GENERALIZED THEORY OF IMPACTS IN PARTICULATE SYSTEMS

K. A. HÄMEEN-ANTTILA and HEIKKI SALO
Department of Astronomy, University of Oulu, Oulu, Finland

(Received 30 January 1993)

Abstract. The theory of collisional systems is generalized for an arbitrary geometry and forces acting in the system, mixtures of different particle types, friction, small deviations from the ideal spherical form, axial rotation, finite size of particles and gravitational interactions. Terms for the formation of new particles and destruction of old ones are also included, and other unspecified parameters can be introduced. Although some approximations are made to simplify the basic equations and to avoid excessive numerical integrations, a comparison with computer simulations shows a good agreement. The tests were continued up to the optical thickness $\tau = 5$.

1. Introduction

The mechanics of planetary rings and other similar systems is a natural extension of statistical mechanics, but it differs from the theory of gaseous matter in some characteristic features: the finite size of particles, partially elastic collisions and gravitational interactions cannot be neglected for macroscopic bodies, whereas molecules are usually treated as perfectly elastic, non-gravitating point masses. The free paths of the particles as compared with the dimensions of the system can also be much larger in an astronomical context than for molecules.

Due to these differences, the statistical theory of macroscopic bodies is an almost independent area of reseach. Its basic equations can be derived from several starting points. One is standard-type gas mechanics corrected for the inelasticity of impacts. This method was used by Goldreich and Tremaine (1978) for Saturn's rings, and the work was later continued by Borderies *et al.* (1982, 1983) along the same lines. Shu and Stewart (1985) used the Krook model of transitions in the phase space as their starting point. This eliminates the collisional integral and leads to an elegant mathematical theory. Araki and Tremaine (1986) and Araki (1988, 1991) introduced another modification which is based on Enskog's (1922) theory of hard-sphere gases. The finite size of particles then becomes a natural part of equations which also include the axial rotation. The results agree very precisely with computer simulations.

An entirely different approach is provided by the use of Keplerian orbital elements as variables (Hämeen-Anttila, 1978). This theory was more extensive than other early attempts, since the axial rotation of particles was also taken into account. The generalization of this work (Hämeen-Anttila, 1983, 1984, hereafter referred as Papers I and II), included more freely defined variables, an arbitrary geometry for the system instead of a disc shape, finite particles having a size

Earth, Moon, and Planets **62**: 47–84, 1993.

a size distribution, gravitational interactions, and formation or destruction of particles, but axial rotation was only discussed later, by Salo (1987a, 1987b). Gravitational interactions have also been studied in an independent fashion (Hornung *et al.*, 1985; Barge and Pellat, 1990, 1991), using a Fokker–Planck operator in Landau's form.

Collisional systems have also been studied by means of computer methods which in the first phase were direct simulations (Trulsen, 1972a, 1972b; Brahic, 1977; Hämeen-Anttila and Lukkari, 1980; Salo, 1987b). Since the modest number of particles is a serious restriction, Trulsen and Wikan (1980) introduced the Monte Carlo method for studying the Poynting–Robertson effect. An even more powerful procedure was developed by Wisdom and Tremaine (1988), who used a small co-moving cell for which the calculations were carried out. Salo (1991) employed the same method with a larger number of particles, and also included the size distribution and spin.

The following sections are a continuation and improved version of Paper II. Friction, surface irregularities and axial rotation are taken into account without neglecting the other phenomena discussed in Paper II. The finite size of particles, for which the latter only provided a semi-qualitative model, is treated in terms of Taylor expansions. The unconventional variables of Papers I and II are used here in a milder form, all the equations being valid for the position and velocity coordinates of the phase space, although the possibility for applying curvilinear systems or orbital elements is retained, since these can occasionally reduce the computational work, as seen in Section 11.

An improvement is also introduced for the calculation of certain integrals which appear in mean values and are a most inconvenient feature of collisional theories. Numerical integrations provide a straightforward solution to the problem (Goldreich and Tremaine, 1978; Araki and Tremaine, 1986; Araki, 1988; 1991), but they are time-consuming if the evolution of the system is to be calculated and each step implies multi-dimensional integrations. The method of Papers I–II and their predecessor (Hämeen-Anttila, 1978) was to use analytical functions as approximations. This was of limited accuracy, however, although it was later considerably improved by Verronen (1989). Analytical approximations are now used as well, but the error is less than 0.6% for Keplerian systems.

The results are compared with numerical simulations of discs performed with the local code (Salo, 1991). The agreement is excellent if the system is thicker than a few times the diameter of particles or if its optical thickness τ is less than unity. Some compromises between the simplicity and accuracy influence the results for fully flattened systems having $\tau > 1$, but even then the error remains tolerable as far as the simulations are carried out, or up to $\tau = 5$. More general systems than discs were not tested.

Sections 2–9 contain the derivation of basic equations. Section 10 summarizes all the necessary relations in a concise form, and Section 11 contains the corresponding equations for discs as a special case of the general theory. The comparison with numerical simulations is given in Section 12.

2. Coordinates

Let the Cartesian coordinates of the generalized phase space be $X^1, X^2 \ldots$. These include the components of the radius vector $\mathbf{R}$, velocity vector $\dot{\mathbf{R}}$, and a set of unspecified parameters q^i which may represent the rotational state or other internal properties of particles. Hence, with matrix notations,

$$\mathbf{X} = \{\mathbf{R}, \dot{\mathbf{R}}, \mathbf{q}\}. \tag{1}$$

We also introduce a set of classification parameters $\mathbf{p} = \{p^1, p^2 \ldots\}$ such as the mass, radius or chemical composition of particles, but they are assumed to be constants and are therefore not included as coordinates of the phase space.

The time-variations of $\mathbf{X}$ consist of a smooth evolution having the total derivative $\dot{\mathbf{X}} = \mathrm{d}\mathbf{X}/\mathrm{d}t$ (t denotes time) along the particle orbit and of random jumps from $\mathbf{X}$ to another state $\mathbf{X}_1$ as a consequence of collisions, gravitational encounters or other similar processes. The dot above the symbol and the subscript 1 are also used for other quantities in the same manner. The primed and unprimed symbols distinguish the interacting particles. The differences $\mathbf{X}_1 - \mathbf{X}$ and $\mathbf{X}' - \mathbf{X}$ are treated as first-order small quantities.

The curvilinear coordinates $\{\mathbf{r}, \mathbf{C}\}$ are generated by the transformations $\mathbf{r} = \mathbf{r}(\mathbf{X}, t)$ and $\mathbf{C} = \mathbf{C}(\mathbf{X}, t)$, which must be independent of $\mathbf{p}$. The first of these parameters, $\mathbf{r}$, represents the radius vector along a freely chosen reference orbit which approximates the true motion, i.e. $|\mathbf{r} - \mathbf{R}| \ll |\mathbf{R}|$, $|\dot{\mathbf{r}} - \dot{\mathbf{R}}| \ll |\dot{\mathbf{R}}|$. The components of $\partial(\mathbf{R} - \mathbf{r})/\partial\mathbf{r}$ must also be small quantities. The choice of $\mathbf{C}$ is free. Since $\mathbf{r}(\mathbf{X}, t)$ and $\mathbf{C}(\mathbf{X}, t)$ are assumed to be regular functions, $\mathbf{r}_1 - \mathbf{r}$, $\mathbf{r}' - \mathbf{r}$, $\mathbf{C}_1 - \mathbf{C}$, and $\mathbf{C}' - \mathbf{C}$ can be treated as first-order small quantities if $\mathbf{X}_1 - \mathbf{X}$ and $\mathbf{X}' - \mathbf{X}$ have this property.

The elements of matrix $\mathbf{r}$ are used as Cartesian components of the radius vector in a j-dimensional space ($0 \leqslant j \leqslant 3$). For instance, the orbital semi-major axis and mean longitude can be adopted as polar components of $\mathbf{r}$ for a thin Keplerian disc (Hämeen-Anttila, 1978). It is also possible to define $\mathbf{r} = \mathbf{R}$, $\mathbf{C} = \{\dot{\mathbf{R}}, \mathbf{q}\}$ or to identify $\mathbf{r}$ with j components of $\mathbf{R}$, in which case $\mathbf{C}$ contains $\dot{\mathbf{R}}$, $\mathbf{q}$, and $3 - j$ spatial coordinates.

If $j < 3$, the vector space $\mathbf{R}$ is split into a j-dimensional subspace $\mathbf{R}_j$ and its $(3 - j)$-dimensional co-space $\mathbf{s}$. The former contains the $\mathbf{r}$ vectors while $\mathbf{s}$ $(= \mathbf{R} - \mathbf{R}_j)$ is perpendicular to them. The matrix

$$\mathbf{x} = \{\mathbf{s}, \dot{\mathbf{R}}, \mathbf{q}\} \tag{2}$$

represents the Cartesian counterpart of $\mathbf{C}$. For a thin disc, $\mathbf{R}_j$ is the 2-dimensional radius vector on the equatorial plane and $\mathbf{s}$ degenerates to a scalar which measures the vertical distance from it (the coordinate z). If $j = 3$, then $\mathbf{R}_j = \mathbf{R}$ and $\mathbf{s} = 0$. Both $\mathbf{r} - \mathbf{R}_j$ and $\mathbf{s}$ are treated as first-order small quantities.

3. Generalized Boltzmann's Equation

The mean value of a quantity ξ for the fixed $\mathbf{r}$ and $\mathbf{p}$ is defined by the relations

$$n\bar{\xi} = \int N\xi D \, \mathrm{d}\mathbf{C} \,, \qquad D = |\partial(\mathbf{R}, \dot{\mathbf{R}}, \mathbf{q})/\partial(\mathbf{r}, \mathbf{C})| \,, \tag{3}$$

n and N standing for the particle densities in the vector spaces $\mathbf{r}$ and $\mathbf{X}$; respectively. Equations (3) also determine n, since $\xi = 1$ must yield $\bar{\xi} = 1$. If $\mathbf{r}$ is 0-dimensional, n represents the total number of particles in the system.

The generalized Boltzmann's equation for curvilinear coordinates is (cf. Papers I and II)

$$\left[\partial(ND)/\partial t + \nabla \cdot (N\dot{\mathbf{r}}D) + \sum_j \partial(N\dot{C}^i D)/\partial C^i\right]/D = \epsilon N + (\mathrm{d}N/\mathrm{d}t)_R \,, \tag{4}$$

where ∇ stands for the operator $\partial/\partial\mathbf{r}$ and ϵN for the rate of creation of new particles per unit volume of the phase space. The last term is a generalization of the collisional integral for arbitrary random jumps.

Equation (4) gives

$$\partial(N\xi D)/\partial t + \nabla \cdot (N\dot{\mathbf{r}}\xi D) + \sum_i \partial(N\dot{C}^i \xi D)/\partial C^i = = [N(\dot{\xi} + \epsilon\xi) + \xi(\mathrm{d}N/\mathrm{d}t)_R]D \,, \tag{5}$$

and if $N\xi$ vanishes rapidly enough for $|\mathbf{C}| \to \infty$, Equations (3) and (5) yield

$$\partial(n\bar{\xi})/\partial t + \nabla \cdot (n\overline{\dot{\mathbf{r}}\xi}) = n(\overline{\epsilon\xi} + \bar{\dot{\xi}}) + \int \xi(\mathrm{d}N/\mathrm{d}t)_R D \, \mathrm{d}\mathbf{C} \,. \tag{6}$$

If $w(\mathbf{r}, \mathbf{C}, \Delta\mathbf{r}, \Delta\mathbf{C}) \, \mathrm{d}\Delta\mathbf{r} \, \mathrm{d}\Delta\mathbf{C} \, \mathrm{d}\mathbf{C}$ represents the probability of a particle jumping from $\{\mathbf{r}, \mathbf{C}\}$ to $\{\mathbf{r} + \Delta\mathbf{r}, \mathbf{C} + \Delta\mathbf{C}\}$, the last term of Equation (6) contains a negative contribution $-\iiint N\xi w D \, \mathrm{d}\Delta\mathbf{r} \, \mathrm{d}\Delta\mathbf{C} \, \mathrm{d}\mathbf{C}$ for jumps away from $\{\mathbf{r}, \mathbf{C}\}$ and a positive one for the transitions $\{\mathbf{r} - \Delta\mathbf{r}, \mathbf{C} - \Delta\mathbf{C}\} \to \{\mathbf{r}, \mathbf{C}\}$. The latter is calculated from a similar integral, but ξ and w are replaced by ξ_1 and $w(\mathbf{r} - \Delta\mathbf{r}, \mathbf{C} - \Delta\mathbf{C}, \Delta\mathbf{r}, \Delta\mathbf{C})$. The integration for $\mathbf{C}$ suppresses the difference between $\mathbf{C} - \Delta\mathbf{C}$ and $\mathbf{C}$, while $\mathbf{r} - \Delta\mathbf{r}$ is treated in terms of power expansions (Paper II). Hence,

$$\partial(n\bar{\xi})/\partial t + \nabla\cdot\Bigg(n\overline{\dot{\mathbf{r}}\xi} + \int\int\int N\,\Delta\mathbf{r}\xi_1 wD\,\mathrm{d}\Delta\mathbf{r}\,\mathrm{d}\Delta\mathbf{C}\,\mathrm{d}\mathbf{C} - $$

$$-\frac{1}{2}\nabla\cdot\int\int\int N\,\Delta\mathbf{r}\,\Delta\mathbf{r}\xi_1 wD\,\mathrm{d}\Delta\mathbf{r}\,\mathrm{d}\Delta\mathbf{C}\,\mathrm{d}\mathbf{C} + \cdots$$

$$= n(\overline{\epsilon\xi} + \bar{\dot{\xi}}) + \int\int\int N(\xi_1 - \xi)wD\,\mathrm{d}\Delta\mathbf{r}\,\mathrm{d}\Delta\mathbf{C}\,\mathrm{d}\mathbf{C}\Bigg)\,. \quad (7)$$

This is the equation of continuity for ξ.

