Z. Wahrscheinlichkeitstheorie verw. Gebiete
45, 35–48 (1978)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

Global Asymptotic Properties of Risk Functions in Estimation

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1. Introduction

Since LeCam in a fundamental paper has established both global as well as local asymptotic lower bounds for risk functions of estimates (LeCam, 1953, Theorems 10 and 14), the usual regularity conditions of estimation theory have been weakened considerably. The local result of LeCam has been proved by Hajek (1972, Theorem 4.1) under weak regularity conditions. The present paper tries to do the same with LeCam's global result.

There are several previous attempts in order to establish a result similar to LeCam's global theorem. In the present context the most important one is an immediate consequence of Hajek's convolution theorem (1970, Theorem). (There is another approach due to Bahadur, 1964, which has been improved by Pfanzagl, 1970.) The main difference (apart from regularity conditions) between LeCam's result and Hajek's is that Hajek considers only such sequences of estimates whose distributions are asymptotically invariant in some sense, whereas LeCam imposes no such restrictions at all.

One possibility of generalizing LeCam's theorem could be giving weak conditions such that a Bernstein-v. Mises result holds. This was the original idea of the author. However, as it was pointed out to the author by Prof. L. LeCam, a suitable combination of Hajek's convolution theorem and of a general result of LeCam on the invariance of limit distributions (LeCam, 1973a, Theorem 1) leads to better results.

This method is used in the present paper. A direct application of Hajek's result is not possible since, roughly spoken, the invariance theorem of LeCam deals with weak convergence in L_∞ where Hajek's result needs convergence a.e. We provide a suitable (and in fact straightforward) generalization of Hajek's result in Corollary 1. It is based on a "strong" convolution theorem which does not rely on invariance properties at all (Theorem 1). So we obtain global asymptotic lower bounds for risk functions of estimates (Proposition 1).

A second application of the invariance theorem under certain conditions yields a more handy version of the convolution theorem (Corollary 2). This

0044-3719/78/0045/0035/$02.80

version implies a characterization of those sequences of estimates whose risk functions attain the lower bound (Proposition 2). This gives also a characterization of asymptotic Bayes estimates.

We assume that the LAN-condition is satisfied in τ-measure where τ is a p-measure on the parameter space. If τ is concentrated at a single point then our condition reduces to the classical LAN-condition which was introduced by LeCam, 1960, and is the central assumption of Hajek's papers, 1970 and 1972.

2. Notations and Results

Let $(\Omega_n, \mathscr{A}_n)$, $n\in\mathbb{N}$, be a sequence of measurable spaces and let $\Theta\subseteq\mathbb{R}^k$ be open. Assume that $\{P_{\vartheta,n}|\mathscr{A}_n: \vartheta\in\Theta\}$, $n\in\mathbb{N}$, is a sequence of families of p-measures (probability measures) such that $\vartheta\mapsto P_{\vartheta,n}(A)$, $A\in\mathscr{A}_n$, $n\in\mathbb{N}$, are Borel measurable.

We use the following notations. If $P|\mathscr{A}$, $Q|\mathscr{A}$ are p-measures on a measurable space $(\Omega, \mathscr{A})$ then dP/dQ denotes the RN-derivative of the Q-continuous part of P with respect to Q. If $X\colon \Omega\to\mathbb{R}^k$ is a measurable function then $\mathscr{L}(X|P)$ denotes the image of P induced by X on $\mathscr{B}^k\cdot\mathscr{C}_{00}(\mathbb{R}^k)$ denotes the space of continuous functions with compact support in $\mathbb{R}^k$ and $\mathscr{C}_0$ denotes the uniform hull of $\mathscr{C}_{00}\cdot\lambda^k|\mathscr{B}^k$ denotes the Lebesgue measure, $\varepsilon_t|\mathscr{B}^k$ the point measure at $t\in\mathbb{R}^k$, and $\nu_{a,A}|\mathscr{B}^k$ the normal distribution with mean $a\in\mathbb{R}^k$ and covariance matrix A. $\mathscr{M}_1(\mathscr{A})$ denotes the family of substochastic measures on $\mathscr{A}$.

The following definition states the central assumption of the present paper. If $\tau|\mathscr{B}^k$ is concentrated at a single point then it reduces to the classical LAN-condition. The classical LAN-condition is defined e.g. in Hajek, 1972.

Definition 1. Let $\tau|\mathscr{B}^k$ be a p-measure satisfying $\tau(\Theta)=1$. The sequence $\{P_{\vartheta,n}\colon \vartheta\in\Theta\}$, $n\in\mathbb{N}$, satisfies LAN in τ-measure if there are

(1) a sequence of positive numbers δ_n, $n\in\mathbb{N}$, decreasing to zero,

(2) a measurable function Γ mapping Θ to the set of symmetric $k\times k$-matrices satisfying $\tau\{\Gamma(\vartheta)$ is positive definite$\}=1$,

(3) measurable functions $\Delta_n\colon \Theta\times\Omega_n\to\mathbb{R}^k$, $n\in\mathbb{N}$, satisfying

$$\lim_{n\in\mathbb{N}}\left|\int f(\cdot,\vartheta)\,d\mathscr{L}(\Delta_n(\vartheta)|P_{\vartheta,n})-\int f(\cdot,\vartheta)\,d\nu_{0,\Gamma(\vartheta)}\right|=0$$

in τ-measure for all measurable functions $f\colon \mathbb{R}^k\times\Theta\to\mathbb{R}$, such that $f(\cdot,\vartheta)\in\mathscr{C}_{00}(\mathbb{R}^k)$, $\vartheta\in\Theta$,

(4) measurable functions $Z_n\colon \Theta\times\mathbb{R}^k\times\Omega_n\to\mathbb{R}$, $n\in\mathbb{N}$, satisfying $\lim_{n\in\mathbb{N}} Z_n(\cdot,t)=0$ in $\int P_{\vartheta,n}\tau(d\vartheta)$-measure for every $t\in\mathbb{R}^k$, such that

$$\frac{dP_{\vartheta+\delta_n t,n}}{dP_{\vartheta,n}}=\exp\left(t'\Delta_n(\vartheta)-\tfrac{1}{2}t'\Gamma(\vartheta)\,t+Z_n(\vartheta,t)\right)$$

for $\vartheta\in\Theta$, $t\in\mathbb{R}^k$, $n\in\mathbb{N}$.

Sufficient conditions for LAN in λ^k-measure are given in LeCam, 1970. In those cases $\delta_n=n^{-1/2}$, $n\in\mathbb{N}$.

