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An Asymptotically Complete Class of Tests

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Summary. This paper investigates sequences of asymptotically similar critical regions $\{S_n > 0\}$, $n \in \mathbb{N}$, under the assumption that the test-statistic S_n admits a certain stochastic expansion. It is shown that for such test-sequences, first order efficiency implies second order efficiency (i.e. efficiency up to an error term $o(n^{-1/2})$). Moreover, the asymptotic power functions of first order efficient test-sequences are determined up to an error term $o(n^{-1})$, and a class of critical regions is specified which is minimal essentially complete up to $o(n^{-1})$.

The results of this paper rest upon the technique of Edgeworth-expansions and are, therefore, restricted to "continuous" probability distributions.

1. Introduction

Let $P_{\theta,\tau}$, $\theta \in \Theta \subset \mathbb{R}$, $\tau \in T \subset \mathbb{R}^p$, be a family of probability measures over a measurable space $(X, \mathscr{A})$. Let $\delta \in \Theta$ be fixed. The problem is to test the hypothesis $\{(\theta,\tau) \in \Theta \times T: \theta = \delta\}$ against alternatives (θ, τ) with $\theta > \delta$ on the basis of an i.i.d. sample of size n. We consider contiguous alternatives $P_{(t,\tau)}^{(n)} = P_{\delta + n^{-1/2}t,\tau}^n$, $t \in \mathbb{R}^+ := [0, \infty)$, $\tau \in T$, and investigate the asymptotic behavior of a test-sequence $\underline{\varphi} = (\varphi_n)_{n \in \mathbb{N}}$ on the basis of its sequence of power functions

$$\Pi_n(\varphi_n)(t,\tau) := P_{(t,\tau)}^{(n)}(\varphi_n), \qquad n \in \mathbb{N}.$$

A test-sequence $\underline{\varphi}$ is "better" than $\underline{\psi}$ of order $o(n^{-r/2})$ at (t,τ) (for short: $\underline{\varphi} \gtrsim_r \underline{\psi}$ *at* (t,τ)) if

$$\Pi_n(\varphi_n)(t,\tau) \geqq \Pi_n(\psi_n)(t,\tau) + o(n^{-r/2}).$$

Test-sequences are comparable in this sense if one is better than the other at every $(t,\tau) \in \mathbb{R}^+ \times T$.

A test-sequence $\underline{\varphi}$ is *as. (asymptotically) of level* $\alpha + o(n^{-r/2})$ [resp. *as. similar of level* $\alpha + o(n^{-r/2})$] for the hypothesis $\{(t,\tau) \in \mathbb{R}^+ \times T: t = 0\}$ if locally uniformly for $\tau \in T$

$$\Pi_n(\varphi_n)(0,\tau) \leqq \alpha + o(n^{-r/2})$$

[resp.

$$\Pi_n(\varphi_n)(0,\tau) = \alpha + o(n^{-r/2})].$$

For a given level α and a given hypothesis, let $\mathscr{T}_r$ [resp. $\mathscr{T}_r^*$] denote the class of all test-sequences which are as. [similar] of level $\alpha + o(n^{-r/2})$.

We introduce the following definitions for an arbitrary subclass $\mathscr{V}_r \subset \mathscr{T}_r$.

A test-sequence $\underline{\varphi} \in \mathscr{V}_r$ is *m.p. (most powerful) in* $\mathscr{V}_r$ *at* (t,τ) if $\underline{\psi} \lesssim_r \underline{\varphi}$ at (t,τ) for all $\underline{\psi} \in \mathscr{V}_r$.

A test-sequence $\underline{\varphi} \in \mathscr{V}_r$ is *u.m.p. (uniformly most powerful) in* $\mathscr{V}_r$ *at* τ if it is m.p. in $\mathscr{V}_r$ at (t,τ) for all $t>0$.

A test-sequence $\underline{\varphi} \in \mathscr{V}_r$ is *u.m.p. in* $\mathscr{V}_r$ if it is m.p. in $\mathscr{V}_r$ at (t,τ) for all $t>0$, $\tau \in T$.

Let $\bar{\mathscr{V}}_r$ denote the class of u.m.p. test-sequences in $\mathscr{V}_r$. Notice that in general u.m.p. test-sequences need not exist. This motivates the introduction of the weaker concept of admissibility.

A test-sequence $\underline{\varphi} \in \mathscr{V}_r$ is *admissible in* $\mathscr{V}_r$ if for any $\underline{\psi} \in \mathscr{V}_r$, $\underline{\psi} \gtrsim_r \underline{\varphi}$ implies $\underline{\psi} \sim_r \underline{\varphi}$.

If u.m.p. test-sequences in $\mathscr{V}_r$ do exist, then a test-sequence is admissible in $\mathscr{V}_r$ iff it is u.m.p. in $\mathscr{V}_r$.

A class $\mathscr{U}_r \subset \mathscr{V}_r$ is *essentially complete in* $\mathscr{V}_r$ if for every $\underline{\psi} \in \mathscr{V}_r$ there exists $\underline{\varphi} \in \mathscr{U}_r$ such that $\underline{\varphi} \gtrsim_r \underline{\psi}$. An essentially complete class $\mathscr{U}_r$ is *minimal* if no proper subclass of $\mathscr{U}_r$ is essentially complete in $\mathscr{V}_r$.

If $\mathscr{U}_r$ is minimal then its elements are admissible in $\mathscr{V}_r$. If $\underline{\varphi}$ is u.m.p. in $\mathscr{V}_r$ then $\{\underline{\varphi}\}$ is minimal essentially complete in $\mathscr{V}_r$.

Keep in mind that all these concepts are *asymptotic*, valid up to an error term $o(n^{-r/2})$.

It is well known that u.m.p. test-sequences in $\mathscr{T}_0^*$ exist, and that these test-sequences are u.m.p. also in the wider class $\mathscr{T}_0$. A corresponding result holds even true for $\mathscr{T}_1^*$ and $\mathscr{T}_1$. By an appropriate studentization procedure, Neyman's $C(\alpha)$-test as well as the test based on the m.l. estimator can be modified so as to become as. similar $o(n^{-1/2})$, i.e. a member of $\mathscr{T}_1^*$ – and both of these tests are u.m.p. in $\mathscr{T}_1$.

For the $C(\alpha)$-test this was shown by Chibisov (1937a, p. 38, Theorem 8.1, and p. 40, Theorem 9.1), for the test based on the m.l. estimator by Pfanzagl (1973, p. 213, Theorem, and p. 261, Proposition).

That both of these tests are u.m.p. not only in $\mathscr{T}_0$ but even in $\mathscr{T}_1$ is no accident. Both tests are based on critical regions $\{S_n>0\}$ with test-statistics S_n admitting a stochastic expansion. Let $\mathscr{S}_r$ denote the class of all such test-sequences based on test-statistics admitting a stochastic expansion of length r. (For a more precise definition see Section 3.) Corollary 1 asserts that all tests in $\mathscr{T}_1^* \cap \mathscr{S}_1$ are u.m.p. in $\mathscr{T}_1$, if they are u.m.p. in $\mathscr{T}_0$. Briefly put, for tests of structure $\mathscr{S}_1$, first order efficiency implies second order efficiency.

If we refine our analysis further by considering the classes $\mathscr{T}_2$ and $\mathscr{T}_2^*$, the situation becomes less transparent – and more interesting. Preliminary results support the following *conjectures*. U.m.p. tests do not exist in $\mathscr{T}_2$ or in $\mathscr{T}_2^*$ (even if further regularity conditions are imposed on the test-sequences). The power achievable in $\mathscr{T}_2^*$ is up to $o(n^{-1})$ the same as the power achievable in $\mathscr{T}_2$. For every test $\underline{\varphi}$ in $\mathscr{T}_2^*$ there exists a test in $\mathscr{T}_2^*$ which is u.m.p. in $\mathscr{T}_0$ and the power of which is up to $o(n^{-1})$ not inferior to the power of $\underline{\varphi}$.

These conjectures provide a justification for restricting ourselves to test-sequences in $\mathscr{T}_2^* \cap \bar{\mathscr{T}}_0$ (where $\bar{\mathscr{T}}_0$ denotes the class of all test-sequences which are u.m.p. in $\mathscr{T}_0$). For technical reasons, we have to impose further regularity conditions on the test-sequences: Instead of all tests in $\mathscr{T}_2^*$, we study the tests in $\mathscr{T}_2^* \cap \mathscr{S}_2$ only. The restriction to such test-sequences seems justifiable since virtually all test-sequences studied in literature are of this type.

Our main result, Corollary 2, provides an essentially complete subclass of $\mathscr{T}_2^* \cap \bar{\mathscr{T}}_0 \cap \mathscr{S}_2$, together with a simple criterion for the admissibility of tests in $\mathscr{T}_2^* \cap \bar{\mathscr{T}}_0 \cap \mathscr{S}_2$. In Section 4 these results are applied to compare the powers of certain common test-sequences in $\bar{\mathscr{T}}_0$.

It turns out that the test based on the m.l. estimator has the advantage of a power directed against alternatives with rejection probability $1-\alpha$.

From the applied point of view, Corollary 1 is by far the more relevant one, because it says that in choosing between sufficiently regular test-sequences we cannot err much: The number of samples wasted by choosing the "wrong" test remains bounded as the sample size increases. Corollary 2 gives us a guide which test to choose among first order efficient ones, but numerical computations suggest that in certain cases power functions are approximated by their asymptotic expansions with sufficient accuracy only for large sample sizes. This means: In order that our formulas reflect the true deficiencies between first order efficient test-sequences sufficiently accurately, the sample sizes have, perhaps, to be so large that these deficiencies become comparably irrelevant.

In Pfanzagl and Wefelmeyer (1978) similar techniques are applied to show that, starting from a m.l. estimator $\theta^{(n)}$ for a vector parameter, the class of all estimators $\theta^{(n)} + n^{-1} q(\theta^{(n)})$ is asymptotically complete of order $o(n^{-1})$ if q runs through all sufficiently regular functions. As a particular consequence we obtain that the m.l. estimator, if made componentwise $o(n^{-1/2})$-median unbiased by an appropriate choice of q, is in the class of all componentwise $o(n^{-1/2})$-median unbiased estimators maximally concentrated up to $o(n^{-1})$ on all convex sets which are symmetric about the true parameter value.

2. Notations

Let $(X, \mathscr{A})$ be a measurable space and $P_{\theta,\tau}|\mathscr{A}$, $(\theta,\tau) \in \Omega = \Theta \times T$ with open $\Theta \subset \mathbb{R}$, $T \subset \mathbb{R}^p$, a family of p-measures (probability measures). Let $P_\omega^n|\mathscr{A}^n$ denote the n-fold independent product of identical components $P_\omega|\mathscr{A}$. For notational convenience we shall consider X^n as a subspace of $X^{\mathbb{N}}$ with elements $\underline{x} = (x_\nu)_{\nu \in \mathbb{N}}$, and $\mathscr{A}^n$ as a sub-σ-field of $\mathscr{A}^{\mathbb{N}}$.

We shall use the following *convention:* if in a product an index occurs at least twice, this means summation over this index starting from 0 in case of a Roman type index, and from 1 in case of a Greek type index.

For a function $f: X \times \Omega \to \mathbb{R}$ and $\omega, \omega' \in \Omega$, indexed from 0 to p, let

$$f^{i_1 \dots i_k}(x, \omega) = (\partial^k / \partial\omega_{i_1} \dots \partial\omega_{i_k}) f(x, \omega),$$
$$P_\omega(f(\cdot, \omega')) = \int f(x, \omega') P_\omega(dx),$$
$$\tilde{f}(\underline{x}, \omega) = n^{-1/2} \sum_{\nu=1}^{n} (f(x_\nu, \omega) - P_\omega(f(\cdot, \omega))).$$

We assume w.l.g. that unspecified functions $f(\cdot, \omega)$ are standardized such that $P_\omega(f(\cdot, \omega)) = 0$ for $\omega \in \Omega$. For typographical reasons we write $\tilde{f}^k$ instead of $\widetilde{f^k}$ etc. f is said to be differentiable if for every $x \in X$ the function $\omega \to f(x, \omega)$ is differentiable on Ω.

Let $\mu | \mathscr{A}$ denote a σ-finite measure dominating $P_\omega | \mathscr{A}$, $\omega \in \Omega$, and $p(\cdot, \omega)$ a density of $P_\omega | \mathscr{A}$ with respect to $\mu | \mathscr{A}$. Define $l = \log p$.

