23}}{\partial L_3} - 2\frac{\partial g_{23}}{\partial L_4}\right) + V_1(4g_{25}g_{62} - g_{26}g_{63})\bigg\} - \\ & -\frac{1}{48}V_1\left\{(4g_{22}g_{23}g_{26} + 8g_{23}^2 g_{25})\left(\frac{\partial V_1}{\partial L_3} - 2\frac{\partial V_1}{\partial L_4}\right) + \right. \\ & +3V_1 g_{23}g_{26}\left(\frac{\partial g_{22}}{\partial L_3} - 2\frac{\partial g_{22}}{\partial L_4}\right) + (3V_1 g_{22}g_{26} + 12V_1 g_{23}g_{25}) \times \\ & \times\left(\frac{\partial g_{23}}{\partial L_3} - 2\frac{\partial g_{23}}{\partial L_4}\right) + 6V_1 g_{23}^2\left(\frac{\partial g_{25}}{\partial L_3} - 2\frac{\partial g_{25}}{\partial L_4}\right) + \\ & \left.\left. +3V_1 g_{22}g_{23}\left(\frac{\partial g_{26}}{\partial L_3} - 2\frac{\partial g_{26}}{\partial L_4}\right)\right\}\right] \end{aligned} \tag{21}$$

$$A = g_3 + g_{41} + 2g_{140}, \tag{22}$$

$$B = \frac{1}{2}g_{11} + \frac{1}{2}g_{46} + g_{144} + g_{33}g_{110} - g_{35}g_{111} - 2V_2 g_{35}g_{75}, \tag{23}$$

$$\begin{aligned} C = {} & \frac{1}{2}g_{13} + \frac{1}{2}g_{48} + g_{145} + g_{27}g_{92} - g_{29}g_{93} + \\ & +g_{34}g_{111} - g_{35}g_{110} + 2V_2 g_{34}g_{75}, \end{aligned} \tag{24}$$

$$D = \frac{1}{2}g_{21} + \frac{1}{2}g_{52} + g_{146} + g_{28}g_{93} - g_{29}g_{92}, \tag{25}$$

$$A' = B, \tag{26}$$

$$\begin{aligned} B' = {} & -g_5 + 4V_2(g_{33}^2 + g_{35}^2) + g_{42} + 2(g_{141} + 2g_{33}g_{108} - \\ & -g_{35}g_{112} + 2V_2(g_{33}g_{73} + g_{35}g_{76})), \end{aligned} \tag{27}$$

$$C' = \frac{1}{2}g_{15} + \frac{1}{2}(g_{50} - 8V_2 g_{35}(g_{34} + g_{35})) + g_{147} + g_{27}g_{94} - \\ -g_{29}g_{95} + g_{112}(g_{33} + g_{34}) - 2g_{35}(g_{109} + g_{108}) + \\ +2V_2\{g_{35}(g_{74} - g_{73}) - g_{76}(g_{33} + g_{34})\}, \tag{28}$$

$$D' = \frac{1}{2}g_{17} + \frac{1}{2}g_{54} + g_{148} + g_{28}g_{95} - g_{29}g_{94} + g_{33}g_{113} - g_{35}g_{114}, \tag{29}$$

$$A'' = C, \tag{30}$$

$$B'' = C', \tag{31}$$

$$C'' = -g_7 + g_{43} + 2V_1(g_{27}^2 + g_{29}^2) + 4V_2(g_{34}^2 + g_{35}^2) + \\ +2(g_{142} + 2g_{27}g_{96} - g_{29}g_{97} + 2g_{34}g_{109} - g_{35}g_{112} + \\ +V_1(g_{27}g_{64} + g_{29}g_{66}) - 2V_2(g_{34}g_{74} - g_{35}g_{76})), \tag{32}$$

$$D'' = \frac{1}{2}g_{19} + \frac{1}{2}(g_{56} - 4V_1 g_{29}(g_{27} + g_{28})) + g_{149} + g_{97}(g_{27} + g_{28}) - \\ -2g_{29}(g_{96} + g_{98}) + g_{34}g_{114} - g_{35}g_{113} - V_1\{g_{29}(g_{64} + g_{65}) + \\ +g_{66}(g_{27} + g_{28})\}, \tag{33}$$

$$A''' = D, \tag{34}$$

$$B''' = D', \tag{35}$$

$$C''' = D'', \tag{36}$$

$$D''' = -g_9 + g_{44} + 2V_1(g_{28}^2 + g_{29}^2) + 2(g_{143} + 2g_{28}g_{98} - g_{29}g_{97} + \\ +V_1(g_{28}g_{65} + g_{29}g_{66})), \tag{37}$$

where the values of g's and V's are given in Part III.

Let a particular solution for Equations (2) and (3) be

$$\begin{aligned} H_s &= M_s \sin(\xi t + u) \\ K_s &= M_s \cos(\xi t + u) \end{aligned} \qquad s = 1, 2, 3, 4, \tag{38}$$

Substituting the particular solution in both sides of Equations (2) we get

$$
\begin{aligned}
&(\xi - a)M_1 - bM_2 - cM_3 - dM_4 = 0,\\
&-a'M_1 + (\xi - b')M_2 - c'M_3 - d'M_4 = 0,\\
&-a''M_1 - b''M_2 + (\xi - c'')M_3 - d''M_4 = 0,\\
&-a'''M_1 - b'''M_2 - c'''M_3 + (\xi - d''')M_4 = 0.
\end{aligned} \tag{39}
$$

Since M_1, M_2, M_3, M_4 are not identically equal to zero, hence the condition that these simultaneous, homogeneous linear equations have a common solution other than $M_1 = M_2 = M_3 = M_4 = 0$ is

$$
\begin{vmatrix}
\xi - a & -b & -c & -d \\
-a' & \xi - b' & -c' & -d' \\
-a'' & -b'' & \xi - c'' & -d'' \\
-a''' & -b''' & -c''' & \xi - d'''
\end{vmatrix} = 0, \tag{40}
$$

which is a well known theorem in the theory of equations.