4. Binary Interactions

If the jumps $\{\Delta\mathbf{r}, \Delta\mathbf{C}\}$ follow from binary interactions, w is proportional to the particie density. Changing the variables, we have for an arbitrary function ζ the relation

$$\int\int\int N\zeta wD\,\mathrm{d}\Delta\mathbf{r}\,\mathrm{d}\Delta\mathbf{C}\,\mathrm{d}\mathbf{C} =$$

$$= \int\int\int\int N(\mathbf{r},\ldots)N(\mathbf{r}',\ldots)\,\psi\Big(\frac{\mathbf{r}+\mathbf{r}'}{2}, \mathbf{r}'-\mathbf{r},\ldots\Big)\times$$

$$\times\,\zeta\,\mathrm{d}(\mathbf{r}'-\mathbf{r})\,\mathrm{d}\mathbf{C}\,\mathrm{d}\mathbf{C}'\,\mathrm{d}\mathbf{p}'\,, \quad (8)$$

in which ψ denotes the probability of interaction. The dots stand for $\mathbf{C}$, $\mathbf{C}'$, $\mathbf{p}$, $\mathbf{p}'$ and t. The Jacobian determinants D and D' are included in ψ. The variables $(\mathbf{r}+\mathbf{r}')/2$ and $\mathbf{r}'-\mathbf{r}$ facilitate the use of power expansions, since ψ varies smoothly with $(\mathbf{r}+\mathbf{r}')/2$ but can diminish abruptly with $\mathbf{r}'-\mathbf{r}$. The collisions are an extreme case in which ψ is discontinuous with respect to the mutual distances between particles.

Let $\mathbf{r}_0 = (\mathbf{r}+\mathbf{r}')/2$. Treating it as an independent variable, we expand $N(\mathbf{r}',\ldots)$ and $\psi\zeta$ according to the powers of $\mathbf{r}'-\mathbf{r}$ and $\mathbf{r}_0-\mathbf{r}$, respectively. Since $\mathbf{r}_0-\mathbf{r} = (\mathbf{r}'-\mathbf{r})/2$, the integrand in Equation (8) formally becomes a Taylor expansion according to the powers of $\mathbf{r}'-\mathbf{r}$, although this difference also appears as an argument of $\psi\zeta$ in the coefficients. The main term is used to define the mean value $\langle\zeta\rangle$ for random jumps: i.e.,

$$n\nu\Big\langle\zeta\Big(\frac{\mathbf{r}+\mathbf{r}'}{2}, \mathbf{r}'-\mathbf{r},\ldots\Big)\Big\rangle = \int\int\int NN'\psi(\mathbf{r}, \mathbf{r}'-\mathbf{r},\ldots)$$

$$\times\,\zeta(\mathbf{r}, \mathbf{r}'-\mathbf{r},\ldots)\,\mathrm{d}(\mathbf{r}'-\mathbf{r})\,\mathrm{d}\mathbf{C}\,\mathrm{d}\mathbf{C}'\,. \quad (9)$$

The notation N' represents $N(\mathbf{r}', \mathbf{C}'\,\mathbf{p}', t)$ at $\mathbf{r}' = \mathbf{r}$. The frequency of jumps, $\nu(\mathbf{r}, \mathbf{p}, \mathbf{p}', t)$, follows from the condition $\langle 1\rangle = 1$. The next terms of the expansion also contribute to Equation (8) and are obtained from Paper II, in which $\langle\ \rangle_r$ corresponds to our $\langle\ \rangle$. Hence

$$\int\int\int N\zeta wD\,\mathrm{d}\Delta\mathbf{r}\,\mathrm{d}\Delta\mathbf{C}\,\mathrm{d}\mathbf{C} = \int n\nu\langle\zeta[1 + (\mathbf{r}' - \mathbf{r})\cdot\nabla\, log\sqrt{N'/N}\rangle\,\mathrm{d}\mathbf{p}' + + \frac{1}{2}\nabla\cdot\int n\nu\langle(\mathbf{r}' - \mathbf{r})\zeta\rangle\,\mathrm{d}\mathbf{p}' + \cdots . \tag{10}$$

Following the same paper with slightly modified notations we define for arbitrary quantities ξ and η the abbreviations

$$S(\xi) = \int \nu\langle(\xi_1 - \xi)[1 + (\mathbf{r}' - \mathbf{r})\cdot\nabla\log\sqrt{N'/N}]\rangle\,\mathrm{d}\mathbf{p}' ,$$

$$W(\xi\eta) = \int (\nu/2)[\langle(\xi_1 - \xi)(\eta_1 - \eta) + (\xi_1 - \xi)\eta + + \xi(\eta_1 - \eta)\rangle - \langle\xi_1 - \xi\rangle\bar{\eta} - \bar{\xi}\langle\eta_1 - \eta\rangle]\,\mathrm{d}\mathbf{p}' \tag{11}$$

and replace Equation (7) by the relation

$$\partial(n\bar{\xi})/\partial t + \nabla\cdot\{n[\overline{\dot{\mathbf{r}}\xi} + S(\mathbf{r})\bar{\xi} + 2W(\mathbf{r}\xi)] - \nabla\cdot[nW(\mathbf{rr})\bar{\xi}]\} = = n[\overline{\epsilon\xi} + \bar{\dot{\xi}} + S(\xi)] , \tag{12}$$

which is obtained from Equations (7)–(11) if one observes that $(\mathbf{r} + \mathbf{r}')/2$ inside $\langle\ \rangle$ is equivalent to $\mathbf{r}$ outside it and $\Delta\mathbf{r} = \mathbf{r}_1 - \mathbf{r}$. The truncation of expansions after the second-order terms implies

$$|\xi_1 - \xi|, |\xi' - \xi| \ll |\xi|, |\xi'| \tag{13}$$

unless ξ is itself a small quantity.

The expressions for S and W depend on the approximative distribution function and interaction mechanism of the particles. The former is introduced in Section 5 and the latter in Sections 6–8.

5. Distribution Function

We assume

$$N = \frac{n\exp[-(\mathbf{C} - \bar{\mathbf{C}})\cdot(\overline{\mathbf{CC}} - \bar{\mathbf{C}}\bar{\mathbf{C}})^{-1}\cdot(\mathbf{C} - \bar{\mathbf{C}})/2]}{D\sqrt{(2\pi)^k\,\mathrm{Det}(\overline{\mathbf{CC}} - \bar{\mathbf{C}}\bar{\mathbf{C}})}} , \tag{14}$$

where k stands for the number of elements in $\mathbf{C}$. This expression agrees with Equations (3), since $\xi = 1$ gives $\bar{\xi} = 1$, and with an appropriate $\mathbf{C}$ it also approximates the empirical distribution function derived from computer simulations for

Keplerian discs (Trulsen, 1972a; Lukkari, 1978) provided that they are not too dense (Wisdom and Tremaine, 1988).

Since the matrix **x** which is defined by Equation (2) contains the same number of elements as **C**, we can use **r** and **x** as free variables instead of **r** and **C**. This involves the transformation $\mathbf{C} = \mathbf{C}(\mathbf{r}, \mathbf{x}, \mathbf{t})$, which is expressed in terms of the Taylor expansion

$$\mathbf{C} = \mathbf{C}_0 + (\mathbf{x} - \mathbf{x}_0) \cdot \left(\frac{\partial \mathbf{C}}{\partial \mathbf{x}}\right)_0 + \cdots . \tag{15}$$

Substitution of $\bar{\mathbf{x}}$ for $\mathbf{x}_0$ gives $\mathbf{C}_0 = \bar{\mathbf{C}}$ and $(\partial \mathbf{C}/\partial \mathbf{x})_0 = \overline{\partial \mathbf{C}/\partial \mathbf{x}}$. Hence,

$$\begin{aligned} \mathbf{C} - \bar{\mathbf{C}} &\cong (\mathbf{x} - \bar{\mathbf{x}}) \cdot \frac{\overline{\partial \mathbf{C}}}{\partial \mathbf{x}}, \\ \overline{\mathbf{C}\mathbf{C}} - \bar{\mathbf{C}}\bar{\mathbf{C}} &\cong \left(\frac{\overline{\partial \mathbf{C}}}{\partial \mathbf{x}}\right)^{\dagger} \cdot (\overline{\mathbf{x}\mathbf{x}} - \bar{\mathbf{x}}\bar{\mathbf{x}}) \cdot \frac{\overline{\partial \mathbf{C}}}{\partial \mathbf{x}}; \end{aligned} \tag{16}$$

and if D is decomposed to the product of $\partial(\mathbf{R}_j, \mathbf{x})/\partial(\mathbf{r}, \mathbf{x})$ and $\partial(\mathbf{r}, \mathbf{x})/\partial(\mathbf{r}, \mathbf{C})$, one finds $D \cong |\mathrm{Det}(\partial \mathbf{x}/\partial \mathbf{C})|$. Equations (14) and (16) then give the expression

$$N = \frac{n \exp[-(\mathbf{x} - \bar{\mathbf{x}}) \cdot (\overline{\mathbf{x}\mathbf{x}} - \bar{\mathbf{x}}\bar{\mathbf{x}})^{-1} \cdot (\mathbf{x} - \bar{\mathbf{x}})/2]}{\sqrt{(2\pi)^k \,\mathrm{Det}(\overline{\mathbf{x}\mathbf{x}} - \bar{\mathbf{x}}\bar{\mathbf{x}})}}, \tag{17}$$

which does not depend on the choice of **C**. This is not only a useful relation, but it shows that Equation (14) has the same content for any system of coordinates **C** in the phase space.

The invariance of N would be lost in the next approximation which implies a distribution function depending on the constants of motion according to a definite (but unknown) law. Hence, if Equation (14) or (17) is adopted, the expansions (7) and (10) must be truncated without trying to calculate new terms for Equation (12). The randomness of jumps usually reduces $\langle \xi_1 - \xi \rangle$ in $S(\xi)$ to a small quantity and justifies the simultaneous use of second-order terms. Otherwise these also ought to be neglected.

The lack of an accurate distribution function is a fundamental problem which appears in other approximation methods as well, e.g., the Krook equation (Shu and Stewart, 1985), which avoids the Taylor expansions (7) and (10), includes the counterpart of Equation (17) as a basic assumption. The consequences are the same as here.

6. Impacts

The particles are assumed to be spherical, although small deviations from this ideal shape are permitted later. The matrix **p** contains the mass m, radius σ and elastic properties of particles, while $\mathbf{q}/\sigma$ represents the vectorial angular velocity

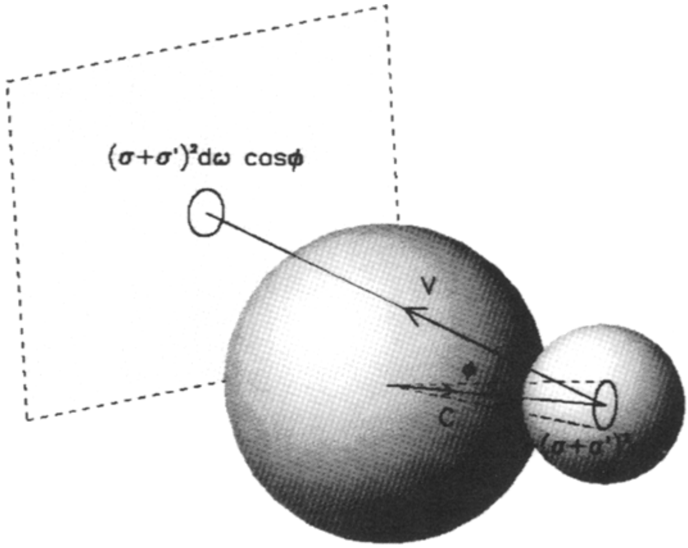

Fig. 1. Two impacting particles.