Moreover,

$$L_{i_{11} \dots i_{1k_1}, \dots, i_{m1} \dots i_{mk_m}}(\omega) = P_\omega \left(\prod_{\nu=1}^{m} l^{i_{\nu 1} \dots i_{\nu k_\nu}}(\cdot, \omega) \right),$$
$$L = (L_{i,j})_{i,j=0,\dots,p},$$
$$L^* = (L_{\alpha,\beta})_{\alpha,\beta=1,\dots,p},$$
$$\Lambda = L^{-1}, \quad \Lambda^* = (L^*)^{-1}$$
$$\lambda_i = \Lambda_{ij} l^j, \quad \lambda^*_\alpha = \Lambda^*_{\alpha\beta} l^\beta.$$

Let $h(\cdot, \omega)$ denote the regression residual with respect to P_ω of $\Lambda_{0i}(\omega) \Lambda_{0j}(\omega)(l^{ij}(\cdot, \omega) - L_{ij}(\omega))$ on $l^0(\cdot, \omega), \dots, l^p(\cdot, \omega)$. We have

(2.1) $\quad h = \Lambda_{0i} \Lambda_{0j} [l^{ij} - L_{ij} - L_{ij,k} \lambda_k].$

Define

$$D(\omega) = \tfrac{1}{4} \Lambda_{00}(\omega)^{-2} P_\omega(h(\cdot, \omega)^2).$$

We have

(2.2) $\quad D = \tfrac{1}{4} \Lambda_{00}^{-2} \Lambda_{0i} \Lambda_{0j} \Lambda_{0k} \Lambda_{0m} (L_{ij,km} - \Lambda_{rs} L_{ij,r} L_{km,s}) - \tfrac{1}{4}$

Furthermore,

$$\varphi(u) = (2\pi)^{-1/2} \exp[-\tfrac{1}{2} u^2],$$
$$\Phi(u) = \int_{r<u} \varphi(r)\, dr,$$
$$N_\beta = \Phi^{-1}(\beta), \quad N = N_\alpha.$$

φ_Σ denotes the Lebesgue density of the multivariate normal distribution with mean vector zero and covariance matrix Σ.

(2.3) *Definition.* Let $f_n: X^{\mathbb{N}} \times \Omega \to \mathbb{R}$, $n \in \mathbb{N}$, be such that $f_n(\cdot, \omega)$ is $\mathscr{A}^n$-measurable for $\omega \in \Omega$ and $n \in \mathbb{N}$. We write $f_n = l_n(r)$ if for every $\tau \in T$ there exist a neighborhood U of (δ, τ) and a constant $a > 0$ such that uniformly for $\omega \in U$ and $\|\omega - \omega'\| \leqq n^{-1/2} \log n$

$$P_\omega^n\{|f_n(\cdot, \omega')| > (\log n)^a\} = o(n^{-r/2}).$$

The relation $f_n = g_n + n^{-s} l_n(r)$ is defined as $n^s(f_n - g_n) = l_n(r)$.

(2.4) *Definition.* An estimator-sequence $\tau^{(n)}$, $n \in \mathbb{N}$, is $o(n^{-r/2})$*-consistent for the nuisance parameter* (given δ) if

$$\tau_\alpha^{(n)} = \tau_\alpha + n^{-1/2}\, l_n(r) \qquad \text{for } \alpha = 1, \ldots, p.$$

(2.5) *Definition.* A $o(n^{-r/2})$-consistent estimator-sequence $\hat{\tau}^{(n)}$, $n \in \mathbb{N}$, for the nuisance parameter is *as. m.l. (asymptotically maximum likelihood)* $o(n^{-r/2})$ (given δ) if

$$\tilde{l}^\alpha(\cdot, \delta, \hat{\tau}^{(n)}) = n^{-r/2}\, l_n(r) \qquad \text{for } \alpha = 1, \ldots, p.$$

The regularity conditions needed inSection 3 are collected in Section 5.

3. The Main Results

We consider the class of test-sequences $\varphi_n = 1_{\{S_n > 0\}}$, $n \in \mathbb{N}$, for the hypothesis $\{(t, \tau) \in \mathbb{R}^+ \times T\colon t = 0\}$ which are based on a sequence of test-statistics S_n, $n \in \mathbb{N}$, admitting a stochastic expansion of the form

$$(3.1) \quad S_n = \tilde{f}_0 + c + \sum_{j=1}^{r} n^{-j/2} Q_j(\tilde{f}_1, \ldots, \tilde{f}_{m_j}, \cdot) + n^{-(r+1)/2}\, l_n(r).$$

(Since δ remains fixed, $f_i(x, \cdot)$, c and $Q_j(r, \cdot)$ are considered as functions of τ only.)

We assume that f_0 fulfills Condition M_3 and D and is differentiable with derivatives $f_0^1, \ldots, f_0^p$ fulfilling Conditions L_2 and M_2, and $P_{\delta,\tau}(f_0(\cdot, \tau)^2) > 0$ for $\tau \in T$.

$\mathscr{S}_0$ is the class of all test-sequences of type (3.1) for $r = 0$ for which the function $c\colon T \to \mathbb{R}$ fulfills a local Lipschitz condition.

$\mathscr{S}_1$ is the class of all test-sequences of type (3.1) for $r = 1$ for which the following regularity conditions are fulfilled.

The functions $\tau \to P_{\delta,\tau}(f_0(\cdot, \tau)^2)$ and c admit partial derivatives fulfilling a local Lipschitz condition on T.

For $\tau \in T$, the functions $f_1(\cdot, \tau), \ldots, f_{m_1}(\cdot, \tau)$ constitute a base for $l^i(\cdot, \delta, \tau)$, $i = 0, \ldots, p$, under $P_{\delta, \tau}$.

The vector consisting of $f_1, \ldots, f_{m_1}$ fulfills Conditions C and U_3; its components fulfill Condition D.

The functions $Q_1, Q_1^{(0)} = (\partial/\partial u_0)\, Q_1$ fulfill Condition B(a).

$\mathscr{S}_2$ is the class of all test-sequences of type (3.1) for $r = 2$ for which the following regularity conditions are fulfilled.

The function c admits second order partial derivatives fulfilling a local Lipschitz condition on T.

For $\tau \in T$, the functions $f_1(\cdot,\tau), \ldots, f_{m_1}(\cdot,\tau)$ constitute a base for $l^i(\cdot,\delta,\tau)$ and $l^{ij}(\cdot,\delta,\tau) - L_{ij}(\delta,\tau)$, $i,j=0,\ldots,p$, under $P_{\delta,\tau}$.

The vector consisting of $f_1,\ldots,f_{m_2}$ fulfills Conditions C and U_4; its components fulfill Condition D.

The functions $f_1,\ldots,f_{m_1}$ admit partial derivatives fulfilling Conditions L_2 and M_2.

Q_1 fulfills Condition $\bar{B}$, and $Q_1^{(0)}$, $Q_1^{(00)}$, Q_2, $Q_2^{(0)}$ fulfill Condition B(a).

Examples of sequences in $\mathscr{S}_r$ are studied in Section 4.

Proposition. *Assume that l admits partial derivatives up to the second order fulfilling Conditions L_3 and M_3. Let Conditions* (i), (ii), (iii), *I_1 be fulfilled. Assume, furthermore, that there exists a $o(n^0)$-consistent estimator-sequence for the nuisance parameter.*

Then the following is true.

(a) *If $\underline{\varphi} \in \mathscr{T}_0$ then uniformly for $0 \leqq t \leqq \log n$ and locally uniformly for $\tau \in T$*

$$\Pi_n(\varphi_n)(t,\tau) \leqq h_0(t,\tau) + o(n^0) \tag{3.2}$$

with $h_0(t,\tau) = \Phi(N + \Lambda_{00}(\delta,\tau)^{-1/2} t)$.

(b) *If $\underline{\varphi} \in \mathscr{S}_0$ is m.p. in $\mathscr{T}_0$ at (t^*,τ^*) then $\underline{\varphi}$ is u.m.p. in $\mathscr{T}_0$ at τ^*.*

More precisely: If for $\underline{\varphi} \in \mathscr{S}_0$ equality holds in (3.2) for some (t^*,τ^*) with $t^* > 0$ then uniformly for $0 \leqq t \leqq \log n$

$$\Pi_n(\varphi_n)(t,\tau^*) = h_0(t,\tau^*) + o(n^0).$$

Theorem 1. *Assume that the family of p-measures fulfills the following regularity conditions. l admits partial derivatives up to the third order fulfilling Conditions L_4 and M_4, and the vector consisting of $l^i, l^{ij} - L_{ij}$, $i,j=0,\ldots,p$, fulfills Condition $\bar{C}$. Conditions* (i), (ii), (iii), *I_2 hold. Assume, furthermore, that there exists a $o(n^{-1/2})$-consistent estimator-sequence for the nuisance parameter.*

Then the following is true.

(a) *If $\underline{\varphi} \in \mathscr{T}_1$ then uniformly for $0 \leqq t \leqq \log n$ and locally uniformly for $\tau \in T$*

$$\Pi_n(\varphi_n)(t,\tau) \leqq H_{1n}(t,\tau) + o(n^{-1/2})$$

with $H_{1n} = h_0 + n^{-1/2} h_1$ as defined in the Appendix.

(b) *If $\underline{\varphi} \in \mathscr{S}_1$ is u.m.p. in $\mathscr{T}_0$ at every τ in a neighborhood of τ^* and has power $\alpha + o(n^{-1/2})$ at $(0,\tau^*)$ then $\underline{\varphi}$ is u.m.p. in $\mathscr{T}_1$ at τ^*. More precisely, we have uniformly for $0 \leqq t \leqq \log n$*

$$\Pi_n(\varphi_n)(t,\tau^*) = H_{1n}(t,\tau^*) + o(n^{-1/2}).$$

Corollary 1. *Under the assumptions of Theorem 1, if $\underline{\varphi} \in \mathscr{T}_1^* \cap \mathscr{S}_1$ is u.m.p. in $\mathscr{T}_0$ then $\underline{\varphi}$ is u.m.p. in $\mathscr{T}_1$. More precisely, we have uniformly for $0 \leqq t \leqq \log n$ and locally uniformly for $\tau \in T$,*

$$\Pi_n(\varphi_n)(t,\tau) = H_{1n}(t,\tau) + o(n^{-1/2}).$$

For the smaller class of test-sequences obtained by as. studentization, this result has earlier been obtained in Pfanzagl (1976, p. 26, Theorem 7.5).

By Corollary 1, the power of a first order efficient test sequence in $\mathscr{T}_1^* \cap \mathscr{S}_1$ is up to to $o(n^{-1/2})$ independent of the $n^{-1/2}$-term Q_1 of the stochastic expansion of its test-statistic. The main content of Theorem 2 is that analogously, the power of a first order efficient test-sequence in $\mathscr{T}_2^* \cap \mathscr{S}_2$ is up to $o(n^{-1})$ independent of Q_2 and, furthermore, that it depends on Q_1 in a simple way.

The proof of Theorem 2 rests on the following Lemma 1 which is proved in Pfanzagl (1976, p. 11, Lemma 4.10) and formulated below for easier reference. The basic idea is to utilize the fact that the test-statistic does not depend on the nuisance parameter. This places a restriction upon the functions Q_j occurring in the stochastic expansion of the test-statistic: The stochastic expansion as a whole must remain invariant under small changes of the nuisance parameter.

Lemma 1. *Let*

$$S_n = \tilde{f}_0 + c + n^{-1/2} Q(\tilde{f}_1, \ldots, \tilde{f}_m, \cdot) + n^{-1} l_n(0),$$

where $f_i(x, \cdot)$, c and $Q(r, \cdot)$ are considered as functions of τ only. Assume that $c: T \to \mathbb{R}$ admits partial derivatives fulfilling a local Lipschitz condition. Let Q fulfill Condition B. Assume that f_0 admits second order and $f_1, \ldots, f_m$ first order partial derivatives fulfilling Conditions L_2 and M_2. Let $f_0, f_1, \ldots, f_m, f_0^1, \ldots, f_0^p$ fulfill Condition M_2.

For $\tau \in T$ let $g(\cdot, \tau)$ denote a vector the components of which constitute a base of $P_{\delta,\tau}$ – a.e. linearly independent functions for $f_1(\cdot, \tau), \ldots, f_m(\cdot, \tau)$, $f_0^1(\cdot, \tau), \ldots, f_0^p(\cdot, \tau)$ under $P_{\delta,\tau}$, and let $M(\tau)$ be a matrix such that $(f_1, \ldots, f_m, f_0^1, \ldots, f_0^p)' = M g$.