By Laplace expansion we can find that the condition is

$$
\begin{aligned}
&\begin{vmatrix} \xi - a & -b \\ -a' & \xi - b' \end{vmatrix} \times \begin{vmatrix} \xi - c'' & -d'' \\ -c''' & \xi - d''' \end{vmatrix} - \\
&-\begin{vmatrix} \xi - a & -b \\ -a'' & -b'' \end{vmatrix} \times \begin{vmatrix} -c' & -d' \\ -c''' & \xi - d''' \end{vmatrix} + \\
&+\begin{vmatrix} \xi - a & -b \\ -a''' & -b''' \end{vmatrix} \times \begin{vmatrix} -c' & -d' \\ \xi - c'' & -d'' \end{vmatrix} + \\
&+\begin{vmatrix} -a' & \xi - b' \\ -a'' & -b'' \end{vmatrix} \times \begin{vmatrix} -c & -d \\ -c''' & \xi - d''' \end{vmatrix} - \\
&-\begin{vmatrix} -a' & \xi - b' \\ -a''' & -b''' \end{vmatrix} \times \begin{vmatrix} -c & -d \\ \xi - c'' & -d'' \end{vmatrix} + \\
&+\begin{vmatrix} -a'' & -b'' \\ -a''' & -b''' \end{vmatrix} \times \begin{vmatrix} -c & -d \\ -c' & -d' \end{vmatrix} = 0,
\end{aligned} \tag{41}
$$

i.e.

$$
\begin{aligned}
&\xi^4 - d'''\xi^3 - c''\xi^3 + c''d'''\xi^2 - b'\xi^3 + \\
&+ b'd'''\xi^2 + b'c''\xi^2 - b'c''d'''\xi - a\xi^3 + ad'''\xi^2 +
\end{aligned}
$$

$$
\begin{aligned}
&+ac''\xi^2 - ac''d'''\xi + ab'\xi^2 - ab'd'''\xi - ab'c''\xi+\\
&+ab'c''d''' - a'b\xi^2 + a'bd'''\xi + a'bc''\xi-\\
&-a'bc''d''' - (b''c'\xi^2 - b''c'd'''\xi - b''c'a\xi + b''c'ad'''+\\
&+b''d'c'''\xi - b''d'c'''a + ba''c'\xi - ba''c'd'''+\\
&+ba''d'c''') + (-b'''c'd''\xi + b'''c'd''a - b'''d'\xi^2+\\
&+b'''d'c''\xi + b'''d'a\xi - b'''d'ac'' - ba'''c'd'' - ba'''d'\xi+\\
&+ba'''d'c'') + (-a'b''c\xi + a'b''cd''' - a'b''dc'''-\\
&-a''c\xi^2 + a''cd'''\xi + a''cb'\xi - a''cb'd''' - a''dc'''\xi+\\
&+a''dc'''b') - (a'b'''cd'' + a'b'''d\xi - a'b'''dc''+\\
&+a'''cd''\xi - a'''cd''b' + a'''d\xi^2 - a'''dc''\xi - a'''db'\xi+\\
&+a'''db'c'') + (a''b'''cd' - a''b'''dc' - b''a'''cd'+\\
&+b''a'''dc' - c'''d''\xi^2 + ac'''d''\xi + b'c'''d'\xi - ab'c'''d'' + a'bc'''d''\\
&-a''bd'c''') \equiv 0, \qquad (42)
\end{aligned}
$$

(Ferrar, 1953), which could be written in the form

$$
\begin{aligned}
0 \equiv\ & \xi^4 + (-d''' - c'' - b' - a)\xi^3 +\\
&+(c''d''' + b'd''' + b'c'' + ad''' + ac'' + ab' -\\
&-a'b - b''c' - b'''d' - a''c - a'''d - c'''d'')\xi^2 +\\
&+(-b'c''d''' - ac''d''' - ab'd''' - ab'c'' + a'bd''' +\\
&+a'bc'' + b''c'd''' + b''c'a - b''d'c''' -\\
&-ba''c' - b'''c'd'' + b'''d'c'' + b'''d'a -\\
&-ba'''d' - a'b''c + a''cd''' + a''cb' -\\
&-a''dc''' - a'b'''d - a'''cd'' + a'''dc'' +\\
&+a'''db' + ac'''d'' + b'c'''d')\xi + (ab'c''d''' - a'bc''d''' -\\
&-b''c'ad''' + b''d'c'''a + ba''c'd''' +\\
&+b'''c'd''a - b'''d'ac'' - ba'''c'd'' + ba'''d'c'' +\\
&+a'b''cd''' - a'b''dc''' - a''cb'd''' + a''dc'''b' -\\
&-a'b'''cd'' + a'b'''dc'' + a'''cd''b' - a'''db'c'' +\\
&+a''b'''cd' - a''b'''dc' - b''a'''cd' + b''a'''dc' -\\
&-ab'c'''d'' + a'bc'''d'' - a''bd'c'''). \qquad (43)
\end{aligned}
$$