Definition 2. Let $\tau|\mathscr{B}^k$ be a p-measure satisfying $\tau(\Theta)=1$ and assume that the sequence $\{P_{\vartheta,n}: \vartheta\in\Theta\}$, $n\in\mathbb{N}$, satisfies LAN in τ-measure. A sequence $(\varphi_n)_{n\in\mathbb{N}}$ of estimates satisfies LIK in τ-measure if for every $\varepsilon>0$

$$\lim_{n\in\mathbb{N}} \int P_{\vartheta,n}\left\{\left|\frac{1}{\delta_n}(\varphi_n-\vartheta)-\Gamma^{-1}(\vartheta)\,\Delta_n(\vartheta)\right|>\varepsilon\right\}\tau(d\vartheta)=0.$$

The name LIK has been chosen because a sequence satisfying this condition behaves similar to logarithms of likelihood ratios.

Sufficient conditions for the existence of sequences of estimates satisfying LIK in τ-measure can be obtained combining Doob's martingale method for consistency of posterior distributions with conditions of LeCam, 1973b, being sufficient for square root consistency of posterior distributions.

Remark 1. It follows from contiguity (cf. the second step of the proof of Lemma 2) that LIK in τ-measure implies

$$\lim_{n\in\mathbb{N}} \int P_{\vartheta+\delta_n t,n}\left\{\left|\frac{1}{\delta_n}(\varphi_n-\vartheta)-\Gamma^{-1}(\vartheta)\,\Delta_n(\vartheta)\right|>\varepsilon\right\}\tau(d\vartheta)=0$$

for every $\varepsilon>0$.

Definition 3. A function $W\colon \mathbb{R}^k\to[0,1]$ is called loss function if $1-W\in\mathscr{C}_0(\mathbb{R}^k)$. A loss function satisfies condition (A) if

(A) for every centered normal distribution $\nu_{0,A}$ and every p-measure $P|\mathscr{B}^k$

$$\int W d\nu_{0,A}\leqq\int W d(\nu_{0,A}*P).$$

A loss function satisfies condition (B) if

(B) for every centered normal distribution $\nu_{0,A}$ and every p-measure $P|\mathscr{B}^k$, $P\neq\varepsilon_0$,

$$\int W d\nu_{0,A}<\int W d(\nu_{0,A}*P).$$

Example 1. It is easy to see that loss functions $W=L\circ\|.\|$ satisfy conditions (A) and (B) if L is increasing and not identically zero.

Throughout the following we denote $N(\vartheta)=\nu_{0,\Gamma^{-1}(\vartheta)}$, $\vartheta\in\Theta$. Proofs of Propositions 1 and 2 are found in Paragraph 3.

Proposition 1. *Assume that $\tau|\mathscr{B}^k$ is a p-measure satisfying $\tau(\Theta)=1$, $\tau\ll\lambda^k$, and that the sequence $\{P_{\vartheta,n}: \vartheta\in\Theta\}$, $n\in\mathbb{N}$, satisfies* LAN *in τ-measure. Then for every sequence of estimates $(\psi_n)_{n\in\mathbb{N}}$*

$$\varliminf_{n\in\mathbb{N}} \iint W\left(\frac{1}{\delta_n}(\vartheta-\psi_n)\right)dP_{\vartheta,n}\,g(\vartheta)\,\tau(d\vartheta)\geqq\iint W dN(\vartheta)\,g(\vartheta)\,\tau(d\vartheta)$$

if W satisfies (A) *and $g\in L_1^+(\tau)$.*

Remark 2. Let the assumptions of Proposition 1 be satisfied. Assume that $(\psi_n)_{n\in\mathbb{N}}$ is a sequence of estimates such that $\mathscr{L}\left(\frac{1}{\delta_n}(\psi_n-\vartheta)|P_{\vartheta,n}\right)$ converges vaguely τ-a.e.

Then Proposition 1 implies

$$\lim_{n\in\mathbb{N}} \int W\left(\frac{1}{\delta_n}(\vartheta-\psi_n)\right) dP_{\vartheta,n} \geqq \int W dN(\vartheta) \qquad \tau\text{-a.e.}$$

If $\tau \sim \lambda^k$ and if the limit distributions are continuous in $\vartheta\in\Theta$, then

$$\lim_{n\in\mathbb{N}} \int W\left(\frac{1}{\delta_n}(\vartheta-\psi_n)\right) dP_{\vartheta,n} \geqq \int W dN(\vartheta), \qquad \vartheta\in\Theta.$$

Remark 3. Let $P_n|\mathscr{B}^k$, $n\in\mathbb{N}$, and $P|\mathscr{B}^k$ be p-measures. The family $\mathscr{F}$ of functions $W\colon \mathbb{R}^k \to \mathbb{R}^+$ such that

$$\varliminf_{n\in\mathbb{N}} \int W dP_n \geqq \int W dP$$

satisfies the following conditions:

(1) $\mathscr{F}$ contains the constants.

(2) $\mathscr{F}$ is a positive convex cone.

(3) If $W_n\in\mathscr{F}$, $n\in\mathbb{N}$, $W_n\uparrow W$, then $W\in\mathscr{F}$.

This implies that the class of loss functions W for which Proposition 1 is true can be extended considerably. It contains all functions $W = L\circ\|.\|$, where $L\colon \mathbb{R}^+ \to \mathbb{R}^+$ is nondecreasing and lower semicontinuous.

Proposition 2. *Assume that $\tau|\mathscr{B}^k$ is a p-measure satisfying $\tau(\Theta)=1$, $\tau \ll \lambda^k$, and that the sequence $\{P_{\vartheta,n}\colon \vartheta\in\Theta\}$, $n\in\mathbb{N}$, satisfies* LAN *in τ-measure. Assume in addition that there exists a sequence of estimates satisfying* LIK *in τ-measure. Then a sequence of estimates $(\psi_n)_{n\in\mathbb{N}}$ satisfies*

$$\lim_{n\in\mathbb{N}} \iint W\left(\frac{1}{\delta_n}(\vartheta-\psi_n)\right) dP_{\vartheta,n}\,\tau(d\vartheta) = \iint W dN(\vartheta)\,\tau(d\vartheta)$$

for some loss function W satisfying (B) *iff $(\psi_n)_{n\in\mathbb{N}}$ satisfies* LIK *in τ-measure.*

Remark 4. Let the assumptions of Proposition 2 be satisfied and let W be a loss function satisfying (B). Propositions 1 and 2 imply that every sequence of Bayes estimates for the prior distribution τ satisfies LIK in τ-measure.