Then for all $(s, u) \in \mathbb{R}^p \times \mathbb{R}^{m+p}$ and $\tau \in T$,

$$\begin{aligned} &Q((Mu)_{i=1,\ldots,m}, \tau) \\ &\quad = Q(((Mu)_i + s_\alpha P_{\delta,\tau}(f_i^\alpha(\cdot, \tau)))_{i=1,\ldots,m}, \tau) \\ &\qquad + s_\alpha (Mu)_{m+\alpha} + \tfrac{1}{2} s_\alpha s_\beta P_{\delta,\tau}(f_0^{\alpha\beta}(\cdot, \tau)) + s_\alpha c^\alpha(\tau). \end{aligned}$$

Remark 1. Assume that for all τ in an open subset of T the components of $f(\cdot, \tau) = (f_1(\cdot, \tau), \ldots, f_m(\cdot, \tau))'$ are $P_{\delta,\tau}$ – a.e. linearly independent and constitute a base for $f_0^1(\cdot, \tau), \ldots, f_0^p(\cdot, \tau)$ under $P_{\delta,\tau}$, say

(3.3) $\quad (f_0^1(\cdot, \tau), \ldots, f_0^p(\cdot, \tau))' = K(\tau) f(\cdot, \tau).$

If the components of f fulfill Condition D then by Lemma 1 we have for $(s, r) \in \mathbb{R}^p \times \mathbb{R}^m$

(3.4)
$$\begin{aligned} Q(r, \tau) &= Q(r - s_\alpha P_{\delta,\tau}(l^\alpha(\cdot, \delta, \tau) f(\cdot, \tau)), \tau) \\ &\quad + s' K(\tau) r + \tfrac{1}{2} s_\alpha s_\beta P_{\delta,\tau}(f_0^{\alpha\beta}(\cdot, \tau)) + s_\alpha c^\alpha(\tau). \end{aligned}$$

If $f_\alpha(\cdot, \tau) = l^\alpha(\cdot, \delta, \tau)$ for $\alpha = 1, \ldots, p$, then we may assume w.l.g. that $f_{p+1}(\cdot, \tau), \ldots, f_m(\cdot, \tau)$ are $P_{\delta,\tau}$-uncorrelated to $l^1(\cdot, \delta, \tau), \ldots, l^p(\cdot, \delta, \tau)$. Then (3.4), applied for $s_\alpha = \Lambda_{\alpha\beta}(\delta, \tau) r_\beta$, $\alpha = 1, \ldots, p$, yields the following "canonical representation" of Q: For $(v, w) \in \mathbb{R}^p \times \mathbb{R}^{m-p}$,

$$(3.5) \quad Q(v, w, \tau) = Q(0, w, \tau) + R(v, w, \tau)$$

where

$$(3.6) \quad \begin{aligned} R(v, w, \tau) = {} & v' \Lambda^*(\delta, \tau) K(\tau) (v, w)' \\ & + \tfrac{1}{2} \Lambda^*_{\alpha\rho}(\delta, \tau) \Lambda^*_{\beta\sigma}(\delta, \tau) P_{\delta, \tau}(f_0^{\alpha\beta}(\cdot, \tau)) v_\rho v_\sigma \\ & + c^\alpha(\tau) \Lambda^*_{\alpha\beta}(\delta, \tau) v_\beta. \end{aligned}$$

Because of (3.3), the function $R(\tilde{l}^1, \ldots, \tilde{l}^p, \tilde{f}_{p+1}, \ldots, \tilde{f}_m, \cdot)$ does not depend on S_n except through the derivatives of f_0 and c.

Remark 2. The formulation of the following Theorem 2 requires some preparations. For notational convenience we omit δ and τ. The leading term of the stochastic expansion of a test-statistic S_n is of the form $\tilde{f}_0 + c$. If $\varphi \in \mathscr{S}_2$ is m.p. in $\mathscr{T}_0$ at τ and has power $\alpha + o(n^{-1})$ at $(0, \tau)$ for all τ in an open subset V of T, then by Lemma 2 (Section 6) for $\tau \in V$ there exists $d(\tau)$ such that $f_0 = d\lambda_0$ P-a.e. and $c = N d \Lambda_{00}^{1/2}$. The function λ_0 may be contained in the space spanned under P by the arguments $f_1, \ldots, f_{m_j}$ of Q_j. It is convenient to separate λ_0 from $f_1, \ldots, f_{m_j}$ and to choose the remaining functions as P-uncorrelated not only to $l^1, \ldots, l^p$ but also to λ_0. (Observe that $l^1, \ldots, l^p$, too, are P-uncorrelated to λ_0.) More specifically, we write the terms $Q_j(\tilde{f}_1, \ldots, \tilde{f}_{m_j}, \cdot)$ of the stochastic expansion of S_n for $j = 1, 2$ as

$$(3.7) \quad d Q_j(\tilde{\lambda}_0 + N \Lambda_{00}^{1/2}, \tilde{l}^1, \ldots, \tilde{l}^p, \tilde{f}_1, \ldots, \tilde{f}_m, \cdot),$$

and we assume that $f_1, \ldots, f_m$ are P-uncorrelated to $l^0, \ldots, l^p$.

Let $J(\tau)$ be a matrix such that

$$(3.8) \quad (\lambda_0^1, \ldots, \lambda_0^p)' = J(\lambda_0, l^1, \ldots, l^p, f_1, \ldots, f_m)'.$$

If $d \equiv 1$ on V then we obtain the following canonical representation: For $(u, v, w) \in \mathbb{R} \times \mathbb{R}^p \times \mathbb{R}^m$,

$$(3.9) \quad Q_1(u + N \Lambda_{00}^{1/2}, v, w, \cdot) = Q_1(u + N \Lambda_{00}^{1/2}, 0, w, \ ) + R_1(u, v, w, \cdot),$$

where

$$(3.10) \quad \begin{aligned} R_1(u, v, w, \cdot) = {} & v' \Lambda^* J(u, v, w)' \\ & + \tfrac{1}{2} \Lambda^*_{\alpha\rho} \Lambda^*_{\beta\sigma} P(\lambda_0^{\alpha\beta}) v_\rho v_\sigma + \tfrac{1}{2} N \Lambda_{00}^{-1/2} \Lambda_{00}^\alpha \Lambda^*_{\alpha\beta} v_\beta. \end{aligned}$$

Because of (3.8), the function $R(\tilde{\lambda}_0, \tilde{l}^1, \ldots, \tilde{l}^p, \tilde{f}_1, \ldots, \tilde{f}_m, \cdot)$ does not depend on the test-sequence.

Let $\Gamma(\theta, \tau)$ denote the covariance matrix of $f(\cdot, \tau) = (f_1(\cdot, \tau), \ldots, f_m(\cdot, \tau))'$ under $P_{\theta, \tau}$. Define

$$\begin{aligned} C(N, \cdot) = {} & \tfrac{1}{6}(1 - N^2) \Lambda_{00}^{-1} \Lambda_{0i} \Lambda_{0j} \Lambda_{0k} L_{i, j, k} \\ & - \int dv\, dw\, \varphi_{L^*}(v) \varphi_\Gamma(w) R_1(-N \Lambda_{00}^{1/2}, v, w, \cdot). \end{aligned}$$

Let $a(\tau)$ be the (unique) vector such that $h(\cdot,\delta,\tau)=a(\tau)'f(\cdot,\tau)\ P_{\delta,\tau}$ – a.e., where h is defined in (2.1).

Theorem 2. *Assume that l admits partial derivatives up to the third order fulfilling Conditions L_4 and M_4. Let Conditions* (i), (ii), (iii), I_3 *be fulfilled. Assume, furthermore, that there exists a $o(n^{-1})$-consistent estimator-sequence for the nuisance parameter.*

Assume that $\varphi \in \mathscr{S}_2$ is m.p. in $\mathscr{T}_0$ at τ and has power $\alpha+o(n^{-1})$ at $(0,\tau)$ for all τ in an open subset V of T.

Then, with the conventions and notations of Remark 2, the following holds true. Uniformly for $0 \leqq t \leqq \log n$ and locally uniformly in $\tau \in V$

$$\Pi_n(\varphi_n)(t,\tau)=H_{2n}(t,\tau) \\ -n^{-1}\varphi(N+t\Lambda_{00}^{-1/2})\tfrac{1}{2}t\Lambda_{00}^{-3/2}\int dw\,\varphi_\Gamma(w)[Q_1(0,0,w,\cdot)-C(N,\cdot) \\ -\tfrac{1}{2}t\Lambda_{00}^{-1}a'w]^2|_{(\delta,\tau)}+o(n^{-1}),$$

where $H_{2n}=h_0+n^{-1/2}h_1+n^{-1}h_2$ depends on the family of p-measures only, and not on the test-sequence. The explicit form of H_{2n} is given in the Appendix.

For the special case of test-sequences obtained by as. studentization of test-statistics admitting a stochastic expansion for which the terms of order $n^{-j/2}$ are polynomials of degree $j+1$ this result was obtained by Bender (1976, p. 78, Theorem 1). His method of proof requires the explicit computation of H_{2n}.

Remark 3. A convenient way to construct test-sequences in $\mathscr{T}_2^*\cap\bar{\mathscr{T}}_0\cap\mathscr{S}_2$ with prescribed power functions is the desensitization and as. studentization procedure described in Section 4. Let $\bar{Q}(w,\tau)$ be an arbitrary (sufficiently regular) function. Observe the conventions of Remark 2. Then the test-sequence obtained from

$$\tilde{\lambda}_0+n^{-1/2}\bar{Q}(\tilde{f}_1,\ldots,\tilde{f}_m,\cdot)$$

by desensitization and as. studentization $o(n^{-1})$ with an as. ml. estimator-sequence $o(n^{-1})$ is in $\mathscr{T}_2^*\cap\bar{\mathscr{T}}_0\cap\mathscr{S}_2$, and the stochastic expansion of the pertinent test-statistic, written as in (3.7), fulfills

$$Q_1(0,0,w,\tau)-C(N,\delta,\tau)=\bar{Q}(w,\tau)-\bar{C}(N,\delta,\tau),$$

where $\bar{C}$ is uniquely determined by $\bar{Q}$.

In particular, admissible test-sequences in $\mathscr{T}_2^*\cap\bar{\mathscr{T}}_0\cap\mathscr{S}_2$ are obtained from $\bar{Q}(w,\tau)=q(\tau)a(\tau)'w$, where q is a sufficiently smooth function of the nuisance parameter (see Corollary 2(c) below).

More specifically, let $\mathfrak{Q}$ denote the class of all functions $q\colon T\to\mathbb{R}^+$ fulfilling a local Lipschitz condition. For $q\in\mathfrak{Q}$, let $\underline{\varphi}(\cdot,q)$ denote the test-sequence obtained from

$$\tilde{\lambda}_0(\cdot,\delta+n^{-1/2}q(\tau)\Lambda_{00}(\delta,\tau),\tau_1+n^{-1/2}q(\tau)\Lambda_{01}(\delta,\tau),\ldots,\tau_p \\ +n^{-1/2}q(\tau)\Lambda_{0p}(\delta,\tau))$$

by desensitization and as. studentization $o(n^{-1})$ with an as. m.l. estimator-sequence $o(n^{-1})$ for the nuisance parameter.

Corollary 2. *Assume that the family of p-measures fulfills the following regularity conditions. l admits partial derivatives up to the fourth order fulfilling Conditions L_5 and M_5, and the vector consisting of $l^i, l^{ij}-L_{ij}, l^{ijk}-L_{ijk}, i,j,k=0,\dots,p$, fulfills Condition $\bar{C}$. Conditions* (i), (ii), (ii), I_3 *hold.*

Assume, furthermore, that there exists a $o(n^{-1})$-consistent estimator-sequence for the nuisance parameter.

Then the following holds true.