$\boldsymbol{\omega}$ of their rotational motion. The relative orbital velocity of the particles is $\mathbf{v} = \dot{\mathbf{R}}' - \dot{\mathbf{R}}$.

To construct Equation (9) for impacts we need the probability of a given particle P of type $\mathbf{p}$ hitting one of those particles P' which belong to the type $\mathbf{p}'$ and are located in the cell $\mathrm{d}\mathbf{v}\,\mathrm{d}\mathbf{q}'$. Since a slab of area A and thickness $\mathrm{d}h$ contains $N'\,\mathrm{d}\mathbf{v}\,\mathrm{d}\mathbf{q}' \cdot A\,\mathrm{d}h$ target particles having the collisional cross-section $\pi(\sigma + \sigma')^2$, the probability of impact, Π_1, is equal to $N'\,\mathrm{d}\mathbf{v}\,\mathrm{d}\mathbf{q}' \cdot A\,\mathrm{d}h \cdot \pi(\sigma + \sigma')^2$ divided by A if P penetrates the slab in the direction of $\mathrm{d}h$. Inserting $\mathrm{d}h = |\mathbf{v}|\,\mathrm{d}t$, we have

$$\Pi_1 = \pi(\sigma + \sigma')^2 N'|\mathbf{v}|\,\mathrm{d}t\,\mathrm{d}\mathbf{v}\,\mathrm{d}\mathbf{q}' \,. \tag{18}$$

If P and P' collide (Figure 1), then $\mathbf{R}' - \mathbf{R} = (\sigma + \sigma')\mathbf{c}$, where $\mathbf{c}$ denotes the unit vector pointing from the centre of P to that of P'. The angle between $-\mathbf{c}$ and $\mathbf{v}$ is ϕ. The probability Π_2 of $\mathbf{c}$ being inside the solid angle $\mathrm{d}\omega$ is given by the ratio of the projected area $(\sigma + \sigma')^2 \cos\phi\; \mathrm{d}\omega$ to the collisional cross-section $\pi(\sigma + \sigma')^2$:

$$\Pi_2 = \cos\phi\,\mathrm{d}\omega/\pi = (-\mathbf{c}) \cdot \mathbf{v}\,\mathrm{d}\omega/\pi|\mathbf{v}| \,. \tag{19}$$

The total number of impacts for those particles of type $\mathbf{p}$ which are located in the cell $\mathrm{d}\mathbf{x}(= \mathrm{d}\mathbf{s}\,\mathrm{d}\dot{\mathbf{R}}\,\mathrm{d}\mathbf{q})$ would be $\Pi_1\Pi_2 N\,\mathrm{d}\mathbf{x}$ if the particles were statistically independent of each other, but having a finite size, they and the target particles occupy some amount of the available volume. If the fraction of free space is $1/g$, the probability of impacts becomes g-fold. The collisional mean value of a function ζ and the impact frequency ν are therefore defined by the expressions

$$\langle \zeta \rangle = \int \zeta g \Pi_1 \Pi_2 N \, d\mathbf{x} \Big/ \int g \Pi_1 \Pi_2 N \, d\mathbf{x} \,, \qquad n\nu \, dt = \int g \Pi_1 \Pi_2 N \, d\mathbf{x} \,. \tag{20}$$

The integrations are carried out for those combinations of variables which give $(-\mathbf{c}) \cdot \mathbf{v} \geqslant 0$. Coagulations and fragmentations of particles introduce additional restrictions, which are included later (Section 10D). Inserting the expressions (18) and (19), one finds

$$n\nu \langle \zeta \rangle = \int_{c \cdot v \leqslant 0} \zeta g N N'(\sigma + \sigma')^2 (-\mathbf{c}) \cdot \mathbf{v} \, d\omega \, d\mathbf{x} \, d\mathbf{v} \, d\mathbf{q}' \,. \tag{21}$$

The evaluation of g leads to a complicated problem, since not all the voids of arbitrary size and shape can be used freely for the motion of particles. Ignoring the details, we assume that

$$1/g = 1 - \int (4\pi \overline{n'_R} \sigma'^3 / 3\delta_{max}) \, d\mathbf{p}' \,, \tag{22}$$

where δ_{max} stands for the maximum attainable packing density and n_R for the number of particles per unit volume in the $\mathbf{R}$ space. The mean value $\overline{n_R}$ can be used, since g is anyway approximative.

The quantities $\dot{\mathbf{R}}_1 - \dot{\mathbf{R}}$ and $\mathbf{q}_1 - \mathbf{q}$ $(= \sigma \boldsymbol{\omega}_1 - \sigma \boldsymbol{\omega})$ are calculated from the conservation laws and properties of collisions.

The conservation of momentum gives

$$m\dot{\mathbf{R}}_1 + m'\dot{\mathbf{R}}'_1 = m\dot{\mathbf{R}} + m'\dot{\mathbf{R}}' \,. \tag{23}$$

If the particles are homogeneous spheres, the rotational angular momentum is $2m\sigma^2\boldsymbol{\omega}/5$, and hence,

$$\begin{aligned} &m\mathbf{R} \times \dot{\mathbf{R}}_1 + m'\mathbf{R}' \times \dot{\mathbf{R}}'_1 + 2(m\sigma\mathbf{q}_1 + m'\sigma'\mathbf{q}'_1)/5 \\ &\quad = m\mathbf{R} \times \dot{\mathbf{R}} + m'\mathbf{R}' \times \dot{\mathbf{R}}' + 2(m\sigma\mathbf{q} + m'\sigma'\mathbf{q}')/5 \,. \end{aligned} \tag{24}$$

The change of the rotational angular momentum, $2m\sigma(\mathbf{q}_1 - \mathbf{q})/5$, follows from the influence of torque during the collision. Since the forces acting on both particles are identical except in their sign, the ratio of torques is $\sigma' : \sigma$. This gives

$$m(\mathbf{q}_1 - \mathbf{q}) = m'(\mathbf{q}'_1 - \mathbf{q}') \,. \tag{25}$$

The velocity differences at the impact point before and after the collision are

$$\mathbf{v}^* = \mathbf{v} - (\mathbf{q} + \mathbf{q}') \times \mathbf{c} \,, \qquad \mathbf{v}_1^* = \mathbf{v}_1 - (\mathbf{q}_1 + \mathbf{q}'_1) \times \mathbf{c} \,, \tag{26}$$

and since $\mathbf{c}$, $\mathbf{c} \times (\mathbf{v}^* \times \mathbf{c})$ and $\mathbf{v}^* \times \mathbf{c}$ are perpendicular to each other, any arbitrary vector can be expressed as a linear combination of these. Introducing the scalar coefficients $-\alpha\mathbf{c} \cdot \mathbf{v}^*$, $1 - \beta$, and γ, we thus find that

$$\mathbf{v}_1^* = -\alpha\mathbf{c}\mathbf{c} \cdot \mathbf{v}^* + (1 - \beta)\mathbf{c} \times (\mathbf{v}^* \times \mathbf{c}) + \gamma\mathbf{v}^* \times \mathbf{c} \tag{27}$$

and, accordingly,

$$\begin{aligned} &\mathbf{c} \cdot \mathbf{v}_1^* = -\alpha\mathbf{c} \cdot \mathbf{v}^* , \\ &\mathbf{c} \times (\mathbf{v}_1^* \times \mathbf{c}) = (1 - \beta)\mathbf{c} \times (\mathbf{v}^* \times \mathbf{c}) + \gamma\mathbf{v}^* \times \mathbf{c} . \end{aligned} \tag{28}$$

Therefore α denotes the coefficient of restitution, and if $\gamma = 0$, $1 - \beta$ corresponds to the friction. If $\gamma \neq 0$, $\mathbf{v}_1^*$ contains a non-zero component in the direction perpendicular to $\mathbf{v}^*$ and $\mathbf{c}$. The resulting asymmetry can only be avoided if γ is a random variable having $\bar{\gamma} = 0$, which represents the influence of surface irregularities. Since these also act in the direction of $\mathbf{c} \times (\mathbf{v}^* \times \mathbf{c})$, a similar random component must be included in β. Two colliding particles always have opposite values of γ. The coefficients α, β, and γ may depend on $\mathbf{v}^*$ and $\mathbf{c}$.

Equations (23)–(27) with $\mathbf{R}' = \mathbf{R} + (\sigma + \sigma')\mathbf{c}$ and $\dot{\mathbf{R}}' = \dot{\mathbf{R}} + \mathbf{v}$ give

$$\begin{aligned} \dot{\mathbf{R}}_1 - \dot{\mathbf{R}} &= [2m'/(m + m')][(1 + \alpha)\mathbf{c}\mathbf{c}/2 + \beta(\mathbf{I}_3 - \mathbf{c}\mathbf{c})/7 + \\ &\quad + \gamma\mathbf{c} \times \mathbf{I}_3/7] \cdot [\mathbf{v} - (\mathbf{q} + \mathbf{q}') \times \mathbf{c}] , \\ \mathbf{q}_1 - \mathbf{q} &= 5\mathbf{c} \times (\dot{\mathbf{R}}_1 - \dot{\mathbf{R}})/2 , \end{aligned} \tag{29}$$

where $\mathbf{I}_3$ stands for the 3-dimensional unit tensor.

Since the rotational energy of a homogeneous sphere is $m\mathbf{q}^2/5$, the change in total energy (translatory + rotational) for a colliding pair of particles is

$$\begin{aligned} \Delta E = -[mm'/(m + m')]\{&(1 - \alpha^2)(\mathbf{c} \cdot \mathbf{v}^*)^2/2 + \\ &+ [1 - (1 - \beta)^2 - \gamma^2](\mathbf{c} \times \mathbf{v}^*)^2/7\} . \end{aligned} \tag{30}$$

This quantity must be either negative or zero.

7. Impacts of Gravitating Particles

The gravitational attraction between mutually colliding particles enlarges their effective cross-sections, increases the impact velocity, reduces the post-collisional velocity and modifies the distribution of $\mathbf{c}$. The last-mentioned effect is not taken into account, and may produce considerable errors for low-velocity impacts, but their treatment is anyway approximative, since we are using the two-body method (for its accuracy, see Wetherill and Cox, 1984, 1985). This approximation is obviously invalidated in the case of dense matter.

The velocities $\mathbf{v}$ and $\mathbf{v}_1$ are now interpreted as asymptotic velocity differences before and after the impact, while $\mathbf{v}_{col}$ and $(\mathbf{v}_{col})_1$ correspond to the instant of collision.

The standard expression for a gravitationally enlarged collision cross-section is

$$[\pi(\sigma+\sigma')^2]_{grav} = (1+2\theta)\pi(\sigma+\sigma')^2\,,$$
$$\theta = G(m+m')/\mathbf{v}^2(\sigma+\sigma')\,, \tag{31}$$

where G stands for Newton's constant of gravitation. This expression is to be substituted for $\pi(\sigma+\sigma')^2$ in Equation (21).

The energy principle as implied by the two-body problem gives the expressions $\mathbf{v}^2(1+2\theta)$ and $\mathbf{v}_1^2+2\theta\mathbf{v}^2$ for $\mathbf{v}_{col}^2$ and $(\mathbf{v}_{col})_1^2$, respectively. Hence, as a rather crude approximation,

$$(\mathbf{c}\cdot\mathbf{v}_{col})^2 \cong (\mathbf{c}\cdot\mathbf{v})^2(1+2\theta)\,,$$
$$(\mathbf{c}\cdot\mathbf{v}_{col})_1^2 \cong (\mathbf{c}\cdot\mathbf{v}_1)^2 + 2\theta(\mathbf{c}\cdot\mathbf{v})^2\,. \tag{32}$$

The error can be large in individual cases, but on average the results are correct for a randomly varying **c**.