3. The Convolution Theorem and its Consequences

We begin with a version of the convolution theorem which is independent of invariance properties of limit distributions. Assume that $\tau|\mathscr{B}^k$ is a p-measure satisfying $\tau(\Theta)=1$ and that the sequence $\{P_{\vartheta,n}\colon \vartheta\in\Theta\}$, $n\in\mathbb{N}$, satisfies LAN in τ-measure. Let $(\psi_n)_{n\in\mathbb{N}}$ be an arbitrary sequence of estimates.

We introduce some notations:

$$R_n(\vartheta, t) = \mathscr{L}\left(t - \frac{1}{\delta_n}(\psi_n - \vartheta)|P_{\vartheta+\delta_n t, n}\right), \qquad \vartheta+\delta_n t\in\Theta,$$

$$S_n(\vartheta, t) = \mathscr{L}\left(\Gamma^{-1}(\vartheta)\,\Delta_n(\vartheta) - \frac{1}{\delta_n}(\psi_n - \vartheta)|P_{\vartheta+\delta_n t, n}\right), \qquad \vartheta+\delta_n t\in\Theta.$$

Both R_n and S_n can trivially be extended to kernels on $\Theta \times \mathbb{R}^k \times \mathscr{B}^k$. Let $D_\alpha = \{t \in \mathbb{R}^k : \|t\| \leqq \alpha\}$, $\alpha \geqq 0$, and define

$$R_{n,\alpha}(\vartheta) = \frac{1}{\lambda^k(D_\alpha)} \int_{D_\alpha} R_n(\vartheta, t)\, dt,$$

$$S_{n,\alpha}(\vartheta) = \frac{1}{\lambda^k(D_\alpha)} \int_{D_\alpha} S_n(\vartheta, t)\, dt.$$

Recall that $N(\vartheta) = \nu_{0,\Gamma^{-1}(\vartheta)}$, $\vartheta \in \Theta$.

Theorem 1. *Assume that $\tau|\mathscr{B}^k$ is a p-measure satisfying $\tau(\Theta)=1$, and that the sequence $\{P_{\vartheta,n}: \vartheta \in \Theta\}$, $n \in \mathbb{N}$, satisfies* LAN *in τ-measure. Then for every sequence $(\psi_n)_{n\in\mathbb{N}}$ of estimates*

$$\lim_{\alpha\to\infty} \overline{\lim_{n\in\mathbb{N}}} \int \|R_{n,\alpha}(\vartheta) - N(\vartheta) * S_{n,\alpha}(\vartheta)\|\, \tau(d\vartheta) = 0.$$

The proof of Theorem 1 is given in Paragraph 4.

Let $(T, \mathscr{T})$ be a locally compact space with countable base and Borel algebra $\mathscr{B}$ and let $(\Omega, \mathscr{A}, \mu)$ be a σ-finite measure space. Consider (substochastic) kernels $P: \Omega \to \mathscr{M}_1(\mathscr{B})$. Let $\mathscr{C}_{00}$ be the space of continuous functions on T vanishing outside compacts. We define the $\mathscr{C}_{00} \otimes L_1(\mu)$-topology of the set of all kernels P to be the coarsest topology such that all functions

$$P \mapsto \iint f(t)\, P(\omega)(dt)\, g(\omega)\, \mu(d\omega), \qquad f \in \mathscr{C}_{00},\ g \in L_1(\mu),$$

are continuous. It is well known that the set of all kernels endowed with this topology is metrizable and compact.

The following result is a generalized version of LeCam's invariance theorem for limit distributions (LeCam, 1973a).

Theorem 2. *Assume that $\Theta \subseteq \mathbb{R}^k$ is a measurable subset and let $\tau|\mathscr{B}^k$ be a p-measure satisfying $\tau(\Theta)=1$, $\tau \ll \lambda^k$. Let $(F_n)_{n\in\mathbb{N}}$ be a sequence of kernels $F_n: \Theta \times \mathbb{R}^k \to \mathscr{M}_1(\mathscr{B}^k)$ being $\mathscr{C}_{00}(\mathbb{R}^k) \otimes L_1(\tau \otimes \lambda^k)$-convergent to a kernel F. If there is a sequence $\delta_n \downarrow 0$ such that*

$$F_n(\vartheta + \delta_n r, t) = F_n(\vartheta, r+t), \qquad \vartheta \in \Theta,\ r, t \in \mathbb{R}^k,\ n \in \mathbb{N},$$

then for τ-almost all $\vartheta \in \Theta$

$$\int F(\vartheta, t)\, h(t)\, dt = \int F(\vartheta, t+r)\, h(t)\, dt$$

for all $r \in \mathbb{R}^k$ and $h \in L_1(\lambda^k)$.

Proof. The proof is similar to the proof of Theorem 1 in LeCam, 1973a.

Throughout the following we assume that $\tau|\mathscr{B}^k$ is a p-measure satisfying $\tau(\Theta) = 1$, $\tau \ll \lambda^k$. We combine the convolution theorem with the invariance theorem.

Corollary 1. *Let the assumptions of Theorem 1 be satisfied and assume in addition that $\tau \ll \lambda^k$. Then for every sequence $(\psi_n)_{n\in\mathbb{N}}$ of estimates and every subsequence $\mathbb{N}_0 \subseteq \mathbb{N}$ such that $(R_n)_{n\in\mathbb{N}_0}$ and $(S_n)_{n\in\mathbb{N}_0}$ are $\mathscr{C}_{00}(\mathbb{R}^k) \otimes L_1(\tau \otimes \lambda^k)$-convergent*

$$\lim_{\alpha\to\infty}\lim_{n\in\mathbb{N}_0}(R_n(.,0)-N*S_{n,\alpha})=0 \qquad (\mathscr{C}_{00}(\mathbb{R}^k)\otimes L_1(\tau)).$$

Proof. In view of Theorem 1 we have only to show that

$$\lim_{n\in\mathbb{N}_0}(R_n(.,0)-R_{n,\alpha})=0 \qquad (\mathscr{C}_{00}(\mathbb{R}^k)\otimes L_1(\tau))$$

for every $\alpha>0$. Let R be the $\mathscr{C}_{00}\otimes L_1(\tau\otimes\lambda^k)$-limit of $(R_n)_{n\in\mathbb{N}_0}$. Then Theorem 2 and Lemma 4 (of paragraph 4) imply that

$$\int R(\vartheta,0)\,g(\vartheta)\,\tau(d\vartheta)=\int R(\vartheta,t)\,g(\vartheta)\,\tau(d\vartheta)$$

for all $g\in L_1(\tau)$ and $t\in\mathbb{R}^k$. This proves the assertion.