An equivalent form to Equation (43) is

$$\xi^4 + D\xi^3 + D'\xi^2 + D''\xi + D''' \equiv 0, \tag{44}$$

Equation (44) is a biquadratic equation where D, D', D'', D''' are functions of the Poincare' linear variables L_u; $u = 1, 2, 3, 4$; and by definition, they are constants. The solution of this quartic equation of the fourth degree gives us the values of ξ_1, ξ_2, ξ_3, ξ_4. There are two methods for the solution of the quartic Equation (44). The first by the procedure of the Italian algebraists Ferrari and Cardan in the first half of the sixteenth century. The quartic equation splits into the two quadratic equations

$$0 \equiv \xi^2 + \left(\frac{D}{2} - \sqrt{\frac{D^2}{4} - D' + y}\right)\xi + \left(\frac{1}{2}y - \sqrt{D''' + \frac{1}{4}y^2}\right),$$

$$0 \equiv \xi^2 + \left(\frac{D}{2} + \sqrt{\frac{D^2}{4} - D' + y}\right)\xi + \left(\frac{1}{2}y + \sqrt{-D''' + \frac{1}{4}y^2}\right), \tag{45}$$

(Uspensky, 1948), y is any root of the resolvent of the quartic equation namely

$$0 \equiv y^3 - D'y^2 + (DD'' - 4D''')y + (4D'D''' - D^2D''' - D''^2). \tag{46}$$

Let the root be that one which is free from imaginary quantities, namely

$$y_1 = \sqrt[3]{A} + \sqrt[3]{B}, \tag{47}$$

where

$$A = -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$$

$$B = -\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} \tag{48}$$

and

$$p = DD'' - 4D''' - \frac{1}{3}D'^2$$

$$q = 4D'D''' - D^2D''' - D''^2 + \frac{1}{3}D'(DD'' - 4D''') - \frac{2}{27}D'^3. \tag{49}$$

So we can assign the value of y_1 which is expressed in terms of coefficients D, D', D'', D''' appearing in the quartic Equation (44), and we are capable of solving the two quadratic Equations (45), which yield the four required roots ξ_u; $u = 1, 2, 3, 4$ of the quartic Equation (44).

The second method to find the values of ξ_u; $u = 1, 2, 3, 4$ is by Jacobi's procedure to solve determinant equations indicated in (Brouwer & Clemence, 1965, p. 521.

Since we have four non-zero values of ξ, we then have correspondingly four sets of equations, each is composed of four linear homogeneous equations. According to Equation (39), the first set is the following

$$\begin{aligned}
&(\xi_1 - a)M_1^{(1)} - bM_2^{(1)} - cM_3^{(1)} - dM_4^{(1)} = 0,\\
&-a'M_1^{(1)} + (\xi_1 - b')M_2^{(1)} - c'M_3^{(1)} - d'M_4^{(1)} = 0,\\
&-a''M_1^{(1)} - b''M_2^{(1)} + (\xi_1 - c'')M_3^{(1)} - d''M_4^{(1)} = 0,\\
&-a'''M_1^{(1)} - b'''M_2^{(1)} - c'''M_3^{(1)} + (\xi_1 - d''')M_4^{(1)} = 0.
\end{aligned} \tag{50}$$

Similarly we can write the second, third and fourth set.

Evidently there is sixteen unknowns M_r^S; $r = 1, 2, 3, 4$; $S = 1, 2, 3, 4$. Multiply the four equations of the first set Equations (50) by $M_1^{(2)}$, $M_2^{(2)}$, $M_3^{(2)}$, $M_4^{(2)}$ respectively; the second set by $M_1^{(1)}$, $M_2^{(1)}$, $M_3^{(1)}$, $M_4^{(1)}$ respectively. Hence from the first equation of set 1, 2 by subtraction, i.e. eliminating (a), we get

$$\begin{aligned}
&(\xi_2 - \xi_1)M_1^{(1)}M_1^{(2)} + (M_1^{(2)}M_2^{(1)} - M_1^{(1)}M_2^{(2)})b + (M_1^{(2)}M_3^{(1)} - M_1^{(1)}M_3^{(2)})c +\\
&+(M_1^{(2)}M_4^{(1)} - M_1^{(1)}M_4^{(2)})d \equiv 0.
\end{aligned} \tag{51}$$

Similarly from the second, third and fourth equations of set (1), (2), we get by subtraction, i.e. by the elimination of b', c'', d'''

$$\begin{aligned}
&(M_2^{(2)}M_1^{(1)} - M_2^{(1)}M_1^{(2)})a' + (\xi_2 - \xi_1)M_2^{(1)}M_2^{(2)} + (M_2^{(2)}M_3^{(1)} - M_2^{(1)}M_3^{(2)})c' +\\
&+(M_2^{(2)}M_4^{(1)} - M_2^{(1)}M_4^{(2)})d' \equiv 0,
\end{aligned} \tag{52}$$

and

$$\begin{aligned}
&(M_3^{(1)}M_1^{(2)} - M_3^{(2)}M_1^{(1)})a'' + (M_3^{(1)}M_2^{(2)} - M_3^{(2)}M_2^{(1)})b'' +\\
&+(\xi_1 - \xi_2)(M_3^{(2)}M_3^{(1)} + (M_3^{(1)}M_4^{(2)} - M_3^{(2)}M_4^{(1)})d'' \equiv 0,
\end{aligned} \tag{53}$$

and

$$(M_4^{(1)}M_1^{(2)} - M_4^{(2)}M_1^{(1)})a''' + (M_4^{(1)}M_2^{(2)} - M_4^{(2)}M_2^{(1)})b''' + (M_4^{(1)}M_3^{(2)} -$$

$$-M_4^{(2)}M_3^{(1)})c''' + (\xi_1 - \xi_2)M_4^{(2)}M_4^{(1)} \equiv 0, \tag{54}$$

i.e.