Since the rotational part of $\mathbf{v}^*$ does not contribute to the first of Equations (28), the true coefficient of restitution, α_{col}, can also be defined in the form $-(\mathbf{c}\cdot\mathbf{v}_{col})_1/\mathbf{c}\cdot\mathbf{v}_{col}$, while the effective coefficient is $\alpha = -\mathbf{c}\cdot\mathbf{v}_1/\mathbf{c}\cdot\mathbf{v}$. Using these expressions in Equations (32) we find that

$$\alpha = \sqrt{(1+2\theta)\alpha_{col}^2 - 2\theta}\,. \tag{33}$$

The imaginary α for $2\theta > \alpha_{col}^2/(1-\alpha_{col}^2)$ corresponds to the gravitational coagulation.

Since any arbitrary vector can be expressed as a linear combination of **c**, $\mathbf{c}\times(\mathbf{v}^*\times\mathbf{c})$, and $\mathbf{v}^*\times\mathbf{c}$ (Section 6), it would also be possible to modify β and γ for the mutual gravitational attraction of particles, but this is not done in the present paper.

8. Gravitational Encounters

A. Gravitational random walk

Slightly modifying the calculations of Papers I and II we derive the influence of encounters from the fluctuating force field and dynamical friction. The former produces a random-walk process which can be described in terms of Equation (7). The mean value $\langle\xi\rangle$ is thereby generalized for encounters according to the relation

$$\int\int\int N\xi w D\,\mathrm{d}\Delta\mathbf{r}\,\mathrm{d}\Delta\mathbf{C}\,\mathrm{d}\mathbf{C} = n\int(\langle\xi\rangle/\Delta t)\,\mathrm{d}\mathbf{p}'\,, \tag{34}$$

in which Δt denotes the average duration of steps and corresponds to the inverse collisional frequency $1/\nu$. The subscript 1 refers to the change from one step to the next, and ξ_1 is calculated from the Taylor expansion

$$\xi_1 = \xi + (\mathbf{X}_1 - \mathbf{X}) \cdot \frac{\partial \xi}{\partial \mathbf{X}} + \frac{1}{2}[(\mathbf{X}_1 - \mathbf{X})(\mathbf{X}_1 - \mathbf{X})] \cdot\cdot \frac{\partial^2 \xi}{\partial \mathbf{X} \partial \mathbf{X}} + \cdots, \tag{35}$$

in which the double scalar product $\cdot\cdot$ denotes the operation $\mathbf{P} \cdot\cdot \mathbf{Q} = \{P_{\ldots ij} Q^{ji \cdots}\}$ for arbitrary tensors $\mathbf{P}$ and $\mathbf{Q}$.

The mean value of $\mathbf{X}_1 - \mathbf{X}$ is assumed to vanish at each point separately. Since the expansion is truncated after the second-order term, $\partial^2\xi/\partial\mathbf{X}\partial\mathbf{X}$ can be replaced by $\overline{\partial^2\xi/\partial\mathbf{X}\partial\mathbf{X}}$. Thereafter $\langle \xi_1 - \xi \rangle$ is reduced to the calculation of $\langle (\mathbf{X}_1 - \mathbf{X})(\mathbf{X}_1 - \mathbf{X}) \rangle$. The expressions $\langle (\mathbf{r}_1 - \mathbf{r})\xi_1 \rangle$ and $\langle (\mathbf{r}_1 - \mathbf{r})(\mathbf{r}_1 - \mathbf{r})\xi_1 \rangle$ are constructed in the same manner, and if we define

$$\begin{aligned} S(\xi) &= \left[\int \frac{\langle (\mathbf{X}_1 - \mathbf{X})(\mathbf{X}_1 - \mathbf{X}) \rangle}{2\Delta t} \,\mathrm{d}\mathbf{p}' \right] \cdot\cdot \overline{\frac{\partial^2 \xi}{\partial \mathbf{X} \partial \mathbf{X}}}, \\ W(\xi\eta) &= \left(\overline{\frac{\partial \xi}{\partial \mathbf{X}}} \right)^{\dagger} \cdot \left[\int \frac{\langle (\mathbf{X}_1 - \mathbf{X})(\mathbf{X}_1 - \mathbf{X}) \rangle}{2\Delta t} \,\mathrm{d}\mathbf{p}' \right] \cdot \overline{\frac{\partial \eta}{\partial \mathbf{X}}}, \end{aligned} \tag{36}$$

Equation (7) becomes identical to Equation (12). Using the same expansions, one can also express S and W in the form (11).

The gravitational field $\mathbf{G}(\mathbf{R}, t)$ acting at $\mathbf{R}$ is

$$\mathbf{G}(\mathbf{R}, t) = \sum_i G m_i (\mathbf{R}_i - \mathbf{R})/|\mathbf{R}_i - \mathbf{R}|^3 , \tag{37}$$

where i numerates the particles. Since $\mathbf{G}\Delta t$ represents the change of velocity during Δt, $W(\dot{\mathbf{R}}\dot{\mathbf{R}})$ can be calculated from $\int \langle \mathbf{G}\mathbf{G} \rangle \Delta t \,\mathrm{d}\mathbf{p}'/2$. The averaging refers to those statistically equivalent distributions of particles which differ only in the random choice of vectors $\mathbf{R}_i - \mathbf{R}$. The mean values of terms $\sim (\mathbf{R}_i - \mathbf{R})(\mathbf{R}_j - \mathbf{R})$ therefore vanish if $i \neq j$, while those having $i = j$ yield the tensor $(\mathbf{R}_i - \mathbf{R})^2 \mathbf{I}_3/3$. Since the summing and averaging correspond to an integration, one finds

$$\langle \mathbf{G}\mathbf{G} \rangle = \int (G^2 m'^2 \mathbf{I}_3 n'_R / 3|\mathbf{R}' - \mathbf{R}|^4) \,\mathrm{d}(\mathbf{R}' - \mathbf{R}) . \tag{38}$$

If $\overline{n'_R}$ is substituted for n'_R, the integration from an unspecified lower bound $|\mathbf{R}' - \mathbf{R}|_{min} = \rho$ to ∞ gives

$$W(\dot{\mathbf{R}}\dot{\mathbf{R}}) = \int \langle \mathbf{G}\mathbf{G} \rangle \Delta t \,\mathrm{d}\mathbf{p}'/2 = \int (2\pi G^2 m'^2 \overline{n'_R} \mathbf{I}_3 \Delta t / 3\rho) \,\mathrm{d}\mathbf{p}' . \tag{39}$$

The same method also provides $\langle \dot{\mathbf{G}}^2 \rangle$ if $\dot{\mathbf{G}}$ is first calculated from Equation (37), and we have

$$\langle \dot{\mathbf{G}}^2 \rangle / \langle \mathbf{G}^2 \rangle = 2 \langle \mathbf{v}^2 \rangle / 3\rho^2 . \tag{40}$$

Another expression for this ratio follows from the Fourier integral of $\mathbf{G}$, $\int \mathbf{A} \exp(2\pi i f t) \,\mathrm{d}f$, and from its derivative with respect to t. Their time-averages $\int \mathbf{A} \cdot \mathbf{A}^* \mathrm{d}f$ and $\int 4\pi^2 f^2 \mathbf{A} \cdot \mathbf{A}^* \mathrm{d}f$ ($\mathbf{A}^*$ is the complex conjugate of $\mathbf{A}$) are essentially identical to $\langle \mathbf{G}^2 \rangle$ and $\langle \dot{\mathbf{G}}^2 \rangle$. If the mean-square frequency $\int f^2 \mathbf{A} \cdot \mathbf{A}^* \, df / \int \mathbf{A} \cdot \mathbf{A}^* \, df$ is used as an estimate for $1/\Delta t^2$, a comparison with Equation (40) gives $\Delta t/\rho = \pi \sqrt{6/\langle \mathbf{v}^2 \rangle}$, and hence, according to Equation (39),

$$W(\dot{\mathbf{R}}\dot{\mathbf{R}}) = \int (2\pi^2/3) G^2 m'^2 \overline{n'_R} \mathbf{I}_3 \sqrt{6/\langle \mathbf{v}^2 \rangle} \, d\mathbf{p}' \,. \tag{41}$$

The result is independent of ρ.

If the encounters are rapid processes, $\mathbf{R}_1 - \mathbf{R}$ is much smaller than the dimensions of the system and can be neglected. Since we also assume $\mathbf{q}_1 - \mathbf{q} = 0$, the transformations (36) determine $S(\xi)$ and $W(\xi\eta)$ as functions of $W(\dot{\mathbf{R}}\dot{\mathbf{R}})$.

B. Dynamical friction

The random walk produces an illusory growth in kinetic energy, which must be balanced by the dynamical friction. This is proportional to the relative velocity of the particle with respect to the local mean motion, but the notations are simpler if we first use the generalized deceleration $\mathbf{\Lambda} \cdot (\langle \mathbf{X}' \rangle - \mathbf{X})$ with a tensorial coefficient $\mathbf{\Lambda}$.

The friction influences the terms $\bar{\dot{\xi}}$ and $\overline{\dot{\mathbf{r}}\xi}$ in Equation (12). Since $\dot{\xi}$ is equivalent to $\partial\xi/\partial t + \dot{\mathbf{X}} \cdot (\partial\xi/\partial\mathbf{X})$, the contribution of the friction is

$$\bar{\dot{\xi}}_f = \int \overline{(\langle \mathbf{X}' \rangle - \mathbf{X}) \cdot \mathbf{\Lambda}^\dagger \cdot \frac{\partial \xi}{\partial \mathbf{X}}} \, d\mathbf{p}' \,. \tag{42}$$

The integrand represents a binary interaction, and we can therefore use relations which correspond to Equations (8)–(10). Hence,

$$n\bar{\dot{\xi}}_f = \int n \left\langle (\mathbf{X}' - \mathbf{X}) \cdot \mathbf{\Lambda}^\dagger \cdot \frac{\partial \xi}{\partial \mathbf{X}} \left[1 + (\mathbf{r}' - \mathbf{r}) \cdot \nabla \log \sqrt{\frac{N'}{N}} \right] \right\rangle d\mathbf{p}' + $$
$$+ \frac{1}{2} \nabla \cdot \int n \left\langle (\mathbf{r}' - \mathbf{r})(\mathbf{X}' - \mathbf{X}) \cdot \mathbf{\Lambda}^\dagger \cdot \frac{\partial \xi}{\partial \mathbf{X}} \right\rangle d\mathbf{p}' \,. \tag{43}$$

A similar expression is found for $\overline{\dot{\mathbf{r}}_f \xi}$.

Let $\mathbf{X}_0$ represent the mean value of $\mathbf{X}$ for the point $(\mathbf{r} + \mathbf{r}')/2$. Inserting the power expansion

$$\frac{\partial \xi}{\partial \mathbf{X}} = \left(\frac{\partial \xi}{\partial \mathbf{X}} \right)_0 + (\mathbf{X} - \mathbf{X}_0) \cdot \left(\frac{\partial^2 \xi}{\partial \mathbf{X} \partial \mathbf{X}} \right)_0 + \cdots \tag{44}$$

and taking into account the fact that $(\mathbf{r} + \mathbf{r}')/2$ inside $\langle \quad \rangle$ is equivalent to $\mathbf{r}$ outside it, one can transform the derivatives of ξ and $\mathbf{r}$ to their mean values in the expressions for $\bar{\dot{\xi}}$ and $\overline{\dot{\mathbf{r}}_f \xi}$. For the same reason, if $\mathbf{\Lambda}$ is assumed to depend on $(\mathbf{r} + \mathbf{r}')/2$, t, $\mathbf{p}$ and $\mathbf{p}'$ in Equation (42), it is a function of $\mathbf{r}$, t, $\mathbf{p}$, and $\mathbf{p}'$ outside the operation $\langle \quad \rangle$.