Corollary 2. *Let the assumptions of Theorem 2 be satisfied. Assume that $\tau\ll\lambda^k$ and that there exists a sequence of estimates satisfying* LIK *in τ-measure. Then for every sequence $(\psi_n)_{n\in\mathbb{N}}$ of estimates and every subsequence $\mathbb{N}_0\subseteq\mathbb{N}$ such that $(R_n)_{n\in\mathbb{N}_0}$ and $(S_n)_{n\in\mathbb{N}_0}$ are $\mathscr{C}_{00}(\mathbb{R}^k)\otimes L_1(\tau\otimes\lambda^k)$-convergent*

$$\lim_{n\in\mathbb{N}_0}(R_n(.,0)-N*S_n(.,0))=0 \qquad (\mathscr{C}_{00}(\mathbb{R}^k)\otimes L_1(\tau)).$$

Proof. Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence of estimates satisfying LIK in τ-measure. Define

$$T_n(\vartheta,t)=\mathscr{L}\left(\frac{1}{\delta_n}(\varphi_n-\psi_n)|P_n(\vartheta,t)\right), \qquad \vartheta+\delta_n t\in\Theta.$$

Condition LIK implies that

$$\lim_{n\in\mathbb{N}}(S_n-T_n)=0 \qquad (\mathscr{C}_{00}\otimes L_1(\tau\otimes\lambda^k)).$$

Since T_n, $n\in\mathbb{N}$, satisfy the functional equation of Theorem 2 the $\mathscr{C}_{00}\otimes L_1(\tau\otimes\lambda^k)$-limit of $(S_n)_{n\in\mathbb{N}_0}$ is invariant in the sense of Theorem 2. Now the proof is finished similar to the proof of Corollary 1.

Proof of Proposition 1. We may assume $g\equiv 1$. It is sufficient to prove

$$\varliminf_{n\in\mathbb{N}_0}\iint W\,dR_n(\vartheta,0)\,\tau(d\vartheta)\geqq\iint W\,dN(\vartheta)\,\tau(d\vartheta)$$

for every subsequence $\mathbb{N}_0\subseteq\mathbb{N}$ such that $(R_n)_{n\in\mathbb{N}_0}$ and $(S_n)_{n\in\mathbb{N}_0}$ are $\mathscr{C}_{00}\otimes L_1(\tau\otimes\lambda^k)$-convergent. For such a sequence $\mathbb{N}_0$ condition (A) and Corollary 1 imply

$$\begin{aligned}\iint W\,dN(\vartheta)\,\tau(d\vartheta)&\leqq\lim_{\alpha\to\infty}\lim_{n\in\mathbb{N}_0}\iint W\,d(N*S_{n,\alpha})(\vartheta)\,\tau(d\vartheta)\\&=\lim_{n\in\mathbb{N}_0}\iint W\,dR_n(\vartheta,0)\,\tau(d\vartheta).\end{aligned}$$

Proof of Proposition 2. The assertion is proved if we show that $\lim_{n\in\mathbb{N}}S_n(.,0)=\varepsilon_0$ in τ-measure. It is sufficient to prove it for every subsequence $\mathbb{N}_0\subseteq\mathbb{N}$ such that $(R_n)_{n\in\mathbb{N}_0}$ and $(S_n)_{n\in\mathbb{N}_0}$ are $\mathscr{C}_{00}\otimes L_1(\tau\otimes\lambda^k)$-convergent. Define $D_\alpha=\{z\in\mathbb{R}^k$:

$\|z\| \leqq \alpha\}$, $\alpha \geqq 0$, and

$$M_\alpha(\vartheta) = \inf_{z \notin D_\alpha} \int W(y+z)\, N(\vartheta)(dy), \qquad \vartheta \in \Theta,$$

$$A_{n,\varepsilon} = \{\vartheta \in \Theta : S_n(\vartheta, 0)(D'_\alpha) \geqq \varepsilon\}, \qquad \varepsilon > 0,\ n \in \mathbb{N}.$$

It follows from condition (B) that $\tau\{M_\alpha > M_0\} = 1$ if $\alpha > 0$. For every $n \in \mathbb{N}$, $\varepsilon > 0$, $\alpha > 0$, we have

$$\begin{aligned}
&\iint W d(N(\vartheta) * S_n(\vartheta, 0))\, \tau(d\vartheta) \\
&\quad = \iiint W(y+z)\, N(\vartheta)(dy)\, S_n(\vartheta, 0)(dz)\, \tau(d\vartheta) \\
&\quad \geqq \int_{A_{n,\varepsilon}} [S_n(\vartheta, 0)(D'_\alpha)\, M_\alpha(\vartheta) + S_n(\vartheta, 0)(D_\alpha)\, M_0(\vartheta)]\, \tau(d\vartheta) + \int_{A'_{n,\varepsilon}} M_0(\vartheta)\, \tau(d\vartheta) \\
&\quad \geqq \int_{A_{n,\varepsilon}} [\varepsilon M_\alpha(\vartheta) + (1-\varepsilon)\, M_0(\vartheta)]\, \tau(d\vartheta) + \int_{A'_{n,\varepsilon}} M_0(\vartheta)\, \tau(d\vartheta) \\
&\quad = \int M_0(\vartheta)\, \tau(d\vartheta) + \varepsilon \int_{A_{n,\varepsilon}} (M_\alpha(\vartheta) - M_0(\vartheta))\, \tau(d\vartheta).
\end{aligned}$$

Then

$$\lim_{n \in \mathbb{N}} \iint W dR_n(\vartheta, 0)\, \tau(d\vartheta) = \int M_0\, d\tau$$

and Corollary 2 imply

$$\lim_{n \in \mathbb{N}_0} \int_{A_{n,\varepsilon}} (M_\alpha - M_0)\, d\tau = 0.$$

Since $\tau\{M_\alpha - M_0 > 0\} = 1$, we obtain $\lim_{n \in \mathbb{N}_0} \tau(A_{n,\varepsilon}) = 0$ and therefore

$$\lim_{n \in \mathbb{N}_0} \int S_n(\vartheta, 0)(D'_\alpha)\, \tau(d\vartheta) = 0$$

which proves the assertion.

4. The Proof of the Convolution Theorem

Assume that $\tau|\mathscr{B}^k$ is a p-measure satisfying $\tau(\Theta) = 1$ and that the sequence $\{P_{\vartheta,n} : \vartheta \in \Theta\}$, $n \in \mathbb{N}$, satisfies LAN in τ-measure.

The convolution theorem is known for the particular case when τ is concentrated at a single point. Our proof of the general version follows the pattern of Hajek's proof. In order to increase the readibility the main steps are isolated.