$$\begin{aligned}(\xi_1 - \xi_2)M_1^{(1)}M_1^{(2)} &= b(M_1^{(2)}M_2^{(1)} - M_1^{(1)}M_2^{(2)}) + c(M_1^{(2)}M_3^{(1)} - \\ &\quad -M_1^{(1)}M_3^{(2)}) + d(M_1^{(2)}M_4^{(1)} - M_1^{(1)}M_4^{(2)}),\end{aligned} \tag{55}$$

$$\begin{aligned}(\xi_1 - \xi_2)M_2^{(1)}M_2^{(2)} &= a'(M_1^{(1)}M_2^{(2)} - M_1^{(2)}M_2^{(1)}) + c'(M_2^{(2)}M_3^{(1)} - \\ &\quad -M_2^{(1)}M_3^{(2)}) + d'(M_2^{(2)}M_4^{(1)} - M_2^{(1)}M_4^{(2)}),\end{aligned} \tag{56}$$

$$\begin{aligned}(\xi_1 - \xi_2)M_3^{(2)}M_3^{(1)} &= a''(-M_3^{(1)}M_1^{(2)} + M_3^{(2)}M_1^{(1)}) + b''(-M_3^{(1)}M_2^{(2)} + \\ &\quad +M_3^{(2)}M_2^{(1)}) + d''(-M_3^{(1)}M_4^{(2)} + M_3^{(2)}M_4^{(1)}),\end{aligned} \tag{57}$$

and

$$\begin{aligned}(\xi_1 - \xi_2)M_4^{(2)}M_4^{(1)} &= a'''(-M_4^{(1)}M_4^{(2)} + M_4^{(2)}M_1^{(1)}) + b'''(-M_4^{(1)}M_2^{(2)} + \\ &\quad +M_4^{(2)}M_2^{(1)}) + c'''(-M_4^{(1)}M_3^{(2)} + M_4^{(2)}M_3^{(1)}).\end{aligned} \tag{58}$$

After addition of L.H.S. and R.H.S. of Equations (55), (56), (57) and (58), we find that

$$\begin{aligned}&(\xi_1 - \xi_2)M_1^{(1)}M_1^{(2)} + (\xi_1 - \xi_2)M_2^{(1)}M_2^{(2)} + (\xi_1 - \xi_2)M_3^{(1)}M_3^{(2)} + \\ &\quad +(\xi_1 - \xi_2)M_4^{(1)}M_4^{(2)} \equiv 0,\end{aligned} \tag{59}$$

i.e.

$$(\xi_1 - \xi_2)[M_1^{(1)}M_1^{(2)} + M_2^{(1)}M_2^{(2)} + M_3^{(1)}M_3^{(2)} + M_4^{(1)}M_4^{(2)}] \equiv 0. \tag{60}$$

Since $\xi_1 \neq \xi_2$, whence

$$M_1^{(1)}M_1^{(2)} + M_2^{(1)}M_2^{(2)} + M_3^{(1)}M_3^{(2)} + M_4^{(1)}M_4^{(2)} = 0. \tag{61}$$

Repeating this process for $(\xi_1 - \xi_3)$; $(\xi_1 - \xi_4)$; $(\xi_2 - \xi_3)$; $(\xi_2 - \xi_4)$; $(\xi_3 - \xi_4)$, i.e. for sets (1, 3), (1, 4); (2, 3); (2, 4); (3, 4) we get-assuming that there is no pairs of equal roots

$$M_1^{(1)}M_1^{(3)} + M_2^{(1)}M_2^{(3)} + M_3^{(1)}M_3^{(3)} + M_4^{(1)}M_4^{(3)} \equiv 0,$$

$$M_1^{(1)}M_1^{(4)} + M_2^{(1)}M_2^{(4)} + M_3^{(1)}M_3^{(4)} + M_4^{(1)}M_4^{(4)} \equiv 0,$$

$$M_1^{(2)}M_1^{(3)} + M_2^{(2)}M_2^{(3)} + M_3^{(2)}M_3^{(3)} + M_4^{(2)}M_4^{(3)} \equiv 0,$$

$$M_1^{(2)}M_1^{(4)} + M_2^{(2)}M_2^{(4)} + M_3^{(2)}M_3^{(4)} + M_4^{(2)}M_4^{(4)} \equiv 0,$$

$$M_1^{(3)}M_1^{(4)} + M_2^{(3)}M_2^{(4)} + M_3^{(3)}M_3^{(4)} + M_4^{(3)}M_4^{(4)} \equiv 0, \tag{62}$$

or generally for any number n of planets

$$M_1^{(r)}M_1^{(s)} + M_2^{(r)}M_2^{(s)} + \cdots + M_n^{(r)}M_n^{(s)} \equiv 0, \tag{63}$$

with $r = 1, 2, \ldots, n$; $s = 1, 2, \ldots, n$; $r \neq s$, (Brouwer and Clemence, 1965).

For instance for $n = 4$, we have for r, s the combinations (1, 2); (1, 3); (1, 4); (2, 3); (2, 4); (3, 4).