Introducing the notations

$$\Delta S(\mathbf{X}) = \int \mathbf{\Lambda} \cdot \langle(\mathbf{X}' - \mathbf{X})[1 + (\mathbf{r}' - \mathbf{r}) \cdot \nabla \log \sqrt{N'/N}]\rangle \, d\mathbf{p}' ,$$

$$\Delta W(\mathbf{XX}) = \frac{1}{2}\int \{\mathbf{\Lambda} \cdot [\langle(\mathbf{X}' - \mathbf{X})\mathbf{X}\rangle - \langle \mathbf{X}' - \mathbf{X}\rangle\bar{\mathbf{X}}] + [\langle \mathbf{X}(\mathbf{X}' - \mathbf{X})\rangle - \bar{\mathbf{X}}\langle \mathbf{X}' - \mathbf{X}\rangle] \cdot \mathbf{\Lambda}^{\dagger}\} \, d\mathbf{p}' \tag{45}$$

one finds that the contributions of $\bar{\xi}_f$ and $\overline{\dot{\mathbf{r}}_f \xi}$ to Equation (12) are correctly taken into account if we add to $S(\xi)$ and $W(\xi\eta)$ the quantities

$$\Delta S(\xi) = \Delta S(\mathbf{X}) \cdot \overline{\frac{\partial \xi}{\partial \mathbf{X}}} + \Delta W(\mathbf{XX}) \cdot \cdot \overline{\frac{\partial^2 \xi}{\partial \mathbf{X} \partial \mathbf{X}}} ,$$

$$\Delta W(\xi\eta) = \left(\overline{\frac{\partial \xi}{\partial \mathbf{X}}}\right)^{\dagger} \cdot \Delta W(\mathbf{XX}) \cdot \overline{\frac{\partial \eta}{\partial \mathbf{X}}} . \tag{46}$$

One must thereby observe the identity

$$[\langle(\mathbf{X}' - \mathbf{X})(\mathbf{X} + \mathbf{X}')\rangle - 2\langle \mathbf{X}' - \mathbf{X}\rangle\bar{\mathbf{X}}] \cdot \overline{\frac{\partial \mathbf{r}}{\partial \mathbf{X}}} = 0 , \tag{47}$$

which follows from the power expansions of $\mathbf{r}$ and $\mathbf{r}'$ at $\mathbf{X}_0$. The result also implies a symmetrization of the tensors if they appear in double products of types $\cdot\cdot (\partial \xi / \partial \mathbf{X} \partial \mathbf{X})$ and $\nabla \cdot (\nabla \cdot)$.

C. Net result

Equations (36), (41), (45), and (46) determine the net result for encounters. If the generalized friction $\mathbf{\Lambda} \cdot (\langle \mathbf{X}' \rangle - \mathbf{X})$ is replaced by the expression $\lambda(\langle \dot{\mathbf{R}}' \rangle - \dot{\mathbf{R}})$ with a scalar coefficient λ, the non-zero components of the resulting $S(\mathbf{X})$ and $W(\mathbf{XX})$ will be

$$S_G(\dot{\mathbf{R}}) = \int \lambda \langle \mathbf{v}[1 + (\mathbf{r}' - \mathbf{r}) \cdot \nabla \log \sqrt{N'/N}]\rangle \, d\mathbf{p}' , \tag{48a}$$

$$W_G(\dot{\mathbf{R}}\mathbf{R}) = [W_G(\mathbf{R}\dot{\mathbf{R}})]^{\dagger} = \int (\lambda/2)(\langle \mathbf{v}\mathbf{R}\rangle - \langle \mathbf{v}\rangle\bar{\mathbf{R}}) \, d\mathbf{p}' , \tag{48b}$$

$$W_G(\dot{\mathbf{R}}\dot{\mathbf{R}}) = \int [(2\pi^2/3)G^2 m'^2 \overline{n'_R} \mathbf{I}_3 \sqrt{6/\langle \mathbf{v}^2\rangle} + (\lambda/2) \times (\langle \mathbf{v}\dot{\mathbf{R}} + \dot{\mathbf{R}}\mathbf{v}\rangle - \langle \mathbf{v}\rangle\bar{\dot{\mathbf{R}}} - \bar{\dot{\mathbf{R}}}\langle \mathbf{v}\rangle)] \, d\mathbf{p}' . \tag{48c}$$

The transformation rule (46) provides $S(\xi)$ and $W(\xi\eta)$.

The coefficient λ is derived from the conservation of energy. Since Equation (12) is only valid if the condition (13) is satisfied, it is not possible to construct the source function for kinetic energy directly, but one can insert $\xi = \dot{\mathbf{R}}^2/2$ and thereafter multiply by m. If $\epsilon = 0$, the resulting energy production is $mn[\dot{\mathbf{R}} \cdot \ddot{\mathbf{R}} + S(\dot{\mathbf{R}}^2/2)]$. The first term represents the mechanical work, while the second one with $S = S_G$ corresponds to the encounters, so that the integral $\int mnS_G(\dot{\mathbf{R}}^2/2)\, d\mathbf{p}$ must vanish. The transformation of type (46) for S_G then gives

$$\int mn[S_G(\dot{\mathbf{R}}) \cdot \bar{\dot{\mathbf{R}}} + \text{trace } W_G(\dot{\mathbf{R}}\dot{\mathbf{R}})]\, d\mathbf{p} = 0\,. \tag{49}$$

To avoid a non-zero net force as a consequence of dynamical friction, we assume

$$mn\lambda(\mathbf{p}, \mathbf{p}') = m'n'\lambda(\mathbf{p}', \mathbf{p})\,. \tag{50}$$

Hence, if $\bar{\dot{\mathbf{R}}}$ is expressed in terms of $(\bar{\dot{\mathbf{R}}}' \pm \bar{\dot{\mathbf{R}}})/2$ and if the third-order quantities are omitted, Equations (48) and (49) give

$$\int\int mn(2\pi^2 G^2 m'^2 \overline{nn'_R} \sqrt{6/\langle \mathbf{v}^2 \rangle} - \lambda \langle \mathbf{v}^2 \rangle /2)\, d\mathbf{p}\, d\mathbf{p}' = 0\,, \tag{51}$$

because the antisymmetric terms vanish in the double integration.

The space density n'_R at point $\{\mathbf{r}, \mathbf{s}\}$ is $\int N(\mathbf{r}, \mathbf{s}, \dot{\mathbf{R}}', \mathbf{q}', \mathbf{p}')\, d\dot{\mathbf{R}}'\, d\mathbf{q}'$, and if we substitute $\mathbf{x}$ for $\mathbf{C}$ in Equations (3), these give

$$\overline{nn'_R} = \int N(\mathbf{r}, \mathbf{s}, \dot{\mathbf{R}}, \mathbf{q}, \mathbf{p}) N(\mathbf{r}, \mathbf{s}, \dot{\mathbf{R}}', \mathbf{q}', \mathbf{p}')\, d\mathbf{s}\, d\dot{\mathbf{R}}\, d\mathbf{q}\, d\dot{\mathbf{R}}'\, d\mathbf{q}'\,. \tag{52}$$

Since $\overline{nn'_R}$ is thus symmetric with respect to $\mathbf{p}$ and $\mathbf{p}'$, the expression $mm'^2\overline{nn'_R}$ in Equation (51) can be replaced by $mm'(m + m')\overline{nn'_R}/2$. Therefore, the solution for λ which satisfies Equations (50) and (51) for any particular choice of other quantities is

$$\lambda = (\pi^2/3) G^2 m'(m + m')\overline{n'_R}(6/\langle \mathbf{v}^2 \rangle)^{3/2}\,. \tag{53}$$

The result deviates from Chandrasekhar's (1960) coefficient in two respects: it does not explicitly depend on the velocity of the moving body and the logarithmic coefficient is absent. The former difference follows from the systematic use of mean values, on account of which λ is actually an averaged coefficient of friction. Chandrasekhar's expression can be modified in the corresponding manner and provides the order-of-magnitude estimate

$$\lambda_{Ch} = \frac{2\pi G^2 m'(m + m')\overline{n'_R}}{\langle \mathbf{v}^2 \rangle^{3/2}} \ln\left[\frac{L\langle \mathbf{v}^2 \rangle}{G(m + m')}\right] = 0.30\lambda \log_{10}\left[\frac{L\langle \mathbf{v}^2 \rangle}{G(m + m')}\right], \tag{54}$$

in which L denotes the mean mutual distance between particles. To illustrate this relation in terms of galactic motions, let us assume $m = m' = 1$ solar mass, $L = 1$ pc, and $\sqrt{\langle \mathbf{v}^2 \rangle} = 20$ km/s. Equation (54) then takes the form $\lambda_{Ch}/\lambda = 1.40$. If L is reduced to 40 AU, the result is 0.29, while $L = 10$ kpc corresponds to $\lambda_{Ch}/\lambda =$

2.60. The absence of the logarithmic factor (which in itself is based on an approximation) therefore has no dramatic consequences.

9. Calculation of Collisional Mean Values

The quantities (11) are most easily calculated for the Cartesian coordinates $\mathbf{X}$, and a subsequent transformation (Section 10A) provides the general case. For the sake of simplicity, $\nabla \log\sqrt{N'/N}$ is replaced by its mean value, which is approximate to $\nabla \log \sqrt{n'/n}$. Using the power expansion of $\mathbf{r}' - \mathbf{r}$ and re-arranging the terms in W, we thus have

$$\mathbf{S}(\mathbf{X}) = \int \nu\left\langle(\mathbf{X}_1 - \mathbf{X})\left[1 + (\mathbf{X}' - \mathbf{X}) \cdot \frac{\overline{\partial \mathbf{r}}}{\partial \mathbf{X}} \cdot \nabla \log\sqrt{\frac{n'}{n}}\right]\right\rangle \mathrm{d}\mathbf{p}' , \tag{55a}$$

$$W(\mathbf{XX}) = \int (\nu/4)\langle 2(\mathbf{X}_1 - \mathbf{X})(\mathbf{X}_1 - \mathbf{X}) - (\mathbf{X}_1 - \mathbf{X})(\mathbf{X}' - \mathbf{X}) - $$

$$- (\mathbf{X}' - \mathbf{X})(\mathbf{X}_1 - \mathbf{X})\rangle\, \mathrm{d}\mathbf{p}' + W^*(\mathbf{XX}) + W^{*\dagger}(\mathbf{XX}) , \tag{55b}$$

$$W^*(\mathbf{XX}) = \int (\nu/4)[\langle(\mathbf{X}_1 - \mathbf{X})(\mathbf{X} + \mathbf{X}')\rangle - 2\langle\mathbf{X}_1 - \mathbf{X}\rangle\bar{\mathbf{X}}]\, \mathrm{d}\mathbf{p}' . \tag{55c}$$

The gravitational parameter θ and the relevant functions of α, β, and γ are also replaced by the appropriate averaged expressions. These are discussed in Section 10D. The collisional mean values in S and W can then be calculated from Equations (9), (17), (21), (22), (29) and (31). The corresponding gravitational quantities $\langle\xi\rangle$ in Equations (48) and (53) follow as a generalization of these results.

The distribution of $\mathbf{R}_j (= \mathbf{R} - \mathbf{s})$ is not defined by Equation (17), but this vector only appears in the difference $\mathbf{R}' - \mathbf{R}$, which is equal to $(\sigma + \sigma')\mathbf{c}$, and in the expression $\langle(\mathbf{X}_1 - \mathbf{X})(\mathbf{R}_j + \mathbf{R}_j')\rangle$. The latter is obtained from the identity

$$[\langle(\mathbf{X}_1 - \mathbf{X})(\mathbf{X} + \mathbf{X}')\rangle - 2\langle\mathbf{X}_1 - \mathbf{X}\rangle\bar{\mathbf{X}}] \cdot \frac{\overline{\partial \mathbf{r}}}{\partial \mathbf{X}} = 0, \tag{56}$$

which corresponds to Equation (47).

A non-zero correlation between $\mathbf{q}$ and $\mathbf{R}$ or $\dot{\mathbf{R}}$ can only be maintained in exceptional cases, which are not discussed in this paper. The integrations for $\mathbf{q}$ and $\mathbf{q}'$ in Equation (21) are therefore trivial. After that the integral (21) contains the variables $\mathbf{s}$, $\mathbf{s}'$, $\dot{\mathbf{R}}$, $\dot{\mathbf{R}}'$ and $\mathbf{c}$. Defining

$$\begin{aligned} \mathbf{y} &= \{\mathbf{s}, \dot{\mathbf{R}}\} , \\ \mathbf{Y} &= \overline{\mathbf{y}\mathbf{y}} - \bar{\mathbf{y}}\bar{\mathbf{y}} \end{aligned} \tag{57}$$

one can change them to $\mathbf{y}$, $\mathbf{y}' - \mathbf{y}$, and $\mathbf{c}$. The first one only appears in linear terms, in which the integration transforms it to $\bar{\mathbf{y}} - (\mathbf{y}' - \mathbf{y} - \bar{\mathbf{y}}' + \bar{\mathbf{y}}) \cdot (\mathbf{Y} + \mathbf{Y}')^{-1} \cdot \mathbf{Y}$.