Lemma 2. *Assume that $\tau|\mathscr{B}^k$ is a p-measure satisfying $\tau(\Theta) = 1$ and that the sequence $\{P_{\vartheta,n} : \vartheta \in \Theta\}$, $n \in \mathbb{N}$, satisfies* LAN *in τ-measure. Then there exist*

(1) *a sequence $k_n \uparrow \infty$,*

(2) *measurable functions $C_n(\cdot, t) : \Theta \to \mathbb{R}$ such that*

$$\lim_{n \in \mathbb{N}} \int \sup_{\|t\| \leqq a} |C_n(\vartheta, t) - 1|\, \tau(d\vartheta) = 0 \qquad \textit{for every } a > 0,$$

such that the measures $Q_n(\vartheta, t)|\mathscr{A}_n$, $Q_n(\vartheta, t) \ll P_{\vartheta,n}$, $\vartheta \in \Theta$, *defined by*

$$\frac{dQ_n(\vartheta, t)}{dP_{\vartheta,n}} = C_n(\vartheta, t) \exp[t' \Delta_n^*(\vartheta) - \tfrac{1}{2} t' \Gamma(\vartheta) t],$$

and $\Delta_n^* = \Delta_n 1_{\{\|\Delta_n\| \leqq k_n\}}$, $\vartheta \in \Theta$, $t \in \mathbb{R}^k$, *are p-measures and satisfy*

$$\lim_{n\in\mathbb{N}} \int \|P_{\vartheta+\delta_n t, n} - Q_n(\vartheta, t)\| \, \tau(d\vartheta) = 0$$

for every $t \in \mathbb{R}^k$.

Proof. For convenience we introduce the following notations:

$$f_{n,t}(\omega_n, \vartheta) = \frac{dP_{\vartheta+\delta_n t, n}}{dP_{\vartheta,n}}(\omega_n), \qquad \omega_n \in \Omega_n,\ \vartheta \in \Theta,$$

$$g_{n,t}(\omega_n, \vartheta) = \frac{dQ_n(\vartheta, t)}{dP_{\vartheta,n}}(\omega_n), \qquad \omega_n \in \Omega_n,\ \vartheta \in \Theta,$$

$$P_n = \int P_{\vartheta,n} \tau(d\vartheta),$$

$$\mu_t = \int \mathscr{L}(\exp | \nu_{-\frac{1}{2}t'\Gamma(\vartheta)t,\, t'\Gamma(\vartheta)t}) \, \tau(d\vartheta),$$

for $t \in \mathbb{R}^k$, $n \in \mathbb{N}$.

1st step: We show that the truncation can be chosen in such a way that for every $a > 0$

$$\lim_{n\in\mathbb{N}} \int \sup_{\|t\| \leqq a} |C_n(\vartheta, t) - 1| \tau(d\vartheta) = 0.$$

Note that the family of functions

$$s \mapsto f_i(s, t, \vartheta) = \begin{cases} \exp(t's - \tfrac{1}{2} t' \Gamma(\vartheta) t) & \text{if } \|s\| \leqq i, \\ \exp(-\tfrac{1}{2} t' \Gamma(\vartheta) t) & \text{if } \|s\| > i, \end{cases}$$

$\vartheta \in \Theta$, $\|t\| \leqq i$, is uniformly bounded and equicontinuous on $\{\|s\| \leqq i\}$ for every $i \in \mathbb{N}$. This implies that

$$\lim_{n\in\mathbb{N}} \int \sup_{\|t\| \leqq i} \left| \int f_i(s, t, \vartheta) \, \mathscr{L}(\Delta_n(\vartheta)|P_{\vartheta,n})(ds) - \int f_i(s, t, \vartheta) \, \nu_{0,\Gamma(\vartheta)}(ds) \right| \tau(d\vartheta) = 0$$

for every $i \in \mathbb{N}$. A standard argument yields a sequence $(k_n)_{n\in\mathbb{N}}$, $k_n \uparrow \infty$, such that

$$\lim_{n\in\mathbb{N}} \int \sup_{\|t\| \leqq k_n} \left| \int f_{k_n}(s, t, \vartheta) \, \mathscr{L}(\Delta_n(\vartheta)|P_{\vartheta,n})(ds) - \int f_{k_n}(s, t, \vartheta) \, \nu_{0,\Gamma(\vartheta)}(ds) \right| \tau(d\vartheta) = 0.$$

Defining $\Delta_n^* = \Delta_n 1_{\{\|\Delta_n\| \leqq k_n\}}$ we obtain

$$\lim_{n\in\mathbb{N}} \int \sup_{\|t\| \leqq a} \left| \int \exp(t' \Delta_n^* - \tfrac{1}{2} t' \Gamma t) \, dP_{\cdot,n} - \nu_{\Gamma^{-1}t,\Gamma}\{\|s\| \leqq k_n\} \right| d\tau = 0$$

for every $a > 0$. Obviously, this implies

$$\lim_{n\in\mathbb{N}} \int \sup_{\|t\| \leqq a} \left| \int \exp(t' \Delta_n^* - \tfrac{1}{2} t' \Gamma t) \, dP_{\cdot,n} - 1 \right| d\tau = 0.$$

Since

$$C_n(\vartheta, t) = 1/\int \exp(t'\Delta_n^*(\vartheta) - \tfrac{1}{2} t'\Gamma(\vartheta)\, t)\, dP_{\vartheta,n},$$

the assertion is proved.

2nd step: We prove that

$$\overline{\lim_{n\in\mathbb{N}}} \int \|P_{\vartheta+\delta_n t, n} - Q_n(\vartheta, t)\|\, \tau(d\vartheta) \leqq \tfrac{1}{2} \overline{\lim_{n\in\mathbb{N}}} \int |f_{n,t} - g_{n,t}|\, dP_n.$$

To this end we have only to show that

$$\lim_{n\in\mathbb{N}} \int P_{\vartheta+\delta_n t, n}(A_n)\, \tau(d\vartheta) = 0 \quad \text{if } \lim_{n\in\mathbb{N}} P_n(A_n) = 0.$$

Standard reasoning yields for every $a < \infty$

$$P_{\vartheta+\delta_n t, n}(A_n) \leqq e^a P_{\vartheta,n}(A_n) + 1 - \int_{|s|<a} s\mathscr{L}(f_{n,t}|P_{\vartheta,n})(ds)$$

which implies

$$\overline{\lim_{n\in\mathbb{N}}} \int P_{\vartheta+\delta_n t, n}(A_n)\, \tau(d\vartheta) \leqq 1 - \int_{|s|<a} s\mu_t(ds).$$

Since $\int s\mu_t(ds) = 1$ the assertion is proved.