The general solution of Equations (2) and (3) is the following

$$H_s = \sum_{j=1}^{4} M_s^{(j)} \sin(\xi_j t + u_j),$$

$$K_s = \sum_{j=1}^{4} M_s^{(j)} \cos(\xi_j t + u_j), \tag{64}$$

To evaluate the constants of integrations, we put $t = 0$ in Equations (64), then

$$(H_s)_0 = \sum_{j=1}^{4} M_s^{(j)} \sin u_j,$$

$$(K_s)_0 = \sum_{j=1}^{4} M_s^{(j)} \cos u_j, \tag{65}$$

where $(H_s)_0$, $(K_s)_0$ are the Poincare' canonical variables known from the orbital elements at $t = 0$. For instance for $(H_s)_0$

$$(H_1)_0 = M_1^{(1)} \sin u_1 + M_1^{(2)} \sin u_2 + M_1^{(3)} \sin u_3 + M_1^{(4)} \sin u_4,$$

$$(H_2)_0 = M_2^{(1)} \sin u_1 + M_2^{(2)} \sin u_2 + M_2^{(3)} \sin u_3 + M_2^{(4)} \sin u_4,$$

$$(H_3)_0 = M_3^{(1)} \sin u_1 + M_3^{(2)} \sin u_2 + M_3^{(3)} \sin u_3 + M_3^{(4)} \sin u_4,$$

$$(H_4)_0 = M_4^{(1)} \sin u_1 + M_4^{(2)} \sin u_2 + M_4^{(3)} \sin u_3 + M_4^{(4)} \sin u_4. \tag{66}$$

Multiply Equations (66) by $M_1^{(r)}$, $M_2^{(r)}$, $M_3^{(r)}$, $M_4^{(r)}$ respectively, we obtain after the addition of all R.H.S. and L.H.S.

$$\begin{aligned}
&(M_1^{(r)}M_1^{(1)} + M_2^{(r)}M_2^{(1)} + M_3^{(r)}M_3^{(1)} + M_4^{(r)}M_4^{(1)}) \sin u_1 + \\
&\quad +(M_1^{(r)}M_1^{(2)} + M_2^{(r)}M_2^{(2)} + M_3^{(r)}M_3^{(2)} + M_4^{(r)}M_4^{(2)}) \sin u_2 + \\
&\quad +(M_1^{(r)}M_1^{(3)} + M_2^{(r)}M_2^{(3)} + M_3^{(r)}M_3^{(3)} + M_4^{(r)}M_4^{(4)}) \sin u_3 + \\
&\quad +(M_1^{(r)}M_1^{(4)} + M_2^{(r)}M_2^{(4)} + M_3^{(r)}M_3^{(4)} + M_4^{(r)}M_4^{(4)}) \sin u_4 \equiv \\
&\qquad \equiv (H_1)_0 M_1^{(r)} + (H_2)_0 M_2^{(r)} + (H_3)_0 M_3^{(r)} + (H_4)_0 M_4^{(r)}.
\end{aligned} \tag{67}$$

Substituting $r = 1, 2, 3, 4$ in the L.H.S. of Equations (67) and using the relations (63), we can obtain

$$\begin{aligned}
&[\{M_1^{(1)}\}^2 + \{M_2^{(1)}\}^2 + \{M_3^{(1)}\}^2 + \{M_4^{(1)}\}^2] \sin u_1 + \\
&\quad +[\{M_1^{(2)}\}^2 + \{M_2^{(2)}\}^2 + \{M_3^{(2)}\}^2 + \{M_4^{(2)}\}^2] \sin u_2 + \\
&\quad +[\{M_1^{(3)}\}^2 + \{M_2^{(3)}\}^2 + \{M_3^{(3)}\}^2 + \{M_4^{(3)}\}^2] \sin u_3 + \\
&\quad +[\{M_1^{(4)}\}^2 + \{M_2^{(4)}\}^2 + \{M_3^{(4)}\}^2 + \{M_4^{(4)}\}^2] \sin u_4 \equiv \\
&\quad \equiv (M_1^{(1)} + M_1^{(2)} + M_1^{(3)} + M_1^{(4)})(H_1)_0 + \\
&\quad +(M_2^{(1)} + M_2^{(2)} + M_2^{(3)} + M_2^{(4)})(H_2)_0 + \\
&\quad +(M_3^{(1)} + M_3^{(2)} + M_3^{(3)} + M_3^{(4)})(H_3)_0 + \\
&\quad +(M_4^{(1)} + M_4^{(2)} + M_4^{(3)} + M_4^{(4)})(H_4)_0.
\end{aligned} \tag{68}$$

Equation (68) could be written in the general form

$$\begin{aligned}
&[\{M_1^{(r)}\}^2 + \{M_2^{(r)}\}^2 + \{M_3^{(r)}\}^2 + \{M_4^{(r)}\}^2] \sin u_r \\
&\quad \equiv M_1^{(r)}(H_1)_0 + M_2^{(r)}(H_2)_0 + M_3^{(r)}(H_3)_0 + M_4^{(r)}(H_4)_0,
\end{aligned} \tag{69}$$

where $r = 1, 2, 3, 4$.

Divide both sides by $\{M_1^{(r)}\}^2$, then we get

$$\left[1 + \left(\frac{M_2^{(r)}}{M_1^{(r)}}\right)^2 + \left(\frac{M_3^{(r)}}{M_1^{(r)}}\right)^2 + \left(\frac{M_4^{(r)}}{M_1^{(r)}}\right)^2\right] \sin u_r \equiv$$

$$\equiv \frac{1}{M_1(\mathrm{r})}\left[(H_1)_0 + \left(\frac{M_2^{(r)}}{M_1^{(r)}}\right)(H_2)_0 + \left(\frac{M_3^{(r)}}{M_1^{(r)}}\right)(H_3)_0 + \left(\frac{M_4^{(r)}}{M_1^{(r)}}\right)(H_4)_0\right], \quad (70)$$

i.e.