Equations (21) and (31) are now reduced to the expression

$$\nu\langle\zeta\rangle = \frac{gn'(\sigma+\sigma')^2(1+2\bar{\theta})}{\sqrt{(2\pi)^{k-3}\,\mathrm{Det}(\mathbf{Y}+\mathbf{Y}')}}\int\int \zeta\,\mathrm{e}^{-u/2}(-\mathbf{c})\cdot\mathbf{v}\,\mathrm{d}\omega\,\mathrm{d}\mathbf{v}\,,$$
$$u = (\mathbf{y}'-\mathbf{y}-\bar{\mathbf{y}}'+\bar{\mathbf{y}})\cdot(\mathbf{Y}+\mathbf{Y}')^{-1}\cdot(\mathbf{y}'-\mathbf{y}-\bar{\mathbf{y}}'+\bar{\mathbf{y}})\,. \tag{58}$$

The vector $\mathbf{s}'-\mathbf{s}$ consists of certain components of $\mathbf{R}'-\mathbf{R}=(\sigma+\sigma')\mathbf{c}$. This quantity is significant if the mutual distances between particles in the direction of $\mathbf{s}$ are not much larger than $\sigma+\sigma'$, but the Gaussian distribution for density can hardly be valid in this case. Since $\mathbf{s}'-\mathbf{s}$ complicates the calculations without leading to any definitely better results, it is neglected in u. The difference $\bar{\mathbf{s}}'-\bar{\mathbf{s}}$ for all the particles can obviously be much larger than $\mathbf{s}'-\mathbf{s}$ for impacts.

According to the definition (9) N' is constructed for the same point $\mathbf{r}$ as N. Therefore $\dot{\mathbf{R}}'-\dot{\mathbf{R}}$ in the exponential part of the integral (58) only consists of the random motion, while ζ and $(-\mathbf{c})\cdot\mathbf{v}$ also include the differential velocity due to the finite size of particles. In the first approximation this difference is not taken into account. Hence, for the calculation of u, $\mathbf{y}'-\mathbf{y}=\{0,\mathbf{v}\}$.

The integration for $\mathbf{c}$ in Equations (58) is trivial. It is carried out for those solid angles in which $(-\mathbf{c})\cdot\mathbf{v}>0$. Thereafter Equations (58) are only needed for $\zeta=\zeta(\mathbf{v})$, and constructing $\overline{n_R}$ according to Equations (17) and (52) we find

$$\nu\langle\zeta(\mathbf{v})\rangle = g\overline{n_R'}\pi(\sigma+\sigma')^2(1+2\bar{\theta})\sqrt{\mathrm{Det}[(\mathbf{Y}+\mathbf{Y}')^{-1}/2\pi]_{vv}}\times$$
$$\times\int\zeta\exp\{-(\mathbf{v}-\mathbf{V})\cdot[(\mathbf{Y}+\mathbf{Y}')^{-1}]_{vv}\cdot(\mathbf{v}-\mathbf{V})/2\}|\mathbf{v}|\,\mathrm{d}\mathbf{v}\,, \tag{59}$$

$$V = \bar{\dot{\mathbf{R}}}'-\bar{\dot{\mathbf{R}}}-(\bar{\mathbf{s}}'-\bar{\mathbf{s}})\cdot(\mathbf{Y}_{ss}+\mathbf{Y}_{ss}')^{-1}\cdot(\mathbf{Y}_{sv}+\mathbf{Y}_{sv}')\,.$$

in which the indices s and v refer to the corresponding subspaces: $\mathbf{Y}_{ss}=\overline{\mathbf{ss}}-\bar{\mathbf{s}}\bar{\mathbf{s}}$, $\mathbf{Y}_{sv}=\overline{\mathbf{s}\dot{\mathbf{R}}}-\bar{\mathbf{s}}\bar{\dot{\mathbf{R}}}$, and $[(\mathbf{Y}+\mathbf{Y}')^{-1}]_{vv}$ contains the components of $(\mathbf{Y}+\mathbf{Y}')^{-1}$ in the velocity space. It is to be observed that $\mathbf{Y}'$, $\bar{\dot{\mathbf{R}}}'$, and $\bar{\mathbf{s}}'$ are constructed for the point $\mathbf{r}$ instead of $\mathbf{r}'$.

The coagulation and fragmentation of particles are included if the integral (59) is calculated for a restricted interval $v_{min}<|\mathbf{v}|<v_{max}$ outside which the number of particles is not conserved.

An improved approximation follows if the differential velocity $\mathbf{v}_D\cong\sigma\mathbf{c}\cdot\nabla\bar{\dot{\mathbf{R}}}+\sigma'\mathbf{c}\cdot\nabla\bar{\dot{\mathbf{R}}}'$ due to the finite particle size is taken into account. Since $\bar{\dot{\mathbf{R}}}'-\bar{\dot{\mathbf{R}}}$ is treated as a small quantity, $\mathbf{v}_D$ is also approximated by $(\sigma+\sigma')\mathbf{c}\cdot\nabla\bar{\dot{\mathbf{R}}}$. Using $\mathbf{v}$ as a notation for the random velocity, we must substitute $\mathbf{v}+\mathbf{v}_D$ for $\mathbf{v}$ in $\zeta(-\mathbf{c})\cdot\mathbf{v}$ when calculating the integral (58). The Taylor expansion of $\zeta(-\mathbf{c})\cdot(\mathbf{v}+\mathbf{v}_D)$ then gives the linearized correction term

$$\Delta(\nu\langle\zeta\rangle) = (\sigma+\sigma')\nu\left\langle\mathbf{c}\cdot(\nabla\bar{\dot{\mathbf{R}}})\cdot\left(\frac{\partial\zeta}{\partial\mathbf{v}}+\frac{\mathbf{c}\zeta}{\mathbf{c}\cdot\mathbf{v}}\right)\right\rangle. \tag{60}$$

The condition $(-\mathbf{c}) \cdot \mathbf{v} > 0$ also ought to be modified, but this turns out to produce a second-order correction and is therefore not taken into account.

The gravitational mean values which appear in Equations (48) and (53) are assumed to coincide with the corresponding collisional expressions $\langle \zeta \rangle$ in the limit $\sigma + \sigma' \to 0$. No attempt is made to include the finite encounter distances.

10. Summary of Basic Equations

A. Equation of continuity

If $\xi = 1$, Equation (12) gives the conservation law

$$\begin{aligned} &\partial n/\partial t + \nabla \cdot \mathbf{F} = n\bar{\epsilon}, \\ &\mathbf{F} = n[\bar{\dot{\mathbf{r}}} + S(\mathbf{r})] - \nabla \cdot [nW(\mathbf{rr})]. \end{aligned} \tag{61}$$

The particle flux $\mathbf{F}$ is used to define the operation

$$\mathrm{d}/\mathrm{d}t = \partial/\partial t + (\mathbf{F}/n) \cdot \nabla, \tag{62}$$

and Equation (12) can now be expressed in the form

$$\begin{aligned} n\,\mathrm{d}\,\bar{\xi}/\mathrm{d}t = n[\overline{\epsilon\xi} - \bar{\epsilon}\bar{\xi} + \bar{\dot{\xi}} + S(\xi)] - \nabla \cdot \{\mathbf{n}[\overline{\dot{\mathbf{r}}\xi} - \\ - \bar{\dot{\mathbf{r}}}\bar{\xi} + 2W(\mathbf{r}\xi) - W(\mathbf{rr}) \cdot \nabla\bar{\xi}]\}. \end{aligned} \tag{63}$$

All the quantities S and W which appear in Equations (61)–(63) can be calculated from the relations (Hämeen-Anttila, 1984)

$$\begin{aligned} &S(\xi) = S(\mathbf{X}) \cdot \overline{\frac{\partial \xi}{\partial \mathbf{X}}} + W(\mathbf{XX}) \cdot\cdot \overline{\frac{\partial^2 \xi}{\partial \mathbf{X} \partial \mathbf{X}}}, \\ &W(\xi\eta) = \left(\overline{\frac{\partial \xi}{\partial X}}\right)^{\dagger} \cdot W(\mathbf{XX}) \cdot \overline{\frac{\partial \eta}{\partial \mathbf{X}}}, \end{aligned} \tag{64}$$

in which the difference in notations is taken into account (W instead of the original $W/2$). The same transformations are also valid for encounters (Section 8).

Since Equations (61)–(63) determine the derivatives of $n\overline{\xi\eta}$, $n\bar{\xi}$, $n\bar{\eta}$ and n, it is also possible to calculate $\mathrm{d}[n(\overline{\xi\eta} - \bar{\xi}\bar{\eta})]/\mathrm{d}t$. Omitting the high-order terms and using Equations (64), one finds that

$$\begin{aligned} \mathrm{d}(\overline{\xi\eta} - \bar{\xi}\bar{\eta})/\mathrm{d}t = \overline{(\epsilon - \bar{\epsilon})(\xi - \bar{\xi})(\eta - \bar{\eta})} + \overline{\dot{\xi}\eta} + \overline{\xi\dot{\eta}} - \\ - \bar{\dot{\xi}}\bar{\eta} - \bar{\xi}\bar{\dot{\eta}} - (\overline{\xi\dot{\mathbf{r}}} - \bar{\xi}\bar{\dot{r}}) \cdot \nabla\bar{\eta} - (\nabla\bar{\xi})^{\dagger} \cdot (\overline{\dot{\mathbf{r}}\eta} - \bar{\dot{\mathbf{r}}}\bar{\eta}) + \\ + 2\left(\overline{\frac{\partial \xi}{\partial \mathbf{X}}} - \overline{\frac{\partial r}{\partial \mathbf{X}}} \cdot \nabla\bar{\xi}\right)^{\dagger} \cdot W(\mathbf{XX}) \cdot \left(\overline{\frac{\partial \eta}{\partial \mathbf{X}}} - \overline{\frac{\partial \mathbf{r}}{\partial \mathbf{X}}} \cdot \nabla\bar{\eta}\right). \end{aligned} \tag{65}$$

Equations (61)–(65) imply conditions of type (13) for ξ and η unless these can be treated as small quantifies.

B. Notations

The tensor **Y** is decomposed to its components in the subspaces **s** and $\dot{\mathbf{R}}$ according to the schema

$$\begin{aligned} &\overline{\mathbf{ss}} - \bar{\mathbf{s}}\bar{\mathbf{s}} = \mathbf{H}\,,\\ &\overline{\mathbf{s}\dot{\mathbf{R}}} - \bar{\mathbf{s}}\bar{\dot{\mathbf{R}}} = \mathbf{K}\,,\\ &\overline{\dot{\mathbf{R}}\dot{\mathbf{R}}} - \bar{\dot{\mathbf{R}}}\bar{\dot{\mathbf{R}}} = \mathbf{T} + \mathbf{K}^{\dagger}\cdot\mathbf{H}^{-1}\cdot\mathbf{K}\,. \end{aligned} \tag{66}$$

The inverse tensor is found to have the component $(\mathbf{Y}^{-1})_{\nu\nu} = \mathbf{T}^{-1}$ in the velocity space. The term $\mathbf{K}^{\dagger}\cdot\mathbf{H}^{-1}\cdot\mathbf{K}$ arises from the difference between the mean values for all the particles at **r** regardless of their **s** and for those particles which have a common 3-dimensional position. The former corresponds to $\overline{\dot{\mathbf{R}}\dot{\mathbf{R}}} - \bar{\dot{\mathbf{R}}}\bar{\dot{\mathbf{R}}}$, while **T** stands for the (averaged) dispersion of velocities in the latter sense. An alternative representation is

$$\mathbf{T}^{-1} = \overline{\frac{\partial\mathbf{C}}{\partial\dot{\mathbf{R}}}}\cdot(\overline{\mathbf{CC}} - \bar{\mathbf{C}}\bar{\mathbf{C}})^{-1}\cdot\left(\overline{\frac{\partial\mathbf{C}}{\partial\dot{\mathbf{R}}}}\right)^{\dagger}. \tag{67}$$

This expression was derived by Verronen (1989) for non-rotating particles, but it is also valid in this paper.