3rd step: We prove that $\lim_{n\in\mathbb{N}} |f_{n,t} - g_{n,t}| = 0$ (P_n) for every $t \in \mathbb{R}^k$. Note that for every $\varepsilon > 0$ there exists $c_\varepsilon < \infty$ such that

$$\lim_{n\in\mathbb{N}} P_n\{\|\Delta_n\| \geqq c_\varepsilon \|\Gamma^{1/2}\|\} \leqq \varepsilon.$$

Let $d(\varepsilon, \vartheta) = \exp(c_\varepsilon \|t\|\, \|\Gamma^{1/2}(\vartheta)\| - \tfrac{1}{2} t'\Gamma(\vartheta)\, t)$. Then

$$\begin{aligned} |f_{n,t} - g_{n,t}| &\leqq |f_{n,t} - \exp(t'\Delta_n^* - \tfrac{1}{2} t'\Gamma t)| + |\exp(t'\Delta_n^* - \tfrac{1}{2} t'\Gamma t) - g_{n,t}| \\ &\leqq \exp(t'\Delta_n - \tfrac{1}{2} t'\Gamma t) |\exp Z_n(t,.) - \exp(t'\Delta_n^* - t'\Delta_n)| \\ &\quad + \exp(t'\Delta_n - \tfrac{1}{2} t'\Gamma t) |C_n(t,.) - 1| \end{aligned}$$

implies

$$\begin{aligned} P_n\{|f_{n,t} - g_{n,t}| > \varepsilon\} &\leqq P_n\{\exp(t'\Delta_n - \tfrac{1}{2} t'\Gamma t) > d(\varepsilon,.)\} \\ &\quad + P_n\{|\exp Z_n(t,.) - \exp(t'\Delta_n^* - t'\Delta_n)| > \varepsilon/2d(\varepsilon,.)\} \\ &\quad + P_n\{|C_n(t,.) - 1| > \varepsilon/2d(\varepsilon,.)\}. \end{aligned}$$

This proves the assertion in view of $\tau\{d(\varepsilon,.) < \infty\} = 1$.

4th step: We prove that $(f_{n,t})_{n\in\mathbb{N}}$ is uniformly (P_n)-integrable for every $t \in \mathbb{R}^k$. To this end we show that

$$\lim_{n\in\mathbb{N}} \mathscr{L}(f_{n,t}|P_n) = \mu_t \quad \text{vaguely}$$

$$\lim_{n\in\mathbb{N}} \int s\mathscr{L}(f_{n,t}|P_n)(ds) = \int s\mu_t(ds).$$

The first equation is obvious. Moreover we know that $\int s\mu_t(ds)=1$. Then the second assertion follows from

$$\lim_{n\in\mathbb{N}} \int s\mathscr{L}(f_{n,t}|P_n)(ds)=\lim_{n\in\mathbb{N}} \int\int \frac{dP_{\vartheta+\delta_n t,n}}{dP_{\vartheta,n}}\, dP_{\vartheta,n}\tau(d\vartheta)$$

for reasons of contiguity.

5^{th} step: We prove that $(g_{n,t})_{n\in\mathbb{N}}$ is uniformly (P_n)-integrable for every $t\in\mathbb{R}^k$. We do it in same way as in the 4^{th} step. In the present case this is almost immediate.

Lemma 3. *Assume that* $\tau|\mathscr{B}^k$ is a *p-measure satisfying* $\tau(\Theta)=1$ *and that the sequence* $\{P_{\vartheta,n}: \vartheta\in\Theta\}$, $n\in\mathbb{N}$, *satisfies* LAN *in* τ*-measure. Then*

$$\lim_{n\in\mathbb{N}} \int |\mathscr{L}(\Delta_n(\vartheta)|P_{\vartheta+\delta_n t,n})-\nu_{\Gamma(\vartheta)t,\Gamma(\vartheta)}|\,\tau(d\vartheta)=0$$

vaguely for every $t\in\mathbb{R}^k$.

Proof. The proof is completely analogous to the case when τ is concentrated at a single point.

Lemma 4. *Assume that* $\tau|\mathscr{B}^k$ *is a p-measure satisfying* $\tau(\Theta)=1$ *and that the sequence* $\{P_{\vartheta,n}: \vartheta\in\Theta\}$, $n\in\mathbb{N}$, *satisfies* LAN *in* τ*-measure. Then for every* $t\in\mathbb{R}^k$, $\varepsilon>0$, *there exists* $\eta(\varepsilon,t)>0$ *such that*

$$\overline{\lim_{n\in\mathbb{N}}} \int \|P_{\vartheta+\delta_n s,n}-P_{\vartheta+\delta_n t,n}\|\,\tau(d\vartheta)\leqq\varepsilon \quad \text{ if } \|s-t\|\leqq\eta(\varepsilon,t).$$

Proof. Lemma 2 and Lemma 3 imply that

$$\lim_{n\in\mathbb{N}} \int \mathscr{L}(\Delta_n^*|Q_n(.,s))\,d\tau=\int \nu_{\Gamma s,\Gamma}d\tau \quad \text{ vaguely}$$

for every $s\in\mathbb{R}^k$. Fix $t\in\mathbb{R}^k$, $\varepsilon>0$. Since

$$\lim_{b\to\infty} \int \nu_{\Gamma t,\Gamma}(D_b')\,d\tau=0$$

there exists some $b>0$ such that

$$\int \nu_{\Gamma t,\Gamma}(D_b')\,d\tau<\frac{\varepsilon}{6}.$$

Since

$$s\mapsto\int \nu_{\Gamma s,\Gamma}(D_b')\,d\tau, \quad s\in\mathbb{R}^k,$$

is continuous there exists $a>0$ such that

$$\int \nu_{\Gamma s,\Gamma}(D_b')\,d\tau<\frac{\varepsilon}{3} \quad \text{ if } \|s-t\|<a.$$

It follows that for $\|s-t\|<a$

$$\overline{\lim_{n\in\mathbb{N}}} \int \|Q_n(\vartheta, s) - Q_n(\vartheta, t)\| \, \tau(d\vartheta) \leq \overline{\lim_{n\in\mathbb{N}}} \int \sup_{A\in\mathscr{A}_n} |Q_n(\vartheta, s)(A\cap\{\Delta_n^*(\vartheta)\in D_b\}) - Q_n(\vartheta, t)(A\cap\{\Delta_n^*(\vartheta)\in D_b\})| \, \tau(d\vartheta) + \frac{2\varepsilon}{3}.$$