(a) *For every $q\in\mathfrak{Q}$, the test-sequence $\underline{\varphi}(\cdot,q)$ belongs to $\mathscr{T}_2^*\cap\bar{\bar{\mathscr{T}}}_0\cap\mathscr{S}_2$, and $Q_1(0,0,w,\tau)=q(\tau)\,a(\tau)'\,w+C(N,\delta,\tau)$.*

In case $D(\delta,\tau)>0$ for $\tau\in T$ (for definition see (2.2)*), the following assertions hold.*

(b) *The class of test-sequences $\underline{\varphi}(\cdot,q)$, $q\in\mathfrak{Q}$, is minimal essentially complete in $\mathscr{T}_2^*\cap\bar{\bar{\mathscr{T}}}_0\cap\mathscr{S}_2$.*

(c) *A test-sequence $\underline{\varphi}\in\mathscr{T}_2^*\cap\bar{\bar{\mathscr{T}}}_0\cap\mathscr{S}_2$ is admissible in $\mathscr{T}_2^*\cap\bar{\bar{\mathscr{T}}}_0\cap\mathscr{S}_2$ iff there exists $q\in\mathfrak{Q}$ such that*

$$Q_1(0,0,w,\tau)=q(\tau)\,a(\tau)'\,w+C(N,\delta,\tau).$$

In this case, uniformly for $0\leq t\leq\log n$ and locally uniformly for $\tau\in T$,

$$\Pi_n(\varphi_n)(t,\tau)=H_{2n}(t,\tau)$$
$$-n^{-1}\varphi(N+t\Lambda_{00}^{-1/2})\tfrac{1}{2}t\Lambda_{00}^{-3/2}\,D(2\Lambda_{00}q-t)^2|_{(\delta,\tau)}+o(n^{-1}).$$

Remark 4. The deficiency of admissible test-sequences in $\mathscr{T}_2^*\cap\bar{\bar{\mathscr{T}}}_0\cap\mathscr{S}_2$ is proportional to D. We remark that D, as it ought to, remains unchanged under all (sufficiently smooth) transformations of the nuisance parameter.

D is proportional to the variance of the regression residual of $\Lambda_{0i}\Lambda_{0j}(l^{ij}-L_{ij})$ on $l^0,\dots,l^p$. Hence D vanishes for $(p+1)$-dimensional exponential families. In a certain sense, D indicates how far a given family deviates from an exponential family. Notice that D is not a function of the "curvature" of a multivariate curved exponential family defined by Reeds in the discussion of Efron (1975, p. 1237). Hence the expectation of Efron (1975, p. 1241) that in hypothesis testing this curvature plays the same role as in the one-dimensional case, does not seem to materialize.

Remark 5. If $D(\delta,\tau)=0$ for all $\tau\in T$ then every admissible test-sequence is u.m.p. in $\mathscr{T}_2^*\cap\bar{\bar{\mathscr{T}}}_0\cap\mathscr{S}_2$ and hence constitutes in itself a minimal essentially complete class.

Remark 6. The essentially complete class presented in Corollary 2 is still rather large, because it allows the value of t against which the power is maximized to vary arbitrarily with the nuisance parameter: For any function $\tau\to t(\tau)$, the essentially complete class contains a test-sequence (namely $\underline{\varphi}(\cdot,\frac{1}{2}t\Lambda_{00}(\delta,\cdot)^{-1})$) which maximizes the power at $(t(\tau),\tau)$, simultaneously for all $\tau\in T$.

It suggests itself to restrict considerations to a suitable subclass of functions $\tau\to t(\tau)$. One reasonable criterion is to choose test-sequences in such a way that a given power β is reached as soon as possible (i.e. for values (t, τ) as close as possible to $(0, \tau)$). This is achieved for $t(\tau) = \Lambda_{00}(\delta, \tau)^{1/2}(N_\beta - N_\alpha)$. Hence $\underline{\varphi}(\cdot, q)$ with $q(\tau) = \frac{1}{2}\Lambda_{00}(\delta, \tau)^{-1/2}(N_\beta - N_\alpha)$ is a reasonable choice out of the essentially complete class.

Remark 7. For the case without nuisance parameter, the class $\underline{\varphi}(\cdot, q)$, $q \in \mathfrak{Q}$, reduces to the class of critical regions of level $\alpha + o(n^{-1})$ based on the class of test-statistics $\bar{l}^0(\cdot, \delta + n^{-1/2}t)$, $t \geqq 0$. This class is essentially complete in $\mathscr{T}_2^*$, not in $\mathscr{T}_2^* \cap \bar{\mathscr{T}}_0 \cap \mathscr{S}_2$ only. Without nuisance parameters, the restriction to $\bar{\mathscr{T}}_0$ becomes meaningless, and the technical reasons enforcing the restriction to $\mathscr{S}_2$ vanish because of a different method of proof available in the case without nuisance parameters. (See Pfanzagl (1975, p. 5, Theorem 2).)

Strasser (1977, p. 25, Theorem 3) computes the power functions for a class of critical regions based on quantiles of posterior distributions with respect to a particular prior. By Pfanzagl (1975, p. 5, Theorem 2, and p. 14, Theorem 3) this class is essentially complete in $\mathscr{T}_2^*$.

4. Applications

In this section we shall apply our results to discuss third order properties of some test-sequences considered in the literature.

Let $V_n: X^n \to \mathbb{R}$, $n \in \mathbb{N}$, be a sequence of functions and $\tau^{(n)}$, $n \in \mathbb{N}$, an estimator-sequence for the parameter τ in the family $P_{\delta, \tau}$, $\tau \in T$, both admitting a stochastic expansion up to $n^{-(r+1)/2} l_n(r)$. Then we can use the following *as. studentization procedure* $o(n^{-r/2})$ to obtain a test-sequence in $\mathscr{T}_r^*$: First we determine $c_n^{(0)}(\tau)$ such that

$$P_{(0,\tau)}^{(n)}\{V_n - c_n^{(0)}(\tau) > 0\} = \alpha + o(n^0).$$

Since δ remains fixed, the dependence of $c_n^{(0)}(\tau)$ on δ is suppressed.

Let $S_n^{(0)} = V_n - c_n^{(0)}(\tau^{(n)})$. We have

$$P_{(0,\tau)}^{(n)}\{S_n^{(0)} > 0\} = \alpha + o(n^0).$$

Under suitable regularity conditions there exists $c_n^{(1)}$ such that

$$P_{(0,\tau)}^{(n)}\{S_n^{(0)} - n^{-1/2} c_n^{(1)}(\tau) > 0\} = \alpha + o(n^{-1/2}).$$

Now we define $S_n^{(1)} = S_n^{(0)} - n^{-1/2} c_n^{(1)}(\tau^{(n)})$. Proceeding in this way we obtain a sequence of "studentized" test-statistics $S_n^{(r)}$, $n \in \mathbb{N}$, such that

$$P_{(0,\tau)}^{(n)}\{S_n^{(r)} > 0\} = \alpha + o(n^{-r/2}).$$

Regularity conditions under which the pertinent test-sequence is in $\mathscr{T}_r^* \cap \mathscr{S}_r$ can be obtained for $r = 1$ from Pfanzagl (1976, p. 17, Theorem 5.8). An analogous studentization procedure, based on an estimator-sequence for (θ, τ) instead of τ, is described in Pfanzagl (1973, p. 254, Lemma 9).

A natural choice for V_n is $n^{1/2}(\theta^{(n)} - \delta)$, where $\theta^{(n)}$ is any estimator for θ. Another possibility is the following: If for every $\tau \in T$ the statistic $U_n(\cdot, \tau)$ is reasonable for testing the hypothesis $\theta \leqq \delta$ if τ is known, then one might try the *desensitized* statistic $V_n = (U_n(\cdot, \tau^{(n)})$.

Let $\hat{\omega}^{(n)} = (\hat{\theta}^{(n)}, \hat{\tau}^{(n)})$ denote the m.l. estimator for $\omega = (\theta, \tau)$ and $\hat{\tau}^{(n)}(\cdot, \delta)$ the m.l. estimator for τ in the family $P_{\delta,\tau}$, $\tau \in T$.

Remark 8. If an estimator-sequence $\tau^{(n)}$, $n \in \mathbb{N}$, with stochastic expansion

$$\tau_\alpha^{(n)} = \tau_\alpha + n^{-1/2} \tilde{g}_\alpha + n^{-1} l_n(2)$$

is used for as. studentization, then $\tilde{g}$ appears in the stochastic expansion of the critical bound and hence of the test-statistic through the term

$$c_n^{(0)}(\tau^{(n)}) = c_n^{(0)} + n^{-1/2} (\partial/\partial \tau_\alpha) c_n^{(0)} \tilde{g}_\alpha + n^{-1} l_n(2).$$

By Theorem 2 the power function of the pertinent test-sequence depends up to $o(n^{-1})$ on $Q_1(0, 0, w, \cdot)$ only. Since w represents regression residuals on $l^0, \ldots, l^p$, the power function is the same for all studentizing estimators for which g is a linear combination of l^i, $i = 0, \ldots, p$. This holds in particular true for $\hat{\tau}^{(n)}(\cdot, \delta)$, $\hat{\tau}^{(n)}$, and $\hat{\omega}^{(n)}$.

In the following we shall discuss the power of certain test-sequences obtained from statistics V_n by as. studentization $o(n^{-1})$ with $\hat{\tau}^{(n)}(\cdot, \delta)$, $\hat{\tau}^{(n)}$, or $\hat{\omega}^{(n)}$. According to Remark 8, in order to evaluate $Q_1(0, 0, w, \cdot)$ it suffices to compute the stochastic expansion of V_n. Observe that Q_1 has to be written in the form (3.7).

(i) By Pfanzagl (1973, p. 253, Lemma 8), the statistic $V_n = n^{1/2}(\hat{\theta}^{(n)} - \delta)$ admits a stochastic expansion of the form

$$V_n = \tilde{\lambda}_0 + n^{-1/2} [\Lambda_{00}^{-1} \Lambda_{0i} \Lambda_{0j} \tilde{l}^{ij} \tilde{\lambda}_0 + \Lambda_{0i} \tilde{l}^{i\alpha} \tilde{\lambda}_\alpha^* + \tfrac{1}{2} \Lambda_{0i} L_{ijk} \tilde{\lambda}_j \tilde{\lambda}_k] + n^{-1} l_n(2).$$

Hence the test-sequence obtained from V_n fulfills

$$Q_1(0, 0, w, \cdot) = -\Lambda_{00}^{-1/2} N a' w + C.$$

According to Remark 6, this test-sequence reaches the power $1 - \alpha$ as soon as possible and is therefore a reasonable option.

For the case without nuisance parameter this result has been obtained earlier by Chibisov (1973b, p. 66, Theorem 8) and Pfanzagl (1974, p. 28, Theorem 4). By Strasser (1977, p. 25, Theorem 3) the same result holds true for the test based on the posterior median with respect to a particular prior.

An explicit formula for the test-sequence based on as. studentization with $\hat{\omega}^{(n)}$ is given in Pfanzagl (1973, p. 213, Theorem, and p. 264).

(ii) Starting from $U_n(\underline{x}, \tau) = a_i(\tau) \tilde{l}^i(\underline{x}, \delta, \tau)$ with sufficiently smooth functions a_i, desensitization with $\hat{\tau}^{(n)}(\cdot, \delta)$ leads to the efficient version of Neyman's $C(\alpha)$-test. We have

$$U_n(\cdot, \hat{\tau}^{(n)}(\cdot, \delta)) = \Lambda_{00}^{-1} a_0 \tilde{\lambda}_0 + n^{-1/2} [\tfrac{1}{2} \Lambda_{00}^{-1} a_0 \Lambda_{0i} L_{i\alpha\beta} \tilde{\lambda}_\alpha^* \tilde{\lambda}_\beta^* + \Lambda_{00}^{-1} a_0^\alpha \tilde{\lambda}_\alpha^* \tilde{\lambda}_0 + \tfrac{1}{2} \Lambda_{00}^{-1} a_0 \Lambda_{0i} \tilde{l}^{i\alpha} \tilde{\lambda}_\alpha^*] + n^{-1} l_n(2).$$

Hence

$$Q_1(0, 0, w, \cdot) = C.$$

According to Remark 6, this test-sequence wastes its power at rejection probabilities near α. It is therefore less recommendable than the test-sequence based on the m.l. estimator discussed in (i).

For the case without nuisance parameter a corresponding result has been obtained by Chibisov (1973b, p. 57) and Pfanzagl (1974, p. 27, Theorem 3).