$$M_1^{(r)} \sin u_r \equiv \frac{(H_1)_0 + \left(\frac{M_2^{(r)}}{M_1^{(r)}}\right)(H_2)_0 + \left(\frac{M_3^{(r)}}{M_1^{(r)}}\right)(H_3)_0 + \left(\frac{M_4^{(r)}}{M_1^{(r)}}\right)(H_4)_0}{\left[1 + \left(\frac{M_2^{(r)}}{M_1^{(r)}}\right)^2 + \left(\frac{M_3^{(r)}}{M_1^{(r)}}\right)^2 + \left(\frac{M_4^{(r)}}{M_1^{(r)}}\right)^2\right]}. \quad (71)$$

The same procedure is repeated for the Poincare' variables $(K_s)_0$, and we obtain

$$M_1^{(r)} \cos u_r \equiv \frac{(K_1)_0 + \left(\frac{M_2^{(r)}}{M_1^{(r)}}\right)(K_2)_0 + \left(\frac{M_3^{(r)}}{M_1^{(r)}}\right)(K_3)_0 + \left(\frac{M_4^{(r)}}{M_1^{(r)}}\right)(K_4)_0}{\left[1 + \left(\frac{M_2^{(r)}}{M_1^{(r)}}\right)^2 + \left(\frac{M_3^{(r)}}{M_1^{(r)}}\right)^2 + \left(\frac{M_4^{(r)}}{M_1^{(r)}}\right)^2\right]}. \quad (72)$$

From Equations (71), (72) we can easily deduce that

$$M_1^{(r)} \equiv \sqrt{\left[(H_1)_0 + \left(\frac{M_2^{(r)}}{M_1^{(r)}}\right)(H_2)_0 + \left(\frac{M_3^{(r)}}{M_1^{(r)}}\right)(H_3)_0 + \left(\frac{M_4^{(r)}}{M_1^{(r)}}\right)(H_4)_0\right]^2 + \left[(K_1)_0 + \left(\frac{M_2^{(r)}}{M_1^{(r)}}\right)(K_2)_0 + \left(\frac{M_3^{(r)}}{M_1^{(r)}}\right)(K_3)_0 + \left(\frac{M_4^{(r)}}{M_1^{(r)}}\right)(K_4)_0\right]^2} \Bigg/ \left[1 + \left(\frac{M_2^{(r)}}{M_1^{(r)}}\right)^2 + \left(\frac{M_3^{(r)}}{M_1^{(r)}}\right)^2 + \left(\frac{M_4^{(r)}}{M_1^{(r)}}\right)^2\right]$$

$$\tan u_r \equiv \frac{(H_1)_0 + \left(\frac{M_2^{(r)}}{M_1^{(r)}}\right)(H_2)_0 + \left(\frac{M_3^{(r)}}{M_1^{(r)}}\right)(H_3)_0 + \left(\frac{M_4^{(r)}}{M_1^{(r)}}\right)(H_4)_0}{(K_1)_0 + \left(\frac{M_2^{(r)}}{M_1^{(r)}}\right)(K_2)_0 + \left(\frac{M_3^{(r)}}{M_1^{(r)}}\right)(K_3)_0 + \left(\frac{M_4^{(r)}}{M_1^{(r)}}\right)(K_4)_0}, \quad (73)$$

whence we can find the values of $M_1^{(r)}$ and the phase constants u_r where $r =$ 1, 2, 3, 4, namely $M_1^{(1)}$, $M_1^{(2)}$, $M_1^{(3)}$, $M_1^{(4)}$; u_1, u_2, u_3, u_4 which are the eight constants of integration. The ratios $M_2^{(1)}/M_1^{(1)}$, $M_3^{(1)}/M_1^{(1)}$, $M_4^{(1)}/M_1^{(1)}$ are known since they are the ratios of the appropriate minors of the determinant Equation (40) for $\xi = \xi_1$.

Similarly the ratios $M_2^{(2)}/M_1^{(2)}$, $M_3^{(2)}/M_1^{(2)}$, $M_4^{(2)}/M_1^{(2)}$, $M_2^{(3)}/M_1^{(3)}$, $M_3^{(3)}/M_1^{(3)}$, $M_4^{(3)}/M_1^{(3)}$, $M_2^{(4)}/M_1^{(4)}$, $M_3^{(4)}/M_1^{(4)}$, $M_4^{(4)}/M_1^{(4)}$ could be acquired as the ratios of the appropriate minors of the determinant Equation (40) for $\xi = \xi_2$; $\xi = \xi_3$; $\xi = \xi_4$ respectively. Whence the sixteen M_r^s are known where $r = 1, 2, 3, 4$ and $s = 1, 2, 3, 4$.

Similarly for the Poincare' variables P, Q the particular solution is

$$P_s = N_s \sin(\zeta t + v)$$

(74)

$$Q_s = N_s \cos(\zeta t + v)$$

Evidently we have a corresponding determinant equation with eigenvalues ζ_u, $u = 1, 2, 3, 4$. The general solution for the four major planets J-S-U-N is

$$P_s = N_s^{(1)} \sin v_1 + \sum_{j=2}^{4} N_s^{(j)} \sin(\zeta_j t + v_j),$$

(75)

$$Q_s = N_s^{(1)} \cos v_1 + \sum_{j=2}^{4} N_s^{(j)} \cos(\zeta_j t + v_j).$$

Both determinant equations for H, K; P, Q have all real roots.

This fact was demonstrated by Sylvester. However the four values of ξ are positive and different and all four values of ζ are negative and different except for the zero root. Similar corresponding expressions may be obtained for the evaluations of N_1^r sin v_r & N_1^r cos v_r (Brouwer and Clemence, 1965).