Furthermore we have

$$\begin{aligned}
&\sup_{A\in\mathscr{A}_n} |Q_n(\vartheta, s)(A\cap\{\Delta_n^*(\vartheta)\in D_b\}) - Q_n(\vartheta, t)(A\cap\{\Delta_n^*(\vartheta)\in D_b\})| \\
&\quad \leq \int_{\Delta_n^*(\vartheta)\in D_b} \left| 1 - \frac{dQ_n(\vartheta, s)}{dQ_n(\vartheta, t)} \right| dQ_n(\vartheta, t) \\
&\quad = \int_{\Delta_n^*(\vartheta)\in D_b} \left| 1 - \frac{C_n(\vartheta, s)}{C_n(\vartheta, t)} \exp[(s-t)'\Delta_n^*(\vartheta) - \tfrac{1}{2}(s'\Gamma(\vartheta)\, s - t'\Gamma(\vartheta)\, t)] \right| dQ_n(\vartheta, t) \\
&\quad \leq \left| 1 - \frac{C_n(\vartheta, s)}{C_n(\vartheta, t)} \exp[-\tfrac{1}{2}(s'\Gamma(\vartheta)\, s - t'\Gamma(\vartheta)\, t)] \right| \\
&\qquad + \frac{C_n(\vartheta, s)}{C_n(\vartheta, t)} \exp[-\tfrac{1}{2}(s'\Gamma(\vartheta)\, s - t'\Gamma(\vartheta)\, t)] \\
&\qquad \cdot \int_{D_b} |1 - \exp[(s-t)'u]| \, \mathscr{L}(\Delta_n^*(\vartheta)|Q_n(\vartheta, t))(du).
\end{aligned}$$

This implies that

$$\begin{aligned}
\overline{\lim_{n\in\mathbb{N}}} \int \|Q_n(\vartheta, s) - Q_n(\vartheta, t)\| \, \tau(d\vartheta) &\leq \int |1 - \exp[-\tfrac{1}{2}(s'\Gamma(\vartheta)\, s - t'\Gamma(\vartheta)\, t)]| \, \tau(d\vartheta) \\
&\quad + \int \exp[-\tfrac{1}{2}(s'\Gamma(\vartheta)\, s - t'\Gamma(\vartheta)\, t)] \\
&\quad \cdot \int_{D_b} |1 - \exp[(s-t)'u]| \, \nu_{\Gamma(\vartheta)t, \Gamma(\vartheta)}(du) \, \tau(d\vartheta).
\end{aligned}$$

Choosing $\eta(\varepsilon, t)$ sufficiently small proves the assertion.

The following lemmas are the key for proving the convolution theorem. They provide the technical essence of Hajek's method.

Lemma 5. *Let $P|\mathscr{B}^k$ be a p-measure and let $D_\alpha = \{s\in\mathbb{R}^k\colon \|s\| \leq \alpha\}$, $\alpha > 0$. Then*

$$\lim_{\alpha\to\infty} \frac{1}{\lambda^k(D_\alpha)} \int_{D_\alpha} (P * \varepsilon_t)(D'_\alpha) \, dt = 0.$$

Proof. Define $E_\alpha = \{s\in\mathbb{R}^k\colon \|s\| \leq \alpha - \sqrt{\alpha}\}$. Then we have $D_{\sqrt{\alpha}} + E_\alpha \subseteq D_\alpha$ which implies $D_{\sqrt{\alpha}} \subseteq D_\alpha - t$ for every $t\in E_\alpha$. It follows that

$$\frac{1}{\lambda^k(D_\alpha)} \int_{D_\alpha} P(D_\alpha - t) \, dt \geq \frac{1}{\lambda^k(D_\alpha)} \int_{E_\alpha} P(D_\alpha - t) \, dt \geq \frac{\lambda^k(E_\alpha)}{\lambda^k(D_\alpha)} P(D_{\sqrt{\alpha}}) \geq \frac{\lambda^k(E_\alpha)}{\lambda^k(D_\alpha)} - P(D'_{\sqrt{\alpha}}).$$

Hence

$$\lim_{\alpha\to\infty} \frac{1}{\lambda^k(D_\alpha)} \int_{D_\alpha} (P * \varepsilon_t)(D'_\alpha) \, dt \leq \lim_{\alpha\to\infty} P(D'_{\sqrt{\alpha}}) + \lim_{\alpha\to\infty} \left(1 - \frac{\lambda^k(E_\alpha)}{\lambda^k(D_\alpha)}\right).$$

This proves the assertion.

Before we state Lemma 6 we have to introduce some notations.

Notations. Let $(\Omega, \mathscr{A}, \mu)$ be a σ-finite measure space. Let $N|\mathscr{B}^k$ be a p-measure with Lebesgue density h. If X: $\Omega \to \mathbb{R}^k$ is a random vector then define a stochastic kernel Q: $\mathbb{R}^k \to \mathscr{M}_1(\mathscr{A})$ by

$$Q_t(A) = C(t) \int_A h(t - X(\omega))\, \mu(d\omega).$$

Let Y: $\Omega \to \mathbb{R}^k$ be another random vector and let

$$R_t = \mathscr{L}(t - Y|Q_t), \qquad t \in \mathbb{R}^k,$$
$$S_t = \mathscr{L}(X - Y|Q_t), \qquad t \in \mathbb{R}^k.$$

If $D_\alpha = \{s \in \mathbb{R}^k \colon \|s\| \leqq \alpha\}$, $\alpha > 0$, define

$$\bar{R}_\alpha = \frac{1}{\lambda^k(D_\alpha)} \int_{D_\alpha} R_t\, dt, \qquad \alpha > 0,$$

$$\bar{S}_\alpha = \frac{1}{\lambda^k(D_\alpha)} \int_{D_\alpha} S_t\, dt, \qquad \alpha > 0.$$

Lemma 6 gives an estimate of $\|\bar{R}_\alpha - N * \bar{S}_\alpha\|$. The remarkable feature of the result is that the estimate is independent of Y.