(iii) The test-sequences obtained from

$$U_n(\underline{x}, \tau) = \prod_{\nu=1}^{n} \frac{p(x_\nu, \delta + n^{-1/2} t, \tau)}{p(x_\nu, \delta, \tau)}$$

by desensitization with $\hat{\tau}^{(n)}(\cdot, \delta)$ or $\hat{\tau}^{(n)}$ are inadmissible, since $Q_1(0, 0, \tilde{f}, \cdot)$ depends on the second order derivatives l^{ij} of l not only through the function h defined in (2.2). An admissible test-sequence is obtained from the test-statistic

$$\prod_{\nu=1}^{n} \frac{p(x_\nu, \delta + n^{-1/2} t, \hat{\tau}^{(n)}(\cdot, \delta + n^{-1/2} t))}{p(x_\nu, \delta, \hat{\tau}^{(n)}(\cdot, \delta))}.$$

or if U_n is desensitized with $\hat{\tau}^{(n)}(\cdot, \delta + n^{-1/2} \frac{1}{2} t)$. Both test-sequences are m.p. against (t, τ) for all $\tau \in T$. These results are obtained as in (i) and (ii) by computing the stochastic expansions of the test-statistics. For example,

$$\begin{aligned} \log U_n(\cdot, \hat{\tau}^{(n)}(\cdot, \delta)) = {} & t \Lambda_{00}^{-1} \tilde{\lambda}_0 \\ & + n^{-1/2} [\tfrac{1}{2} t^2 \tilde{l}^{00} - \tfrac{1}{2} t^2 L_{00,\alpha} \tilde{\lambda}_\alpha^* + \tfrac{1}{2} t \Lambda_{00}^{-1} \Lambda_{0i} L_{i\alpha\beta} \tilde{\lambda}_\alpha^* \tilde{\lambda}_\beta^* \\ & + \tfrac{1}{2} t \Lambda_{00}^{-1} \Lambda_{0i} \tilde{l}^{i\alpha} \tilde{\lambda}_\alpha^*] + n^{-1} l_n(2). \end{aligned}$$

The test-sequence is inadmissible because of the term $\frac{1}{2} t^2 \tilde{l}^{00}$.

5. Regularity Conditions and Definitions

In this section we collect the assumptions and definitions which are needed in Section 3.

(i) $P_\omega | \mathscr{A}$, $\omega \in \Omega$, are mutually absolutely continuous.

(ii) $L_i(\delta, \tau) = 0$ for $i = 0, \ldots, p$ and $\tau \in T$.

(iii) $L(\delta, \tau)$ is positive definite for $\tau \in T$.

Conditions M_r, L_r, D refer to a function $f: X \times \Omega \to \mathbb{R}$ such that $f(\cdot, \omega)$ is $\mathscr{A}$-measurable for $\omega \in \Omega$.

Condition $\mathbf{M}_r$. For every $\tau \in T$ there exists a neighborhood U of (δ, τ) such that

$$\sup_{\omega, \omega' \in U} P_\omega(|f(\cdot, \omega')|^r) < \infty.$$

Condition L_r. For every $\tau \in T$ there exist a neighborhood U of (δ, τ) and an $\mathscr{A}$-measurable function k: $X \to \mathbb{R}$ such that

(a) $|f(x,\omega) - f(x,\omega')| \leqq \|\omega - \omega'\| \, k(x)$

for $x \in X$ and $\omega, \omega' \in U$,

(b) k fullfills Condition M_r.

Condition D. For $p(\cdot,\omega) f(\cdot,\omega)$, the order of integration with respect to $\mu|\mathscr{A}$ and differentiation with respect to ω is interchangeable at $\omega = (\delta, \tau)$, for every $\tau \in T$.

If f depends only on the nuisance parameter τ then differentiation is understood to refer only to τ.

Condition I_r. The function l admits partial derivatives to the order $r+1$, and the products $l^{i_1 \dots i_{k_1}} \dots l^{i_{k_m+1} \dots i_s}$ fulfill Condition D for $1 \leqq m \leqq s \leqq r$.

Conditions C, $\bar{\mathrm{C}}$, U_r refer to a vector-valued function $g: X \times \Omega \to \mathbb{R}^q$ such that $g(\cdot,\omega)$ is $\mathscr{A}$-measurable for $\omega \in \Omega$.

Condition C. For every $\tau \in T$ there exists a neighborhood U of (δ, τ) such that

$$\limsup_{\|r\| \to \infty} \sup_{\omega \in U} |P_\omega(\exp[i r' g(\cdot,\omega)]| < 1.$$

Condition C implies that the components of $g(\cdot,\omega)$ are linearly independent under P_ω for $\omega \in U$.

Condition $\bar{\mathrm{C}}$. There exists a subvector $\bar{g}$ of g such that for $\omega \in \Omega$ the components of $\bar{g}(\cdot,\omega)$ constitute a base for the components of $g(\cdot,\omega)$ under P_ω, and $\bar{g}$ fulfills Condition C.

Condition C is a uniform version of Cramer's condition. The following useful sufficient condition is given by Bhattacharya and Ghosh (1978, p. 446, Lemma 2.3):

If P is a p-measure on the Borel algebra of $\mathbb{R}^m$ the Lebesgue density of which is positive on an open set V, if g_j: $\mathbb{R}^m \to \mathbb{R}$ are continuously differentiable and $1, g_1, \dots, g_q$ are linearly independent on V, then the induced measure $P * g$ fulfills Cramér's condition. A uniform version can be proven similarly.

Assume that l admits partial derivatives l^i such that $\omega \to P_\omega(|l^i(\cdot,\omega)|)$ are locally bounded on Ω. If g fulfills Condition C then for every $\tau \in T$ there exists a neighborhood U of (δ, τ) such that

$$\limsup_{\|r\| \to \infty} \sup_{\omega, \omega' \in U} |P_\omega(\exp[i r' g(\cdot,\omega')])| < 1. \tag{5.1}$$

(For the proof of (5.1), use

$$p(\cdot,\omega) = p(\cdot,\omega') + (\omega - \omega')_i \int_0^1 p^i(\cdot, \omega' + t(\omega - \omega'))\, dt.)$$

Condition U_r. For every $\tau \in T$ there exists a neighborhood U of (δ, τ) such that

$$\lim_{a \to \infty} \sup_{\omega, \omega' \in U} P_\omega(\|g(\cdot,\omega')\|^r 1_{\{\|g(\cdot,\omega')\| > a\}}) = 0.$$

Condition B. A function $Q: \mathbb{R}^q \times \Omega \to \mathbb{R}$ fulfills Condition B if for every $\tau \in T$ there exist a neighborhood U of (δ, τ) and constants $a, b > 0$ such that

(a) for $r, r' \in \mathbb{R}^q$ with $\|r'\| \leqq \|r\|$

$|Q(r,\delta,\tau) - Q(r',\delta,\tau)| \leqq \|r - r'\| (a + \|r\|^b)$,

(b) for $r \in \mathbb{R}^q$ and $\omega, \omega' \in U$

$|Q(r,\omega) - Q(r,\omega')| \leqq \|\omega - \omega'\| (a + \|r\|^b)$.

Let $\mathscr{B}^q$ denote the Borel algebra of $\mathbb{R}^q$ and $\mathscr{C}$ the class of all convex sets in $\mathscr{B}^q$.

(5.2) *Definition.* A sequence $Q_\omega^{(n)} | \mathscr{B}^q$, $\omega \in \Omega$, $n \in \mathbb{N}$, of families of probability measures admits an *Edgeworth-expansion* $o(n^{-r/2})$ *with Lebesgue-density*

$$\varphi_\Sigma \left(1 + \sum_{j=1}^{r} n^{-j/2} G_j \right)$$

[*locally uniformly in* $\omega \in \Omega$] if uniformly for $C \in \mathscr{C}$ [and locally uniformly in $\omega \in \Omega$]

$$Q_\omega^{(n)}(C) = \int_C \varphi_{\Sigma(\omega)}(r) \left(1 + \sum_{j=1}^{r} n^{-j/2} G_j(r,\omega) \right) dr + o(n^{-r/2}).$$

6. Lemmas and Proofs

Remark 9. In order to explain why the remainder terms of the stochastic expansions derived in the proofs are of the form $n^{-s/2} l_n(r)$, it seems sufficient to treat a typical example, say $\tilde{f}(\cdot, \delta, \hat{\tau}^{(n)})$, where $\hat{\tau}^{(n)}$ is as. m.l. $o(n^{-1/2})$ for the nuisance parameter. By a Taylor expansion there exists $\bar{\tau}$ between $\hat{\tau}^{(n)}(\underline{x})$ and τ, depending on $n, \underline{x}, \tau$, such that

$$\tilde{f}(\underline{x}, \delta, \hat{\tau}^{(n)}(\underline{x})) = \tilde{f}(\underline{x}, \delta, \tau) + (\hat{\tau}_\alpha^{(n)}(\underline{x}) - \tau_\alpha) n^{-1/2} \sum_{\nu=1}^{n} f^\alpha(x_\nu, \delta, \bar{\tau}).$$

Write

$$n^{-1/2} \sum_{\nu=1}^{n} f^\alpha(x_\nu, \delta, \bar{\tau}) = n^{-1/2} \sum_{\nu=1}^{n} (f^\alpha(x_\nu, \delta, \bar{\tau}) - f^\alpha(x_\nu, \delta, \tau))$$
$$+ \tilde{f}^\alpha(\underline{x}, \delta, \tau) + n^{1/2} P_{\delta,\tau}(f^\alpha(\cdot, \delta, \tau)).$$

With Conditions L_2 and M_3 for f^α the first and second righthand terms are $l_n(1)$ by Pfanzagl (1973, p. 248, Lemma 4) and Nagaev (1965, p. 215, Corollary 2). By Pfanzagl (1973, p. 254, Lemma 9) the as. m.l. estimator admits a stochastic expansion

$$\hat{\tau}_\alpha^{(n)} = \tau_\alpha + n^{-1/2} \tilde{\lambda}_\alpha^*(\cdot, \delta, \tau) + n^{-1} l_n(1).$$

Hence

$$\tilde{f}(\cdot, \delta, \hat{\tau}^{(n)}) = \tilde{f}(\cdot, \delta, \tau) + P_{\delta,\tau}(f^\alpha(\cdot, \delta, \tau)) \tilde{\lambda}_\alpha^*(\cdot, \delta, \tau)$$
$$+ n^{-1/2} l_n(1).$$

Remark 10. "Good" test-sequences in $\mathscr{S}_r$ are obtained from appropriate statistics by desensitization and as. studentization $o(n^{-r/2})$ with an as. m.l. estimator-sequence $o(n^{-r/2})$. In order to ensure the existence of as. m.l. estimator-sequences we make use of an inductive *improvement procedure* which is defined as follows:

$$T_{(0)}^{(n)}(\cdot,\delta,\tau)=\tau,$$
$$T_{(k)}^{(n)}=T_{(k-1)}^{(n)}+n^{-1/2}\tilde{\lambda}^*(\cdot,\delta,T_{(k-1)}^{(n)}).$$

If $\tau^{(n)}$, $n\in\mathbb{N}$, is a $o(n^{-r/2})$-consistent estimator-sequence for the nuisance parameter then $T_{(r)}^{(n)}(\cdot,\delta,\tau^{(n)})$ is as. m.l. $o(n^{-r/2})$ and admits a stochastic expansion about τ up to $n^{-(r+1)/2}\,l_n(r)$. See Pfanzagl (1973, p. 249, Lemma 6, and p. 253, Lemma 8). For $r=0,1,2$, this holds true under the assumptions of the Proposition, Theorem 1, and Corollary 2, respectively.

Lemma 2. *Let the assumptions of the Proposition be fulfilled. Assume that* $\underline{\varphi}\in\mathscr{S}_0$. Then

$$P_{\delta,\tau}(f_0^\alpha(\cdot,\tau))=-P_{\delta,\tau}(l^\alpha(\cdot,\delta,\tau)f_0(\cdot,\tau))=0.$$

Moreover, if $\underline{\varphi}\in\mathscr{S}_0$ is u.m.p. in $\mathscr{T}_0$ at τ^* and has power $\alpha+o(n^0)$ at $(0,\tau^*)$ then there exists $d(\tau^*)$ such that

$$f_0(\cdot,\tau^*)=d(\tau^*)\,\lambda_0(\cdot,\delta,\tau^*)\qquad P_{\delta,\tau^*}-\text{a.e.}$$

This implies

$$c(\tau^*)=N d(\tau^*)\,\Lambda_{00}(\delta,\tau^*)^{1/2}.$$

Proof of the Proposition and of Lemma 2. In order to simplify our notations we omit the parameter (δ,τ^*) whenever convenient. Part (a) of the assertion of the Proposition is straightforward. For the proof of (b), observe first that there exists a test-sequence $\underline{\varphi}\in\mathscr{T}_0^*\cap\mathscr{S}_0$ with $\Pi_n(\varphi_n)=\mathrm{h}_0+o(n^0)$. Such a test-sequence is obtained by desensitization and as. studentization of the test-statistic $\tilde{\lambda}_0(\cdot,\delta,\tau)$ with a $o(n^0)$-consistent estimator-sequence for the nuisance parameter.