The systematic relative velocity defined by Equations (59) and the differential motion of colliding particles introduce the quantities

$$\begin{aligned} &\mathbf{V} = \bar{\dot{\mathbf{R}}}' - \bar{\dot{\mathbf{R}}} - (\bar{\mathbf{s}}' - \bar{\mathbf{s}})\cdot(\mathbf{H}+\mathbf{H}')^{-1}\cdot(\mathbf{K}+\mathbf{K}')\,,\\ &\mathbf{D} = (\sigma+\sigma')\nabla\bar{\dot{\mathbf{R}}}\,. \end{aligned} \tag{68}$$

The difference between **V** and $\bar{\dot{\mathbf{R}}}' - \bar{\dot{\mathbf{R}}}$ corresponds to that between **T** and $\overline{\dot{\mathbf{R}}\dot{\mathbf{R}}} - \bar{\dot{\mathbf{R}}}\bar{\dot{\mathbf{R}}}$. The axial rotation is described by the vector $\bar{\mathbf{q}} = \overline{\sigma\omega}$ and tensors

$$\begin{aligned} &\mathbf{Q} = \overline{\mathbf{qq}} - \bar{\mathbf{q}}\bar{\mathbf{q}}\,,\\ &\mathbf{J} = \mathbf{Q} + \mathbf{Q}' + (\bar{\mathbf{q}} + \bar{\mathbf{q}}')(\bar{\mathbf{q}} + \bar{\mathbf{q}}')\,. \end{aligned} \tag{69}$$

The gravitational expressions (48) are incorporated into S and W in terms of the modified coefficients of collisional mean values. For this purpose we define

$$\Gamma_1 = \frac{m'}{m+m'}\left[\frac{1+\bar{\alpha}}{2} + \frac{\bar{\beta}}{7} + \frac{\pi^2G^2(m+m')^2}{3\nu}\left(\frac{6}{\langle\mathbf{v}^2\rangle}\right)^{3/2}\overline{n'_R}\right], \tag{70a}$$

$$\Gamma_2 = \left(\frac{m'}{m+m'}\right)^2\left[\overline{\left(\frac{1+\alpha}{2} - \frac{\beta}{7}\right)^2} + 3\overline{\left(\frac{\gamma}{7}\right)^2} + \frac{2\pi^2G^2(m+m')^2}{3\nu}\left(\frac{6}{\langle\mathbf{v}^2\rangle}\right)^{3/2}\overline{n'_R}\right], \tag{70b}$$

$$\Gamma_3 = \Gamma_1 - \left(\frac{m'}{m+m'}\right)^2\left[\overline{\left(\frac{1+\alpha}{2} + \frac{\beta}{7}\right)^2} - \overline{\left(\frac{\gamma}{7}\right)^2}\right], \tag{70c}$$

$$\Gamma_4 = \frac{m'\bar{\beta}}{7(m+m')}, \tag{70d}$$

$$\Gamma_5 = \frac{2}{3}\Gamma_4 - \frac{8}{15}\left(\frac{m'}{m+m'}\right)^2\left[\overline{\left(1+\alpha+\frac{3\beta}{7}\right)\frac{\beta}{7}} - \overline{\left(\frac{\gamma}{7}\right)^2}\right], \tag{70e}$$

$$\Gamma_6 = \left(\frac{m'}{m+m'}\right)^2 \overline{\left(\frac{\beta}{7}\right)^2}, \tag{70f}$$

$$\Gamma_7 = \left(\frac{m'}{m+m'}\right)^2 \overline{\left(\frac{\gamma}{7}\right)^2}, \tag{70g}$$

$$\Gamma_8 = \frac{16}{35}\left[\frac{7(1+\bar{\alpha})m'}{12(m+m')} - \Gamma_1 + \Gamma_3 + \frac{4}{3}\Gamma_4 + \frac{5}{8}\Gamma_5 + \Gamma_6 + \frac{1}{3}\Gamma_7\right], \tag{70h}$$

$$\Gamma_9 = \frac{2}{5}\left[\frac{(1+\bar{\alpha})m'}{3(m+m')} + \Gamma_4 + 4\Gamma_6 + \frac{4}{3}\Gamma_7\right] + \Gamma_5 - \Gamma_8\,. \tag{70i}$$

C. Expressions for S and W

Since the viscosity soon dissipates rapid internal motions, we assume $|\langle\mathbf{v}\rangle| \ll \langle|\mathbf{v}|\rangle$. Therefore, $\langle\mathbf{v}\rangle$ and $\langle\mathbf{vvv}/\mathbf{v}^2\rangle$ are treated as second-order quantities; while $\langle\mathbf{v}|\mathbf{v}|\rangle$ can be ignored.

The terms $\sim\mathbf{K}^\dagger\cdot\mathbf{H}^{-1}\cdot\mathbf{K}$, $\sigma\mathbf{v}\mathbf{H}^{-1}\cdot\mathbf{K}$ are not taken into account. They represent systematic motion (usually expansion or contraction) in the $\mathbf{s}$ space. If the corresponding velocity is much smaller than $\langle|\mathbf{v}|\rangle$, $\mathbf{K}^\dagger\cdot\mathbf{H}^{-1}\cdot\mathbf{K}$ is negligible as compared with $\mathbf{T}$. The tensor $\sigma\mathbf{v}\mathbf{H}^{-1}\cdot\mathbf{K}$ could be larger in collapsed systems having $\mathbf{H}\sim\sigma^2$, but $\mathbf{K}$ tends to vanish in these. If $\mathbf{K}^\dagger\cdot\mathbf{H}^{-1}\cdot\mathbf{K}$ or $\sigma\mathbf{v}\mathbf{H}^{-1}\cdot\mathbf{K}$ were occasionally significant, one could use a 3-dimensional $\mathbf{r}$ space.

Since Γ_6, $\Gamma_7 < 0.02$, the terms proportional to them are not needed with high precision. This permits the calculation of $\mathbf{J}\cdot\langle\mathbf{cc}\rangle$, $\langle\mathbf{c}\times\mathbf{J}\times\mathbf{c}\rangle$, and $\langle\mathbf{cc}\cdot\mathbf{J}\cdot\mathbf{cc}\rangle$ as if the distribution of $\mathbf{c}$ were isotropic. The same method was used for $\Gamma_4(\bar{\mathbf{q}}+\bar{\mathbf{q}}')\times\langle\mathbf{cc}\rangle$ and $\Gamma_4(\mathbf{I}_3-\langle\mathbf{cc}\rangle)$, except in $S(\mathbf{q})$.

The correction (60) for the differential velocity is not added to the small terms $\sim\Gamma_6$, Γ_7, or $\Gamma_4\bar{\mathbf{q}}$. It is also neglected in $W(\mathbf{qq})$, since this tensor determines $\mathbf{Q}$ which elsewhere has the coefficient Γ_6 or Γ_7. Excluding $S(\mathbf{q})$, which is sensitive to the differential velocity, the correction term was partly calculated for an isotropic distribution of $\mathbf{c}$.

If the radius vector $\mathbf{r}$ depends only on $\mathbf{R}$ and $\dot{\mathbf{R}}$, the expressions for S and W are

$$S(\mathbf{R}) = 0\,, \tag{71a}$$

$$S(\dot{\mathbf{R}}) = \int \nu\Big\{\Gamma_1\langle\mathbf{v}\rangle + \Big[(\sigma+\sigma')\langle(\dot{\mathbf{R}}_1 - \dot{\mathbf{R}})\mathbf{c}\rangle\cdot\overline{\frac{\partial \mathbf{r}}{\partial \mathbf{R}}} +$$

$$+ \langle(\dot{\mathbf{R}}_1 - \dot{\mathbf{R}})\mathbf{v}\rangle\cdot\overline{\frac{\partial \mathbf{r}}{\partial \dot{\mathbf{R}}}}\Big]\cdot\nabla\log\sqrt{\frac{n'}{n}}\Big\}\,d\mathbf{p}', \tag{71b}$$

$$S(\mathbf{q}) = \int \frac{5\nu}{4}\Gamma_4\Big[(\sigma+\sigma')\nabla\times\bar{\dot{\mathbf{R}}} - \Big\langle\frac{\mathbf{v}\times\mathbf{D}\cdot\mathbf{v}}{\mathbf{v}^2}\Big\rangle -$$

$$-\Big(3\mathbf{I}_3 - \Big\langle\frac{\mathbf{v}\mathbf{v}}{\mathbf{v}^2}\Big\rangle\Big)\cdot(\bar{\mathbf{q}}+\bar{\mathbf{q}}')\Big]\,d\mathbf{p}', \tag{71c}$$

$$W(\mathbf{R}\mathbf{R}),\ W(\mathbf{R}\mathbf{q}),\ W(\mathbf{q}\mathbf{R}),\ W(\dot{\mathbf{R}}\mathbf{q}),\ W(\mathbf{q}\dot{\mathbf{R}}) = 0 \tag{71d}$$

$$W(\dot{\mathbf{R}}\mathbf{R}) = W^{\dagger}(\mathbf{R}\dot{\mathbf{R}}) = -\int\frac{\nu}{4}\Big\{(\sigma+\sigma')\langle(\dot{\mathbf{R}}_1-\dot{\mathbf{R}})\mathbf{c}\rangle\cdot$$

$$\cdot[\mathbf{I}_3 - (\mathbf{H}+\mathbf{H}')^{-1}\cdot(\mathbf{H}'-\mathbf{H})] + 2\langle(\dot{\mathbf{R}}_1-\dot{\mathbf{R}})\mathbf{v}\rangle\cdot(\mathbf{T}+\mathbf{T}')^{-1}\cdot$$

$$\cdot\Big[\mathbf{K}^{\dagger}\cdot(\mathbf{H}+\mathbf{H}')^{-1}\cdot\mathbf{H}' - \mathbf{K}'^{\dagger}\cdot(\mathbf{H}+\mathbf{H}')^{-1}\cdot\mathbf{H} +$$

$$+\frac{1}{2}(\mathbf{T}'-\mathbf{T})\cdot\overline{\frac{\partial\mathbf{r}}{\partial\dot{\mathbf{R}}}}\cdot\Big(\overline{\frac{\partial\mathbf{r}}{\partial\mathbf{R}_j}}\Big)^{-1}\Big]\Big\}\,d\mathbf{p}', \tag{71e}$$

$$W(\dot{\mathbf{R}}\dot{\mathbf{R}}) = \int\frac{\nu}{2}\Big\{\frac{1}{3}\Gamma_2\langle\mathbf{v}^2\rangle\mathbf{I}_3 - \Gamma_3\langle\mathbf{v}\mathbf{v}\rangle + \Gamma_5\Big[\Big\langle\frac{\mathbf{v}\mathbf{v}}{|\mathbf{v}|}\Big\rangle\times(\bar{\mathbf{q}}+\bar{\mathbf{q}}') -$$

$$-(\bar{\mathbf{q}}+\bar{\mathbf{q}}')\times\Big\langle\frac{\mathbf{v}\mathbf{v}}{|\mathbf{v}|}\Big\rangle\Big] + \frac{4}{3}\Gamma_6(\mathbf{I}_3\,\mathrm{trace}\,\mathbf{J} - \mathbf{J}) +$$

$$+\frac{4}{15}\Gamma_7(\mathbf{I}_3\,\mathrm{trace}\,\mathbf{J} + 7\mathbf{J}) + \frac{1}{2}\Big[\langle(\dot{\mathbf{R}}_1-\dot{\mathbf{R}})\mathbf{v}\rangle\cdot(\mathbf{T}+\mathbf{T}')^{-1}$$

$$\cdot(\mathbf{T}'-\mathbf{T}) + (\mathbf{T}'-\mathbf{T})\cdot(\mathbf{T}+\mathbf{T}')^{-1}\cdot\langle\mathbf{v}(\dot{\mathbf{R}}_1-\dot{\mathbf{R}})\rangle\Big] +$$

$$+\Gamma_8\Big(\Big\langle\frac{\mathbf{v}\mathbf{v}}{|\mathbf{v}|}\Big\rangle\cdot\mathbf{D} + \mathbf{D}^{\dagger}\cdot\Big\langle\frac{\mathbf{v}\mathbf{v}}{|\mathbf{v}|}\Big\rangle + \Big\langle\frac{\mathbf{v}\mathbf{v}}{|\mathbf{v}|}\Big\rangle\,\mathrm{trace}\,\mathbf{D}\Big) -$$

$$-\Gamma_9\Big[\mathbf{D}\cdot\Big\langle\frac{\mathbf{v}\mathbf{v}}{|\mathbf{v}|}\Big\rangle + \Big\langle\frac{\mathbf{v}\mathbf{v}}{|\mathbf{v}|}\Big\rangle\cdot\mathbf{D}^{\dagger} + \frac{1}{2}(\mathbf{D}+\mathbf{D}^{\dagger})\langle|\mathbf{v}|\rangle\Big] -$$

$$-\frac{8}{35}\Gamma_2\Big(2\Big\langle\frac{\mathbf{v}\mathbf{v}}{|\mathbf{v}|}\Big\rangle\cdot\cdot\,\mathbf{D} + \langle|\mathbf{v}|\rangle\,\mathrm{trace}\,\mathbf{D}\Big)\mathbf{I}_3\Big\}\,d\mathbf{p}', \tag{71f}$$

$$W(\mathbf{qq}) =$$

$$\int \frac{5\nu}{3}\left\{\Gamma_6\left[\frac{15}{8}(\mathbf{I}_3\langle \mathbf{v}^2\rangle - \langle \mathbf{vv}\rangle) + \left\langle\frac{\mathbf{vv}}{|\mathbf{v}|}\right\rangle \times (\bar{\mathbf{q}} + \bar{\mathbf{q}}') - \right.\right.$$

$$\left. - (\bar{\mathbf{q}} + \bar{\mathbf{q}}') \times \left\langle\frac{\mathbf{vv}}{|\mathbf{v}|}\right\rangle + \frac{1}{2}(\mathbf{I}_3 \,\mathrm{trace}\, \mathbf{J} + 7\mathbf{J})\right] +$$

$$+ 3\Gamma_7\left[\frac{5}{8}\left(\frac{1}{3}\mathbf{I}_3\langle \mathbf{v}^2\rangle + \langle \mathbf{vv}\rangle\right) - \left\langle\frac{\mathbf{vv}}{|\mathbf{v}|}\right\rangle \times (\bar{\mathbf{q}} + \bar{\mathbf{q}}') + \right.$$

$$\left.\left. + (\bar{\mathbf{q}} + \bar{\mathbf{q}}') \times \left\langle\frac{\mathbf{vv}}{|\mathbf{v}|}\right\rangle + \frac{5}{6}(\mathbf{I}_3 \,\mathrm{trace}\, \mathbf{J} - \mathbf{J})\right] - 2\Gamma_4\mathbf{Q}\right\} \mathrm{d}\mathbf{p}' . \qquad (71g)$$