Lemma 6. Let $(\Omega, \mathscr{A}, \mu)$ be a measurable space, $N \ll \lambda^k$ a p-measure on $\mathscr{B}^k$ and X: $\Omega \to \mathbb{R}^k$ a random variable. Then for every further random variable Y: $\Omega \to \mathbb{R}^k$, $\alpha > 0$,

$$\|\bar{R}_\alpha - N * \bar{S}_\alpha\| \leqq \min\{1, 2 \sup_{t \in D_\alpha} |C(t) - 1|\} + \frac{1}{\lambda^k(D_\alpha)} \int_{D_\alpha} (N * \mathscr{L}(X|Q_t))(D'_\alpha)\, dt.$$

Proof. Let $\bar{Q}_\alpha|\mathscr{A} \otimes \mathscr{B}^k$ be defined by

$$\bar{Q}_\alpha(A \times B) = \frac{1}{\lambda^k(D_\alpha)} \int_{D_\alpha \cap B} Q_t(A)\, dt, \qquad A \in \mathscr{A},\ B \in \mathscr{B}^k.$$

Then for every $B \in \mathscr{B}^k$ we have

$$\bar{R}_\alpha(B) = \bar{Q}_\alpha\{(\omega, t) \colon t - Y(\omega) \in B\} = \int \bar{Q}_\alpha(B + Y(\omega)|\mathscr{A})(\omega)\, \bar{Q}_\alpha(d\omega).$$

This implies

$$|\bar{R}_\alpha(B) - \int (N * \varepsilon_{X(\omega)})(B + Y(\omega))\, \bar{Q}_\alpha(d\omega)| \leqq \int \|\bar{Q}_\alpha(.\,|\, \mathscr{A})(\omega) - (N * \varepsilon_{X(\omega)})\| \bar{Q}_\alpha(d\omega).$$

Since

$$\int (N * \varepsilon_{X(\omega)})(B + Y(\omega))\, \bar{Q}_\alpha(d\omega) = \int N(B + Y(\omega) - X(\omega))\, \bar{Q}_\alpha(d\omega) = (N * \bar{S}_\alpha)(B),$$

it follows that

$$\|\bar{R}_\alpha - N * \bar{S}_\alpha\| \leqq \int \|\bar{Q}_\alpha(.\,|\,\mathscr{A})(\omega) - (N * \varepsilon_{X(\omega)})\|\, \bar{Q}_\alpha(d\omega).$$

Now the problem is reduced to a Bernstein-v. Mises approximation. We use the following inequality: For any two p-measures $P|\mathscr{A}$, $Q|\mathscr{A}$, $Q \ll P$ and every $A \in \mathscr{A}$ such that $Q(A) > 0$ we have

$$\|Q(.\,|A) - P\| \leqq 2\min\left\{1, \sup_{\omega\in A}\left|\frac{dQ}{dP} - 1\right|\right\} + 2P(A').$$

The particular form of Q_t implies that

$$\bar{Q}_\alpha(B|\mathscr{A})(\omega) = \frac{\int\limits_{B\cap D_\alpha} C(t)(N * \varepsilon_{X(\omega)})(dt)}{\int\limits_{D_\alpha} C(t)(N * \varepsilon_{X(\omega)})(dt)}.$$

Applying the inequality mentioned just before we obtain

$$\|\bar{Q}_\alpha(.|\mathscr{A})(\omega) - N * \varepsilon_{X(\omega)}\| \leqq 2\min\{1, \sup_{t\in D_\alpha}|C(t) - 1|\} + 2(N * \varepsilon_{X(\omega)})(D'_\alpha).$$

Now the assertion follows from

$$\int (N * \varepsilon_{X(\omega)})(D'_\alpha)\, Q_t(d\omega) = \int N(D'_\alpha - X(\omega))\, Q_t(d\omega) = (N * \mathscr{L}(X|Q_t))(D'_\alpha).$$

Proof of Theorem 1. Let $\tau|\mathscr{B}^k$ be a p-measure satisfying $\tau(\Theta) = 1$. We define:

$$U_n(\vartheta, t) = \mathscr{L}\left(t - \frac{1}{\delta_n}(\psi_n - \vartheta)|Q_n(\vartheta, t)\right), \qquad \vartheta\in\Theta,\ t\in\mathbb{R}^k,$$

$$V_n(\vartheta, t) = \mathscr{L}\left(\Gamma^{-1}(\vartheta)\,\Delta_n(\vartheta) - \frac{1}{\delta_n}(\psi_n - \vartheta)|Q_n(\vartheta, t)\right), \qquad \vartheta\in\Theta,\ t\in\mathbb{R}^k.$$

The kernels U_n and V_n correspond to R_n and S_n, the only difference being that $P_{\vartheta+\delta_n t, n}$ is replaced by $Q_n(\vartheta, t)$. Similarly we define

$$U_{n,\alpha}(\vartheta) = \frac{1}{\lambda^k(D_\alpha)} \int\limits_{D_\alpha} U_n(\vartheta, t)\, dt,$$

$$V_{n,\alpha}(\vartheta) = \frac{1}{\lambda^k(D_\alpha)} \int\limits_{D_\alpha} V_n(\vartheta, t)\, dt.$$

Lemma 2 implies that for every $\alpha > 0$

$$\lim_{n\in\mathbb{N}} \int \|R_{n,\alpha}(\vartheta) - U_{n,\alpha}(\vartheta)\|\, \tau(d\vartheta) = 0,$$

$$\lim_{n\in\mathbb{N}} \int \|S_{n,\alpha}(\vartheta) - V_{n,\alpha}(\vartheta)\|\, \tau(d\vartheta) = 0.$$

Hence it is sufficient to show that

$$\lim_{\alpha\to\infty} \overline{\lim_{n\in\mathbb{N}}} \int \|U_{n,\alpha}(\vartheta) - N(\vartheta) * V_{n,\alpha}(\vartheta)\|\, \tau(d\vartheta) = 0.$$

Let $\vartheta\in M$, $n\in\mathbb{N}$ be fixed and apply Lemma 6 to

$$Q_t = Q_n(\vartheta, t)$$
$$X = \Gamma^{-1}(\vartheta)\, \Delta_n(\vartheta)$$
$$Y = \frac{1}{\delta_n}(\psi_n - \vartheta)$$
$$N = N(\vartheta) = \nu_{0, \Gamma^{-1}(\vartheta)}.$$

This yields

$$\|U_{n,\alpha}(\vartheta) - N(\vartheta) * V_{n,\alpha}(\vartheta)\| \leqq \min\{1, 2 \sup_{t \in D_\alpha} |C_n(t, \vartheta) - 1|\}$$
$$+ \frac{1}{\lambda^k(D_\alpha)} \int_{D_\alpha} (N(\vartheta) * \mathscr{L}(\Gamma^{-1}(\vartheta)\, \Delta_n(\vartheta) | Q_n(\vartheta, t))(D'_\alpha)\, dt.$$

From Lemma 2, (2), and Lemma 3 we obtain

$$\overline{\lim_{n \in \mathbb{N}}} \int \|U_{n,\alpha}(\vartheta) - N(\vartheta) * V_{n,\alpha}(\vartheta)\|\, \tau(d\vartheta) \leqq \frac{1}{\lambda^k(D_\alpha)} \int \int_{D_\alpha} (N(\vartheta) * N(\vartheta) * \varepsilon_t)(D'_\alpha)\, dt\, \tau(d\vartheta).$$

Now Lemma 5 proves the assertion.

Acknowledgement. The author wishes to thank Professor L. LeCam for valuable suggestions.

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Received December 8, 1977