If $\underline{\varphi}\in\mathscr{S}_0$ we have $S_n=\tilde{f}_0+c+n^{-1/2}\,l_n(0)$. By a uniform version of the Central Limit Theorem (apply Corollary 18.3 in Bhattacharya and Rao (1976, p. 184) for $k_n=n$, $s=2$, $\varepsilon=n^{-1/4}$) we have uniformly for $0\leq t\leq\log n$

$$\text{(6.1)}\quad \Pi_n(\varphi_n)(t,\tau^*)=P_{(t,\tau^*)}^{(n)}\{\tilde{f}_0>-c\}+o(n^0)$$
$$=\Phi((c+tP(l^0 f_0))\,P(f_0^2)^{-1/2})+o(n^0).$$

If $\underline{\varphi}$ has power $\alpha+o(n^0)$ at $(0,\tau^*)$ we have $c(\tau^*)=NP_{\delta,\tau^*}(f_0(\cdot,\tau^*)^2)^{1/2}$. For every $s\in\mathbb{R}^p$,

$$S_n=\tilde{f}_0(\cdot,\tau+n^{-1/2}s)+c(\tau+n^{-1/2}s)+n^{-1/2}\,l_n(0)$$
$$=\tilde{f}_0+s_\alpha P(f_0^\alpha)+c+n^{-1/2}\,l_n(0).$$

Hence $P(l^\alpha f_0)=-P(f_0^\alpha)=0$ for $\alpha=1,\dots,p$, and therefore $P(l^0 f_0)=\Lambda_{00}^{-1}P(\lambda_0 f_0)$.

If $\underline{\varphi}$ is m.p. in $\mathscr{T}_0$ at (t^*,τ^*), i.e.

$$\Pi_n(\varphi_n)(t^*,\tau^*)=\Phi(N+t^*\Lambda_{00}(\delta,\tau^*)^{-1/2})+o(n^0),$$

we obtain by comparison with (6.1) for $t=t^*$

$$P(\lambda_0 f_0) P(f_0^2)^{-1/2} = \Lambda_{00}^{1/2} = P(\lambda_0^2)^{1/2}.$$

By Hölder's inequality, this implies

$$f_0(\cdot, \tau^*) = d(\tau^*)\, \lambda_0(\cdot, \tau^*) \qquad P_{\delta, \tau^*}-\text{a.e.}$$

Proof of Theorem 2. Fix $\tau^* \in V$ and let U denote a sufficiently small generic neighborhood of τ^*.

Since the test-statistic S_n does not depend on the nuisance parameter, its stochastic expansion is invariant under small changes of τ. We may therefore consider it as depending on t, τ through $\bar{\omega}_n(t,\tau) = (\delta, \tau + n^{-1/2} t \mu(\delta, \tau))$ with $\mu = (\mu_\alpha)_{\alpha=1,\ldots,p}$ and $\mu_\alpha = -\Lambda_{00}^{-1} \Lambda_{0\alpha}$. The technical convenience of this choice will become clear later.

We denote the alternative by $\omega_n(t,\tau) = (\delta + n^{-1/2} t, \tau)$. Observe that $\bar{\omega}_n(t,\tau)$ is an as. least favorable hypothesis for $\omega_n(t,\tau)$ in the sense that the power of a Neyman-Pearson test for (δ, τ') against $\omega_n(t,\tau)$ is minimal up to $o(n^{-1/2})$ for $\tau' = \tau + n^{-1/2} t \mu(\delta, \tau)$.

Whenever convenient, we suppress δ and the argument (t,τ) of $\bar{\omega}_n$ and ω_n.

If the test-sequence is m.p. in $\mathscr{T}_0$ at τ and has power $\alpha + o(n^0)$ at $(0,\tau)$ for $\tau \in V$ then $f_0(\cdot,\tau) = d(\tau)\, \lambda_0(\cdot,\tau)$ $P_{\delta,\tau}$-a.e. for $\tau \in V$ by Lemma 2. Assume first that $d \equiv 1$ on V. With the notations of Remark 2, let

$$g = (g_0, \ldots, g_{p+m})' = (\lambda_0, l^1, \ldots, l^p, f_1, \ldots, f_m)',$$
$$e_0 = (1, 0, \ldots, 0)', \qquad \text{the unit vector in } \mathbb{R}^{1+p+m}.$$

Since the vector g fulfills Conditions C and U_4, we obtain from (5.1) and Pfanzagl (1973, p. 242, Lemma 2) that the sequence of p-measures induced by $P^n_{\omega_n}$ and

$$\begin{aligned}\underline{x} \to\ & \tilde{g}(\underline{x}, \bar{\omega}_n) + N\Lambda_{00}(\bar{\omega}_n)^{1/2} e_0 \\ &= n^{-1/2} \sum_{\nu=1}^{n} (g(x_\nu, \bar{\omega}_n) - P_{\omega_n}(g(\cdot, \bar{\omega}_n))) \\ &\quad + n^{1/2} P_{\omega_n}(g(\cdot, \bar{\omega}_n)) + N\Lambda_{00}(\bar{\omega}_n)^{1/2} e_0\end{aligned}$$

admits uniformly for $|t| \leq \log n$ and $\tau \in U$ an Edgeworth-expansion $o(n^{-1})$ with Lebesgue density χ_n defined for $r \in \mathbb{R} \times \mathbb{R}^p \times \mathbb{R}^m$ by

$$\text{(6.2)} \quad \begin{aligned} \chi_n(r) = &\ \varphi_{\Sigma(\omega_n, \bar{\omega}_n)}(r - n^{1/2} P_{\omega_n}(g(\cdot, \bar{\omega}_n)) - N\Lambda_{00}(\bar{\omega}_n)^{1/2} e_0) \\ &\left(1 + \sum_{j=1}^{2} n^{-j/2} G_j(r - n^{1/2} P_{\omega_n}(g(\cdot, \bar{\omega}_n)) - N\Lambda_{00}(\bar{\omega}_n)^{1/2} e_0, \omega_n, \bar{\omega}_n)\right),\end{aligned}$$

where $\Sigma(\omega, \omega')$ denotes the (positive definite) covariance matrix of $g(\cdot, \omega')$ under P_ω. Define

$$A = (a_{ij})_{i,j=0,\ldots,p+m} = \Sigma^{-1}.$$

Let $\sigma_{ijk}(\omega,\omega')$ denote the third centered moments of $g(\cdot,\omega')$ under P_ω. Then we have

$$G_1(r,\cdot,\cdot)=\tfrac{1}{6}\sigma_{ijk}\,a_{in}r_n(a_{jp}r_p\,a_{kq}r_q-3a_{jk}).$$

By Pfanzagl (1973, p. 243, Corollary; notice that this corollary is easily extended to cover functions Q_j such that $Q_1, Q_1^{(0)}, Q_1^{(00)}, Q_2, Q_2^{(0)}$ fulfill Condition $B(a)$) the sequence of measures

$$P^n_{\omega_n}*\left(\tilde g+N\Lambda_{00}^{1/2}e_0+\sum_{j=1}^{2}n^{-j/2}Q_j(\tilde g+N\Lambda_{00}^{1/2}e_0,\cdot)e_0\right)\Bigg|_{\bar\omega_n}$$

admits uniformly for $|t|\leqq\log n$ and $\tau\in U$ an Edgeworth-expansion $o(n^{-1})$ with Lebesgue-density

$$\text{(6.3)}\quad \chi_n-n^{-1/2}(Q_1(\cdot,\bar\omega_n)\chi_n)^{(0)} +n^{-1}[\tfrac{1}{2}(Q_1(\cdot,\bar\omega_n)^2\chi_n)^{(00)}-(Q_2(\cdot,\bar\omega_n)\chi_n)^{(0)}].$$

Since $Q_j(u,v,w,\bar\omega_n)^k\chi_n(u,v,w)\to 0$ for $u\to\infty$ and $j,k=1,2$ we obtain from (6.3) uniformly for $|t|\leqq\log n$ and $\tau\in U$

$$\begin{aligned}\text{(6.4)}\quad P^n_{\omega_n}\{S_n>0\}=&\int_{u>0}du\,dv\,dw\,\chi_n(u,v,w)\\ &+n^{-1/2}\int dv\,dw\,Q_1(0,v,w,\bar\omega_n)\chi_n(0,v,w)\\ &+n^{-1}\int dv\,dw[-Q_1(0,v,w,\bar\omega_n)Q_1^{(0)}(0,v,w,\bar\omega_n)\chi_n(0,v,w)\\ &-\tfrac{1}{2}Q_1(0,v,w,\bar\omega_n)^2\chi_n^{(0)}(0,v,w)+Q_2(0,v,w,\bar\omega_n)\chi_n(0,v,w)]\\ &+o(n^{-1}).\end{aligned}$$

The Lebesgue-density χ_n defined in (6.2) depends on t,τ through $\bar\omega_n$ and ω_n. We consider $A(\omega_n,\bar\omega_n)=\Sigma^{-1}(\omega_n,\bar\omega_n)$ and $P_{\omega_n}(g(\cdot,\bar\omega_n))$ as functions of ω_n, expand them about $\bar\omega_n(t,\tau)$ and replace ω_n by $\omega_n(t,\tau)$. Then we obtain uniformly for $|t|\leqq\log n$ and $\tau\in U$

$$\begin{aligned}\text{(6.5)}\quad P_{\omega_n}(g(\cdot,\bar\omega_n))=&(P(g)+n^{-1/2}t\Lambda_{00}^{-1}P(\lambda_0 g)\\ &+n^{-1}t^2[\tfrac{1}{2}\Lambda_{00}^{-2}\Lambda_{0i}\Lambda_{0j}P((l^{ij}+l^i l^j)g)\\ &+\Lambda_{00}^{-1}\Lambda_{0\alpha}(\Lambda_{00}^{-1}\Lambda_{0i})^\alpha P(l^i g)])|_{\bar\omega_n}\\ &+o(n^{-1}),\end{aligned}$$

$$\begin{aligned}\text{(6.6)}\quad a_{ij}(\omega_n,\bar\omega_n)=&(a_{ij}(\cdot,\cdot)-n^{-1/2}t\Lambda_{00}^{-1}P(\lambda_0 g_p g_q)a_{ip}(\cdot,\cdot)a_{jq}(\cdot,\cdot))|_{\bar\omega_n}\\ &+o(n^{-1/2}).\end{aligned}$$

Since $g_1(\cdot,\tau),\ldots,g_{p+m}(\cdot,\tau)$ are $P_{\delta,\tau}$-uncorrelated to $g_0(\cdot,\tau)=\lambda_0(\cdot,\delta,\tau)$, the expansion of $P_{\omega_n}(g(\cdot,\bar\omega_n))$ starts with $n^{-1/2}te_0$. This is the motive for expanding about $\bar\omega_n$ and not about (δ,τ).

We may write $\Sigma(\bar\omega_n,\bar\omega_n)$ as a diagonal block matrix with $\Lambda_{00}(\bar\omega_n)$, $L^*(\bar\omega_n)$ and $\Gamma(\bar\omega_n)$ (see Remark 2) in the diagonal. For pairs $(\bar\omega_n,\bar\omega_n)$ as arguments we have

$$(6.7)\quad G_1(r-te_0,\cdot,\cdot)=G_1(r,\cdot,\cdot)
-\tfrac{1}{2}t\Lambda_{00}^{-1}\sigma_{ij0}a_{ip}(r_p-t\delta_{0p})a_{jq}(r_q-t\delta_{0q})
-\tfrac{1}{2}t^2\Lambda_{00}^{-2}\sigma_{i00}a_{ip}(r_p-t\delta_{0p})
+\tfrac{1}{2}t\Lambda_{00}^{-1}\sigma_{ij0}a_{ij}-\tfrac{1}{6}t^3\Lambda_{00}^{-3}\sigma_{000},$$

where δ_{ij} is Kronecker's symbol.