From Equation (1)

$$L_s' = \text{const.} \tag{76}$$

and

$$\lambda_s' = -\frac{\partial}{\partial L_s'} \int (F_0' + F_1' + F_2' + F_3') \\ (L_s', H_s'(t), K_s'(t), P_s'(t), Q_s'(t))\, \mathrm{d}t \quad s = 1, 2, 3, 4, \tag{77}$$

where $H_s'(t), \ldots, Q_s'(t)$ are substituted from Equations (64), (75).

2. The Eigen Value–Eigen Vector Method

A homogeneous linear system differential equations may be represented by

$$\frac{dx}{dt} = Ax, \tag{78}$$

where

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

There exists one and only one solution for this initial value problem

$$\frac{dx}{dt} = Ax \quad x(t_0) = x^0 = \begin{bmatrix} x_1^0 \\ x_2^0 \\ \vdots \\ x_n^0 \end{bmatrix}. \tag{79}$$

Moreover, this solution exists for $-\infty < t < +\infty$

Equations (2), (3) could be represented by

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \\ \frac{dx_4}{dt} \\ \frac{dx_5}{dt} \\ \frac{dx_6}{dt} \\ \frac{dx_7}{dt} \\ \frac{dx_8}{dt} \end{bmatrix} \equiv \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} & A_{16} & A_{17} & A_{18} \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} & A_{26} & A_{27} & A_{28} \\ A_{31} & A_{32} & A_{33} & A_{34} & A_{35} & A_{36} & A_{37} & A_{38} \\ A_{41} & A_{42} & A_{43} & A_{44} & A_{45} & A_{46} & A_{47} & A_{48} \\ A_{51} & A_{52} & A_{53} & A_{54} & A_{55} & A_{56} & A_{57} & A_{58} \\ A_{61} & A_{62} & A_{63} & A_{64} & A_{65} & A_{66} & A_{67} & A_{68} \\ A_{71} & A_{72} & A_{73} & A_{74} & A_{75} & A_{76} & A_{77} & A_{78} \\ A_{81} & A_{82} & A_{83} & A_{84} & A_{85} & A_{86} & A_{87} & A_{88} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix}, \tag{80}$$

i.e.

$$\frac{dx_1}{dt} = A_{15}x_5 + A_{16}x_6 + A_{17}x_7 + A_{18}x_8,$$

$$\frac{dx_2}{dt} = A_{25}x_5 + A_{26}x_6 + A_{27}x_7 + A_{28}x_8,$$

$$\frac{dx_3}{dt} = A_{35}x_5 + A_{36}x_6 + A_{37}x_7 + A_{38}x_8,$$

$$\frac{dx_4}{dt} = A_{45}x_5 + A_{46}x_6 + A_{47}x_7 + A_{48}x_8,$$

$$\frac{dx_5}{dt} = A_{51}x_1 + A_{52}x_2 + A_{53}x_3 + A_{54}x_4,$$

$$\frac{dx_6}{dt} = A_{61}x_1 + A_{62}x_2 + A_{63}x_3 + A_{64}x_4,$$

$$\frac{dx_7}{dt} = A_{71}x_1 + A_{72}x_2 + A_{73}x_3 + A_{74}x_4,$$

$$\frac{dx_8}{dt} = A_{81}x_1 + A_{82}x_2 + A_{83}x_3 + A_{84}x_4. \tag{81}$$

If we put

$$A_{11} = A_{12} = A_{13} = A_{14} = 0,$$

$$A_{21} = A_{22} = A_{23} = A_{24} = 0,$$

$$A_{31} = A_{32} = A_{33} = A_{34} = 0,$$

$$A_{41} = A_{42} = A_{43} = A_{44} = 0,$$

$$A_{55} = A_{56} = A_{57} = A_{58} = 0,$$

$$A_{65} = A_{66} = A_{67} = A_{68} = 0,$$

$$A_{75} = A_{76} = A_{77} = A_{78} = 0,$$

$$A_{85} = A_{86} = A_{87} = A_{88} = 0, \tag{82}$$

and

$$\begin{matrix} x_1 = H_1 & x_2 = H_2 & x_3 = H_3 & x_4 = H_4 \\ x_5 = K_1 & x_6 = K_2 & x_7 = K_3 & x_8 = K_4 \end{matrix} \tag{83}$$

where

$$A_{15} = a; \quad A_{16} = b; \cdots - d''' = A_{84}. \tag{84}$$

Equation (78) has a unique solution X if det $A \neq 0$.

The secular perturbation Equations (2), (3) is of the form

$$\dot{X} = AX, \tag{85}$$

since they are first order linear homogeneous system of ordinary differential equations with constant coefficients, this suggests that we have exponential functions

$X^1(t)$, $X^2(t)$,..., $X^n(t)$ as solution. We try $X(t) = e^{\xi t}\mathrm{v}$ where v is a constant vector.

We find that $X(t) = e^{\xi t}\mathrm{v}$ is a solution if and only if ξ, v satisfy

$$A\mathrm{v} = \xi \mathrm{v}, \tag{86}$$

v is called eigenvector of A with eigenvalue ξ. To find the eigenvector we rewrite

$$0 = A\mathrm{v} - \xi\mathrm{v} = (A - \xi I)\mathrm{v}, \tag{87}$$

where I is the identity matrix.