Applying (6.5), (6.6), (6.7) to (6.2) and inserting the result into (6.4), we obtain uniformly for $|t|\leqq \log n$ and $\tau\in U$,

$$(6.8)\quad P^n_{\omega_n}\{S_n>0\}=P^n_{\omega_n}\{\tilde\lambda_0(\cdot,\bar\omega_n)+N\Lambda_{00}(\bar\omega_n)^{1/2}>0\}
+\Lambda_{00}^{-1/2}\varphi(N+t\Lambda_{00}^{-1/2})(n^{-1/2}\int dv\,dw\,\varphi_{L^*}(v)\varphi_\Gamma(w)Q_1(0,v,w,\cdot)
+n^{-1}\int dv\,dw\,\varphi_{L^*}(v)\varphi_\Gamma(w)[(G_1(-N\Lambda_{00}^{1/2},v,w,\cdot,\cdot)
+t^2b'v+\tfrac{1}{2}t^2\Lambda_{00}^{-2}a'w+tk(\cdot,t))Q_1(0,v,w,\cdot)
-\tfrac{1}{2}(N\Lambda_{00}^{-1/2}+t\Lambda_{00}^{-1})Q_1(0,v,w,\cdot)^2
-Q_1(0,v,w,\cdot)Q_1^{(0)}(0,v,w,\cdot)
+Q_2(0,v,w,\cdot)])|_{\bar\omega_n}+o(n^{-1}).$$

Let $d=\Lambda_{0i}\Lambda_{0j}(l^{ij}-L_{ij})$. By (2.1) we have

$$h=a'f=d|(l^0,\dots,l^p)=d-\Lambda_{0i}\Lambda_{0j}L_{ij,k}\lambda_k.$$

Hence

$$P(dg_i)a_{ij}=P(d\lambda_0)\Lambda_{00}^{-1}\delta_{0j}+P(dl^\alpha)\Lambda^*_{\alpha\beta}\delta_{\beta j}+a_i\delta_{i,\,j-p-1}.$$

Furthermore,

$$(\Lambda_{00}^{-1}\Lambda_{0i})^\alpha P(l^i g_j)a_{jk}=(\Lambda_{00}^{-1}\Lambda_{0\beta})^\alpha\delta_{\beta k}.$$

Hence the functions $b\colon T\to\mathbb{R}^p$ and $k\colon T\times\mathbb{R}\to\mathbb{R}$ do not depend on the test-sequence. b is locally bounded on T, and k is a polynomial in t, the coefficients of which are locally bounded functions of τ.

We shall now make use of the assumption that for $\tau\in V$

$$(6.9)\quad P^{(n)}_{(0,\tau)}\{S_n>0\}=\alpha+o(n^{-1}).$$

By Pfanzagl (1973, p. 242, Lemma 2) there exist $C_1(\tau)$ and $C_2(\tau)$ (not depending on the test-sequence) such that

$$(6.10)\quad P^{(n)}_{(0,\tau)}\{\tilde\lambda_0(\cdot,\delta,\tau)+N\Lambda_{00}(\delta,\tau)^{1/2}>0\}
=\Phi(N)+\varphi(N)[n^{-1/2}C_1(\tau)+n^{-1}C_2(\tau)]+o(n^{-1}).$$

Inserting (6.9) and (6.10) into (6.8) for $t=0$ we obtain the following relations for $\tau\in V$:

$$(6.11)\quad C_1+\Lambda_{00}^{-1/2}\int dv\,dw\,\varphi_{L^*}(v)\varphi_\Gamma(w)Q_1(0,v,w,\cdot)=0,$$

$$(6.12)\quad C_2+\Lambda_{00}^{-1/2}\int dv\,dw\,\varphi_{L^*}(v)\,\varphi_\Gamma(w)\cdot[G_1(-N\Lambda_{00}^{1/2},v,w,\cdot,\cdot)\,Q_1(0,v,w,\cdot)-\tfrac{1}{2}N\Lambda_{00}^{-1/2}\,Q_1(0,v,w,\cdot)^2-Q_1(0,v,w,\cdot)\,Q_1^{(0)}(0,v,w,\cdot)+Q_2(0,v,w,\cdot)]=0.$$

Aplying these relations to (6.8) we obtain uniformly for $|t|\leqq\log n$ and $\tau\in U$

$$(6.13)\quad P_{\omega_n}^n\{S_n>0\}=P_{\omega_n}^n\{\tilde\lambda_0(\cdot,\bar\omega_n)+N\Lambda_{00}(\bar\omega_n)^{1/2}>0\}+\varphi(N+t\Lambda_{00}^{-1/2})(-n^{-1/2}\,C_1+n^{-1}[-C_2-tk(\cdot,t)\,C_1+\int dv\,dw\,\varphi_{L^*}(v)\,\varphi_\Gamma(w)(t^2\Lambda_{00}^{-1/2}b'v\,Q_1(0,v,w,\cdot)+\tfrac{1}{2}t^2\Lambda_{00}^{-5/2}a'w\,Q_1(0,v,w,\cdot)-\tfrac{1}{2}t\Lambda_{00}^{-3/2}Q_1(0,v,w,\cdot)^2)])|_{\bar\omega_n}+o(n^{-1}).$$

By Lemma 2 (see also Remark 2) the function Q_1 may be decomposed into two summands

$$(6.14)\quad Q_1(u,v,w,\cdot)=Q_1(u,0,w,\cdot)+R_1(u-N\Lambda_{00}^{1/2},v,w,\cdot),$$

where $R_1(\tilde\lambda_0,\tilde l^1,\dots,\tilde l^p,\tilde f_1,\dots,\tilde f_m,\cdot)$ is defined in (3.10) and depends on the family of p-measures only and not on the test-sequence. Hence

$$\int dv\,dw\,\varphi_{L^*}(v)\,\varphi_\Gamma(w)\,v\,Q_1(0,v,w,\cdot)=\int dv\,dw\,\varphi_{L^*}(v)\,\varphi_\Gamma(w)\,v\,R_1(-N\Lambda_{00}^{1/2},v,w,\cdot)$$

and does, therefore, not depend on Q_1. The two remaining terms in (6.13) which depend on Q_1 may be rewritten, omitting the factor $-\frac{1}{2}t\Lambda_{00}^{-3/2}$, as

$$(6.15)\quad \int dv\,dw\,\varphi_{L^*}(v)\,\varphi_\Gamma(w)[Q_1(0,v,w,\cdot)^2-t\Lambda_{00}^{-1}a'w\,Q_1(0,v,w,\cdot)]=\int dv\,dw\,\varphi_{L^*}(v)\,\varphi_\Gamma(w)\,[Q_1(0,0,w,\cdot)-C-\tfrac{1}{2}t\Lambda_{00}^{-1}a'w]^2+\int dv\,dw\,\varphi_{L^*}(v)\,\varphi_\Gamma(w)[R_1(-N\Lambda_{00}^{1/2},v,w,\cdot)-2C-t\Lambda_{00}^{-1}a'w+2Q_1(0,0,w,\cdot)]\,R_1(-N\Lambda_{00}^{1/2},v,w,\cdot)-2\Lambda_{00}^{1/2}\,CC_1-t^2D-C^2.$$

It is convenient to introduce

$$C=\int dw\,\varphi_\Gamma(w)\,Q_1(0,0,w,\cdot).$$

Using the canonical representation (6.14) and relation (6.11) we obtain

$$C=-\Lambda_{00}^{1/2}\,C_1-\int dv\,dw\,\varphi_{L^*}(v)\,\varphi_\Gamma(w)\,R_1(-N\Lambda_{00}^{1/2},v,w,\cdot).$$

According to (3.8) and (3.10), the function $R_1(\tilde\lambda_0,\tilde l^1,\dots,\tilde l^p,\tilde f_1,\dots,\tilde f_m,\cdot)$ consists of summands which either do not depend on $\tilde f_1,\dots,\tilde f_m$ or contain one of the functions $\tilde{l}^1, \ldots, \tilde{l}^p$ as a linear factor. Together with relation (6.11) this implies that the second integral on the right side of (6.15) does not depend on Q_1. Straightforward, if tedious, computations show that the terms in (6.13) which do not depend on Q_1 add up to H_{2n}.

The general case $f_0 = d\lambda_0$ reduces to the case $d \equiv 1$ as follows. Let $\tau^{(n)}$, $n \in \mathbb{N}$, be a $o(n^{-1})$-consistent estimator-sequence for the nuisance parameter. By Remark 10, the improved estimator-sequence $T^{(n)}_{(2)}(\cdot, \delta, \tau^{(n)})$, $n \in \mathbb{N}$, is as. m.l. $o(n^{-1})$ and admits a stochastic expansion about τ up to $n^{-3/2}\, l_n(2)$ which depends only on l^i and $l^{ij} - L_{ij}$, $i,j = 0, \ldots, p$, and hence on g. Hence we may apply the above proof to the modified sequence of test-statistics $d(T^{(n)}_{(2)}(\cdot, \delta, \tau^{(n)}))^{-1} S_n$, $n \in \mathbb{N}$, the stochastic expansion of which starts with

$$\tilde{\lambda}_0 + N\Lambda_{00}^{1/2} + n^{-1/2}[d^{-1} d^{\alpha} \tilde{\lambda}^*_{\alpha}(\tilde{\lambda}_0 + N\Lambda_{00}^{1/2}) + Q_1(\tilde{\lambda}_0 + N\Lambda_{00}^{1/2}, \tilde{l}^1, \ldots, \tilde{l}^p, \tilde{f}_1, \ldots, \tilde{f}_m, \cdot)].$$

The term $d^{-1} d^{\alpha} \tilde{\lambda}^*_{\alpha}(\tilde{\lambda}_0 + N\Lambda_{00}^{1/2})$ cancels out (see (6.15)).

Proof of Theorem 1. (a) See Pfanzagl (1973, p. 260, Proposition).

(b) The proof is contained in the proof of Theorem 2. All expansions should be derived for τ^* only instead of $\tau \in U$. Observe that relation (6.11) is needed only for $\tau = \tau^*$, since in the $n^{-1/2}$-term of (6.8) the parameter $\bar{\omega}_n$ may be replaced by (δ, τ^*) without changing the power by more than $o(n^{-1/2})$.

Proof of Corollary 1. The corollary is an immediate consequence of Theorem 1 except for the assertion that the expansion of the power of test-sequences in $\mathscr{T}_1^* \cap \bar{\mathscr{T}}_0 \cap \mathscr{S}_1$ is locally uniform on T. This follows from the proof of Theorem 2.

Proof of Corollary 2. (a) By assumption, there exists a $o(n^{-1})$-consistent estimator-sequence $\tau^{(n)}$, $n \in \mathbb{N}$, for the nuisance parameter. Hence by Remark 10 there exists an as. m.l. estimator-sequence $o(n^{-1})$ for the nuisance parameter, namely $T^{(n)}_{(2)}(\cdot, \delta, \tau^{(n)})$, $n \in \mathbb{N}$. The explicit form of its stochastic expansion is given in Pfanzagl (1973, p. 253, Lemma 8). The computation of the stochastic expansion of the test-statistic pertinent to $\varphi(\cdot, q)$ is straightforward.

(b) Let $\varphi \in \mathscr{T}_2^* \cap \bar{\mathscr{T}}_0 \cap \mathscr{S}_2$, and let Q_1 denote the $n^{-1/2}$-term of the stochastic expansion of the test-statistic. We have to find $q \in \mathfrak{Q}$ such that $\varphi \lesssim_2 \varphi(\cdot, q)$. By (a) and Theorem 2, this is equivalent to finding $q \in \mathfrak{Q}$ such that for all $t \geq 0$

$$(6.16) \quad \int dw\, \varphi_r(w)[q a' w - \tfrac{1}{2} t \Lambda_{00}^{-1} a' w]^2 \leq \int dw\, \varphi_r(w)[Q_1(0,0,w,\cdot) - C - \tfrac{1}{2} t \Lambda_{00}^{-1} a' w]^2.$$

It follows easily from Hölder's inequality that this holds true for

$$(6.17) \quad q := |\int dw\, \varphi_r(w)\, a' w (Q_1(0,0,w,\cdot) - C)| (\int dw\, \varphi_r(w)(a' w)^2)^{-1}.$$

Since Q_1 fulfills Condition B, the function defined by (6.17) fulfills a local Lipschitz condition on T, so that $q \in \mathfrak{Q}$. Hence the class $\varphi(\cdot, q)$, $q \in \mathfrak{Q}$, is essentially complete in $\mathscr{T}_2^* \cap \bar{\mathscr{T}}_0 \cap \mathscr{S}_2$.

If $q_1 \neq q_2$, say $q_1(\tau) < q_2(\tau)$, then by (6.17) we have $\underline{\varphi}(\cdot, q_1) \succ_2 \underline{\varphi}(\cdot, q_2)$ at (t, τ) with $t < \Lambda_{00}(\delta, \tau)(q_1(\tau) + q_2(\tau))$ and $\underline{\varphi}(\cdot, q_1) \prec_2 \underline{\varphi}(\cdot, q_2)$ at (t, τ) with $t > \Lambda_{00}(\delta, \tau)(q_1(\tau) + q_2(\tau))$. Hence the class $\underline{\varphi}(\cdot, q)$, $q \in \mathfrak{Q}$, is minimal.

(c) The admissibility criterion follows immediately from the fact that the inequality in (6.16) with the q defined in (6.17) is strict unless $Q_1(0,0,w,\cdot) - C = q a' w$.

7. Appendix

This section contains the explicit form of

$$H_{2n} = h_0 + n^{-1/2} h_1 + n^{-1} h_2.$$

We have

$$\begin{aligned} H_{2n}(t,\tau) = {} & \Phi(N + \Lambda_{00}^{-1/2} t) \\ & + \varphi(N + \Lambda_{00}^{-1/2} t) \Lambda_{00}^{-1/2} t [n^{-1/2} A(N, \Lambda_{00}^{-1/2} t, \cdot) \\ & + n^{-1} B(N, \Lambda_{00}^{-1/2} t, \cdot)]|_{(\delta,\tau)}, \end{aligned}$$

where

$$A(N, s, \cdot) = a_{10} N + a_{01} s,$$
$$B(N, s, \cdot) = \sum_{i+j=0,\,2,\,4} b_{ij} N^i s^j.$$

The coefficients a_{ij} and b_{ij} are presented in an economic way with the following notations. We standardize expressions $L_i, L_{i,j}, L_{ij,k}$ etc. by multiplying them with a power of $\Lambda_{00}^{1/2}$ according to the number of indices. Examples:

$$(i) = \Lambda_{00}^{1/2} L_i, \quad (i,j) = \Lambda_{00} L_{i,j}, \quad (ij,k) = \Lambda_{00}^{3/2} L_{ij,k}.$$

If in a bracket an index, say i, is replaced by a dot this means multiplication by $\Lambda_{00}^{-1} \Lambda_{0i}$ and summation over $i = 0, \ldots, p$. If a pair of indices i,j is replaced by a pair of asterisks or plus signs, this means multiplication by $\Lambda_{00}^{-1} \Lambda_{ij}$ and summation over $i,j = 0, \ldots, p$. Examples:

$$(0\cdot,\cdot) = \Lambda_{00}^{-2} \Lambda_{0i} \Lambda_{0j} (0i,j) = \Lambda_{00}^{-1/2} \Lambda_{0i} \Lambda_{0j} L_{0i,j}$$
$$\begin{aligned} (*\cdot,\cdot)(*,0,\cdot) &= \Lambda_{00}^{-1} \Lambda_{ij} (i\cdot,\cdot)(j,0,\cdot) \\ &= \Lambda_{00}^{-1} \Lambda_{ij} \Lambda_{0k} \Lambda_{0m} \Lambda_{0q} L_{ik,m} L_{j,0,q}, \end{aligned}$$
$$\begin{aligned} (*,+,\cdot)(*+,\cdot) &= \Lambda_{00}^{-1} \Lambda_{ij} \Lambda_{00}^{-1} \Lambda_{km} (i,k,\cdot)(jm,\cdot) \\ &= \Lambda_{00}^{-1} \Lambda_{ij} \Lambda_{km} \Lambda_{0q} \Lambda_{0r} L_{i,k,q} L_{jm,r}. \end{aligned}$$

With these notations,

$$a_{10} = -\tfrac{1}{6}(\cdot,\cdot,\cdot)$$
$$a_{01} = -\tfrac{1}{3}(\cdot,\cdot,\cdot) - \tfrac{1}{2}(\cdot\cdot,\cdot) + \tfrac{1}{2}(0,\cdot,\cdot) + (0\cdot,\cdot)$$

$$
\begin{aligned}
b_{00}=&\tfrac{1}{8}-\tfrac{2}{9}(\cdot,\cdot,\cdot)^2-\tfrac{1}{2}(\cdot,\cdot,\cdot)(\cdot\cdot,\cdot)-\tfrac{1}{4}(\cdot\cdot,\cdot)^2-\tfrac{1}{24}(\cdot,\cdot,\cdot,\cdot)\\
&+\tfrac{1}{2}(\cdot\cdot,\cdot\cdot)-\tfrac{1}{2}(*\cdot,*\cdot)-\tfrac{1}{2}(*,\cdot\cdot)(*,\cdot\cdot)+\tfrac{1}{2}(*,+\cdot)(*,+\cdot)\\
&+\tfrac{1}{2}(*,\cdot,\cdot)(*,\cdot,\cdot)+(*,\cdot,\cdot)(*\cdot,\cdot)+\tfrac{1}{2}(*\cdot,\cdot)(*\cdot,\cdot)\\
&-\tfrac{1}{4}(*,+,\cdot)(*,+,\cdot)-\tfrac{1}{2}(*,+,\cdot)(*+,\cdot)-\tfrac{1}{4}(*+,\cdot)(*+,\cdot)
\end{aligned}
$$

$$
b_{20}=-\tfrac{1}{8}+\tfrac{5}{72}(\cdot,\cdot,\cdot)^2+\tfrac{1}{24}(\cdot,\cdot,\cdot,\cdot)-\tfrac{1}{8}(*,\cdot,\cdot)(*,\cdot,\cdot)
$$

$$
\begin{aligned}
b_{11}=&-\tfrac{1}{8}+\tfrac{3}{8}(\cdot,\cdot,\cdot)^2+\tfrac{7}{12}(\cdot,\cdot,\cdot)(\cdot\cdot,\cdot)-\tfrac{1}{3}(\cdot,\cdot,\cdot)(0,\cdot,\cdot)\\
&-\tfrac{2}{3}(\cdot,\cdot,\cdot)(0\cdot,\cdot)+\tfrac{1}{8}(\cdot,\cdot,\cdot,\cdot)+\tfrac{1}{4}(\cdot\cdot,\cdot,\cdot)-\tfrac{1}{6}(0,\cdot,\cdot,\cdot)\\
&-\tfrac{1}{2}(0\cdot,\cdot,\cdot)-\tfrac{1}{2}(*,\cdot,\cdot)(*,\cdot,\cdot)-\tfrac{1}{4}(*,\cdot,\cdot)(*,\cdot\cdot)\\
&-\tfrac{1}{2}(*,\cdot,\cdot)(*\cdot,\cdot)+\tfrac{1}{2}(*,\cdot,\cdot)(*,0,\cdot)+\tfrac{1}{2}(*,\cdot,\cdot)(*,0\cdot)\\
&+\tfrac{1}{2}(*,\cdot,\cdot)(*0,\cdot)
\end{aligned}
$$

$$
\begin{aligned}
b_{02}=&\tfrac{4}{9}(\cdot,\cdot,\cdot)^2+\tfrac{4}{3}(\cdot,\cdot,\cdot)(\cdot\cdot,\cdot)+(\cdot\cdot,\cdot)^2-\tfrac{5}{6}(\cdot,\cdot,\cdot)(0,\cdot,\cdot)\\
&-\tfrac{5}{4}(\cdot,\cdot\cdot)(0,\cdot,\cdot)-\tfrac{5}{3}(\cdot,\cdot,\cdot)(0\cdot,\cdot)-\tfrac{5}{2}(\cdot,\cdot\cdot)(0\cdot,\cdot)\\
&+\tfrac{3}{8}(0,\cdot,\cdot)^2+\tfrac{3}{2}(0,\cdot,\cdot)(0\cdot,\cdot)+\tfrac{3}{2}(0\cdot,\cdot)^2\\
&+\tfrac{1}{8}(\cdot,\cdot,\cdot,\cdot)+\tfrac{1}{2}(\cdot\cdot,\cdot,\cdot)+\tfrac{1}{6}(\cdot\cdot\cdot,\cdot)+\tfrac{1}{8}(\cdot\cdot,\cdot\cdot)\\
&-\tfrac{1}{3}(0,\cdot,\cdot,\cdot)-\tfrac{1}{2}(0,\cdot\cdot,\cdot)-(0\cdot,\cdot,\cdot)-\tfrac{1}{2}(0\cdot\cdot,\cdot)\\
&-\tfrac{1}{2}(0\cdot,\cdot\cdot)+\tfrac{1}{4}(0,0,\cdot,\cdot)+\tfrac{1}{4}(00,\cdot,\cdot)+\tfrac{1}{2}(00\cdot,\cdot)\\
&+(0,0\cdot,\cdot)+\tfrac{1}{2}(0\cdot,0\cdot)-\tfrac{1}{2}(*,\cdot,\cdot)(*,\cdot,\cdot)-\tfrac{1}{2}(*,\cdot,\cdot)(*,\cdot\cdot)\\
&-(*,\cdot,\cdot)(*\cdot,\cdot)-\tfrac{1}{8}(*,\cdot\cdot)(*,\cdot\cdot)-\tfrac{1}{2}(*,\cdot\cdot)(*\cdot,\cdot)\\
&-\tfrac{1}{2}(*\cdot,\cdot)(*\cdot,\cdot)+(*,\cdot,\cdot)(*,0,\cdot)+\tfrac{1}{2}(*,\cdot\cdot)(*,0,\cdot)\\
&+(*\cdot,\cdot)(*,0,\cdot)+(*,\cdot,\cdot)(*,0\cdot)+\tfrac{1}{2}(*,\cdot\cdot)(*,0\cdot)\\
&+(*\cdot,\cdot)(*,0\cdot)+(*,\cdot,\cdot)(*0,\cdot)+\tfrac{1}{2}(*,\cdot\cdot)(*0,\cdot)\\
&+(*\cdot,\cdot)(*0,\cdot)-\tfrac{1}{2}(*,0,\cdot)(*,0,\cdot)-(*,0,\cdot)(*,0\cdot)\\
&-(*,0,\cdot)(*0,\cdot)-\tfrac{1}{2}(*,0\cdot)(*,0\cdot)-(*,0\cdot)(*0,\cdot)\\
&-\tfrac{1}{2}(*0,\cdot)(*0,\cdot)
\end{aligned}
$$

$$
b_{40}=0
$$

$$
b_{31}=-\tfrac{1}{72}(\cdot,\cdot,\cdot)^2
$$

$$
\begin{aligned}
b_{22}=&-\tfrac{5}{72}(\cdot,\cdot,\cdot)^2-\tfrac{1}{12}(\cdot,\cdot,\cdot)(\cdot\cdot,\cdot)+\tfrac{1}{12}(\cdot,\cdot,\cdot)(0,\cdot,\cdot)\\
&+\tfrac{1}{6}(\cdot,\cdot,\cdot)(0\cdot,\cdot)
\end{aligned}
$$

$$
\begin{aligned}
b_{13}=&-\tfrac{1}{9}(\cdot,\cdot,\cdot)^2-\tfrac{1}{4}(\cdot,\cdot,\cdot)(\cdot\cdot,\cdot)-\tfrac{1}{8}(\cdot\cdot,\cdot)^2\\
&+\tfrac{1}{4}(\cdot,\cdot,\cdot)(0,\cdot,\cdot)+\tfrac{1}{4}(\cdot\cdot,\cdot)(0,\cdot,\cdot)+\tfrac{1}{2}(\cdot,\cdot,\cdot)(0\cdot,\cdot)\\
&+\tfrac{1}{2}(\cdot\cdot,\cdot)(0\cdot,\cdot)-\tfrac{1}{8}(0,\cdot,\cdot)^2-\tfrac{1}{2}(0,\cdot,\cdot)(0\cdot,\cdot)\\
&-\tfrac{1}{2}(0\cdot,\cdot)^2
\end{aligned}
$$

$$
b_{04}=-\tfrac{1}{2}(\tfrac{1}{3}(\cdot,\cdot,\cdot)+\tfrac{1}{2}(\cdot\cdot,\cdot)-\tfrac{1}{2}(0,\cdot,\cdot)-(0\cdot,\cdot))^2.
$$

For the case without nuisance parameter, H_{2n} reduces to the "envelope power function" obtained in Pfanzagl (1974, p. 18). See also Pfanzagl (1975, p. 13).

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