The condition that Equation (87) has a nonzero solution v is

$$0 \equiv \det(A-\xi I) \equiv \det \begin{bmatrix} -\xi & 0 & 0 & 0 & A_{15} & A_{16} & A_{17} & A_{18} \\ 0 & -\xi & 0 & 0 & A_{25} & A_{26} & A_{27} & A_{28} \\ 0 & 0 & -\xi & 0 & A_{35} & A_{36} & A_{37} & A_{38} \\ 0 & 0 & 0 & -\xi & A_{45} & A_{46} & A_{47} & A_{48} \\ A_{51} & A_{52} & A_{53} & A_{54} & -\xi & 0 & 0 & 0 \\ A_{61} & A_{62} & A_{63} & A_{64} & 0 & -\xi & 0 & 0 \\ A_{71} & A_{72} & A_{73} & A_{74} & 0 & 0 & -\xi & 0 \\ A_{81} & A_{82} & A_{83} & A_{84} & 0 & 0 & 0 & -\xi \end{bmatrix} \equiv p(\xi), (88)$$

and $A_{Sr} = A_{rS}$, $S = 1, 2, 3, 4$; $r = 1, 2, 3, 4$.

It is evident that the eigenvalues ξ of A are the roots of the determinant Equation (88). The eigenvectors of A are the non zero solutions of the equtaions $(A-\xi I) = 0$; $P(\xi)$ is the characteristic polynomial of A which is the determinant of the matrix $A - \xi I$. It is a polynomial of degree n in ξ with leading term $(-1)^n \xi^n$.

We can prove that the general solution of the equation $\dot{X} = AX$ is

$$X(t) = C_1 e^{\xi_1 t}\mathrm{v}^1 + C_2 e^{\xi_2 t}\mathrm{v}^2 + \cdots + \cdots + C_8 e^{\xi_8 t}\mathrm{v}^8, \tag{89}$$

where v is an eigenvector of A with eigenvalues ξ and

$$\mathrm{V} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_8 \end{bmatrix}, \tag{90}$$

$C_1, C_2, \ldots, C_8 \neq 0$ and are constants.

From $P(\xi) = 0$, we can find the eigenvalues of $A : \xi_1, \xi_2, \ldots \xi_8$. From each eigenvalue, we find a corresponding eigenvector of A and a corresponding solution of the differential equation via $(A - \xi I)\mathrm{v} = 0$, i.e.

$$(A - \xi I)\mathrm{V} = \begin{bmatrix} \text{Matrix of Eq. (88)} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{91}$$

From the eigenvalues $\xi_1, \xi_2, \ldots, \xi_8$, the non zero eigenvectors v^1, $v^2, \ldots, v^8$ and the constants $C_1, C_2, \ldots, C_8$, we find the general solution (Braun, 1983)

$$X(t) = C_1 X^1(t) + C_2 X^2(t) + \cdots + C_8 X^8(t). \tag{92}$$

3. Theorem of Hurwitz

For the polynomial $f(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_n$, we define the matrix

$$H = \begin{bmatrix} a_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ a_3 & a_2 & a_1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ a_5 & a_4 & a_3 & a_2 & a_1 & 1 & 0 & 0 & 0 & \cdots \\ a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & 1 & 0 & \cdots \\ . & . & . & . & . & . & . & . & . & \cdots \end{bmatrix}, \tag{93}$$

$a_1, a_2, a_3, \ldots$ are the diagonal terms of this matrix, all $a_k = 0$ for $k > n$.

The principal minors Δ_ν; $\nu = 1, 2, 3, \ldots, \infty$ of H are

$$\Delta_1 = a_1, \quad \Delta_2 = \begin{vmatrix} a_1 & 1 \\ a_3 & a_2 \end{vmatrix}, \quad \Delta_3 = \begin{vmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{vmatrix} \cdots \Delta_n = a_n \Delta_{n-1}. \tag{94}$$

If $\Delta_1 > 0$; $\Delta_2 > 0$; $\Delta_3 > 0$; ... then we have a necessary and sufficient condition that $f(\lambda)$ is a Hurwitz polynomial and have only roots with negative real parts, and H is a Hurwitz matrix.

In our J-S-U-N planetary theory

$$f(\xi) = \xi^4 + D\xi^3 + D'\xi^2 + D''\xi + D''' = 0,$$

whence

$$\begin{matrix} a_1 = D & a_2 = D' & a_3 = D'' & a_4 = D''' \\ a_5 = 0 & a_6 = 0 & a_7 = 0 & \end{matrix}; \tag{95}$$

$$H = \begin{bmatrix} D & 1 & 0 & 0 \\ D'' & D' & D & 1 \\ 0 & D''' & D'' & D' \\ 0 & 0 & 0 & D''' \end{bmatrix}. \tag{96}$$

Then

$$\Delta_1 = D \quad \Delta_2 = \begin{vmatrix} D & 1 \\ D'' & D' \end{vmatrix},$$

$$\Delta_3 = \begin{vmatrix} D & 1 & 0 \\ D'' & D' & D \\ 0 & D''' & D'' \end{vmatrix},$$

$$\Delta_4 = \begin{vmatrix} D & 1 & 0 & 0 \\ D'' & D' & D & 1 \\ 0 & D''' & D'' & D' \\ 0 & 0 & 0 & D''' \end{vmatrix}. \tag{97}$$

For stable motion, we should have (Leipholz, 1987)

$$\Delta_1 > 0 \quad \Delta_2 > 0 \quad \Delta_3 > 0 \quad \Delta_4 > 0,$$

D, D', D'', D''' are functions of L'_s; $S = 1, 2, 3, 4$; where $L'_s = \sqrt{\mu_s a_s}$; μ_S is given by

$$\mu_S = k^2 m_0 \frac{m_0 + m_1 + \cdots + m_S}{m_0 + m_1 + \cdots + m_{S-1}},$$

where k^2 is the Gaussian constant; m_0 is the mass of sun and m_S is the mass of planet S.

We substitute for the numerical values of the a_S's (semi-majoraxes) of the four major planets and for μ_S's, in order that we might use the above condition of stability.

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