The mean values $\langle(\dot{\mathbf{R}}_1 - \dot{\mathbf{R}})\mathbf{c}\rangle$ and $\langle(\dot{\mathbf{R}}_1 - \dot{\mathbf{R}})\mathbf{v}\rangle$ are used here as abbreviations for the expressions

$$\langle(\dot{\mathbf{R}}_1 - \dot{\mathbf{R}})\mathbf{c}\rangle = -4\{(2\Gamma_1 + \Gamma_4)\langle \mathbf{vv}/|\mathbf{v}|\rangle + (\Gamma_1 - 2\Gamma_4)\mathbf{I}_3\langle|\mathbf{v}|\rangle +$$
$$+ 5\Gamma_4[(\bar{\mathbf{q}} + \bar{\mathbf{q}}') \times \mathbf{I}_3 - \mathbf{D}^{\dagger}]/2 -$$
$$- (\Gamma_1 - 3\Gamma_4/2)(\mathbf{D} + \mathbf{D}^{\dagger} + \mathbf{I}_3 \,\mathrm{trace}\, \mathbf{D})\}/15 , \qquad (72a)$$

$$\langle(\dot{\mathbf{R}}_1 - \dot{\mathbf{R}})\mathbf{v}\rangle = \Gamma_1\langle \mathbf{vv}\rangle + 4[5\Gamma_4(\bar{\mathbf{q}} + \bar{\mathbf{q}}') \times \langle \mathbf{vv}/|\mathbf{v}|\rangle - (2\Gamma_1 + \Gamma_4) \cdot$$
$$\cdot (\langle \mathbf{vv}/|\mathbf{v}|\rangle \cdot \mathbf{D} + \mathbf{D}^{\dagger} \cdot \langle \mathbf{vv}/|\mathbf{v}|\rangle + \langle \mathbf{vv}/|\mathbf{v}|\rangle \,\mathrm{trace}\, \mathbf{D}) -$$
$$- (\Gamma_1 - 2\Gamma_4)(2\mathbf{D} \cdot \langle \mathbf{vv}/|\mathbf{v}|\rangle + \langle|\mathbf{v}|\rangle\mathbf{D})]/15 . \qquad (72b)$$

A simpler approximation, $\langle(\dot{\mathbf{R}}_1 - \dot{\mathbf{R}})\mathbf{v}\rangle = \Gamma_1\langle \mathbf{vv}\rangle$, can be used for the terms $\sim\mathbf{T}' - \mathbf{T}$ as was shown by numerical calculations. This is important, because $\langle \mathbf{vv}\rangle \cdot (\mathbf{T} + \mathbf{T}')^{-1}$ is easy to derive from equations of Section D.

D. Mean values, ν, $\overline{n'_R}$ and g

The mean values $\langle\zeta(\mathbf{v})\rangle$ and ν are calculated from Equations (59), in which $[(\mathbf{Y} + \mathbf{Y}')^{-1}]_{\nu\nu}$ is to be replaced by $(\mathbf{T} + \mathbf{T}')^{-1}$ and $\mathbf{V}$ by the expression given in Equations (68). The coagulation and fragmentation of particles introduce the finite bounds $v_{min} < |\mathbf{v}| < v_{max}$ into Equation (59) and are assumed to reduce the integral by the same proportion as in the case of an isotropic $\mathbf{T} + \mathbf{T}'$. This is equivalent to multiplication of $\nu\langle\zeta\rangle$ by the coefficients

$$J_k = \int_{z_{min}}^{z_{max}} z^k \,\mathrm{e}^{-z}\,\mathrm{d}z \Big/ \int_0^{\infty} z^k \,\mathrm{e}^{-z}\,\mathrm{d}z , \qquad (73)$$

$$z = 3\mathbf{v}^2/2 \,\mathrm{trace}(\mathbf{T} + \mathbf{T}') .$$

The calculation of J_k is trivial for integer values of the index, and $J_{3/2}$ follows from Galton's function approximated by the expression

$$\frac{1}{\sqrt{2\pi}}\int_0^{x_{max}} \exp(-x^2/2)\,dx \cong \frac{1}{2}\sqrt{1-\exp(-2x_{max}^2/\pi)} \tag{74}$$

with an accuracy better than 1% (Boll, 1957).

Defining

$$\mathbf{Z} = 3(\mathbf{T}+\mathbf{T}')/\mathrm{trace}(\mathbf{T}+\mathbf{T}')\,, \tag{75}$$

we obtain the approximations

$$\langle \mathbf{v}\rangle = (\mathbf{I}_3 + \langle \mathbf{vv}/\mathbf{v}^2\rangle)\cdot \mathbf{V}\,, \tag{76a}$$

$$\langle \mathbf{vv}\rangle = (16J_2/45J_1)(\mathbf{Z}+\mathbf{Z}\cdot\mathbf{Z}/4)\,\mathrm{trace}(\mathbf{T}+\mathbf{T}')\,, \tag{76b}$$

$$\langle \mathbf{vv}/|\mathbf{v}|\rangle = \langle|\mathbf{v}|\rangle \mathbf{Z}/3\,, \tag{76c}$$

$$\langle|\mathbf{v}|\rangle = (J_{3/2}/J_1)(\mathrm{Det}\,\mathbf{Z})^{-1/30}\sqrt{3\pi\,\mathrm{trace}(\mathbf{T}+\mathbf{T}')/8}\,, \tag{76d}$$

$$\langle \mathbf{vv}/\mathbf{v}^2\rangle = \mathbf{Z}/3 - (\mathrm{Det}\,\mathbf{Z})^{8/15}[(1-\mathbf{Z}\cdot\cdot\,\mathbf{Z}/3)\mathbf{I}_3 - \mathbf{Z} + \mathbf{Z}\cdot\mathbf{Z}]/15\,, \tag{76e}$$

$$\nu = g\overline{n_R'}(\sigma+\sigma')^2(1+2\bar{\theta})J_1(\mathrm{Det}\,\mathbf{Z})^{1/30}\sqrt{8\pi\,\mathrm{trace}(\mathbf{T}+\mathbf{T}')/3}\,. \tag{76f}$$

This expression for ν is used in Equations (71) and (72), while the true frequency of collisions follows from Equations (60) and (76f):

$$\nu + \Delta\nu \cong \nu[1 - \mathrm{trace}\,\mathbf{D}\sqrt{\pi/6\,\mathrm{trace}(\mathbf{T}+\mathbf{T}')}]\,. \tag{77}$$

If $\mathbf{V}\to 0$, Equations (76) are exact for the isotropic distribution of velocities. For Keplerian systems in which the largest eigenvalue of $\mathbf{Z}$ is four times the smallest one, the relative errors are less than 0.6%. This estimate was obtained without including the coagulation and fragmentation of particles. These processes increase the error, but the laws for them are far less certain than the numerical approximations.

The mean space density for particles of type $\mathbf{p}'$ in the zone occupied by those of type $\mathbf{p}$ is calculated from Equations (17) and (52) as

$$\overline{n_R'} = \frac{n'\exp[-(\bar{\mathbf{s}}'-\bar{\mathbf{s}})\cdot(\mathbf{H}+\mathbf{H}')^{-1}\cdot(\bar{\mathbf{s}}'-\bar{\mathbf{s}})/2]}{\sqrt{(2\pi)^{3-j}\,\mathrm{Det}(\mathbf{H}+\mathbf{H}')}}\,. \tag{78}$$

If the $\mathbf{r}$ space is 3-dimensional, then $\overline{n_R} = n$.

The coefficient g is problematic. The maximum packing density for identical spheres, $\delta_{max} = \pi/\sqrt{18}$ (cf. Anderson, 1974), can be used only for gravitationally compressed systems having no differential motion. A more realistic model follows from the cubic arrangement of particles, which is the densest one permitting a non-zero velocity gradient. In this case $\delta_{max} = (4\pi\sigma^3/3):(2\sigma)^3$, and Equation (22) gives

$$1/g = 1 - 8\int \overline{n_R'}\sigma'^3\,d\mathbf{p}'\,. \tag{79}$$

If the system consists of particles having very different radii, Equation (79) is not justified, because the small particles can be located between the larger ones without reducing the free space available for these. A generally valid analytic expression for δ_{max} would therefore depend on **p**, **p**′ and the degree of compression.

If maximum accuracy is sought for, the relevant functions of α, β, γ, and θ ought to be averaged according to the rule $\langle f(\alpha, \beta, \gamma, \theta)(\mathbf{c} \cdot \mathbf{v})^2 \rangle / \langle (\mathbf{c} \cdot \mathbf{v})^2 \rangle$. This is a slight modification of an empirical finding for α (Hämeen-Anttila and Lukkari, 1980) and it is also suggested by the expressions $(\mathbf{c} \cdot \mathbf{v})^2$ and $(\mathbf{vc} + \mathbf{cv})\mathbf{c} \cdot \mathbf{v}$ in the dominating terms of $W(\mathbf{XX})$. This method leads to excessive numerical integrations, however, and we shall see in Section 12 that satisfactory accuracy also follows from a simpler procedure in which $\sqrt{\mathrm{trace}(\mathbf{T} + \mathbf{T}')}$ is substituted for $|\mathbf{c} \cdot \mathbf{v}|$ $(= |\mathbf{c} \cdot \mathbf{v}^*|)$, on which α, β, and γ are assumed to depend. If the particles gravitate, the definition $\langle \theta(\mathbf{c} \cdot \mathbf{v})^2 \rangle / \langle (\mathbf{c} \cdot \mathbf{v})^2 \rangle$ is used for $\bar{\theta}$. Equations (31)–(33) then give

$$\begin{aligned} \bar{\theta} &= G(m + m')/\langle \mathbf{v}^2 \rangle (\sigma + \sigma') , \\ \alpha &= \sqrt{(1 + 2\bar{\theta})[\alpha_{col}(w)]^2 - 2\bar{\theta}} , \\ w &= \sqrt{(1 + 2\bar{\theta})\,\mathrm{trace}(\mathbf{T} + \mathbf{T}')} . \end{aligned} \tag{80}$$

A similar method was employed in Papers I and II, although $|\mathbf{c} \cdot \mathbf{v}|$ was replaced by $\sqrt{\langle (\mathbf{c} \cdot \mathbf{v})^2 \rangle}$. This is approximate to $\sqrt{2\,\mathrm{trace}(\mathbf{T} + \mathbf{T}')/3}$.

11. Equations for a Thin Disc

For thin discs we define $\mathbf{R} = \mathbf{r} + z\mathbf{N}$, $\mathbf{s} = z\mathbf{N}$ and $\mathbf{C} = \{z, \dot{\mathbf{R}}, \mathbf{q}\}$, where **N** stands for the unit vector perpendicular to the equatorial plane (the xy plane) and $\mathbf{r} \cdot \mathbf{N} = 0$. The notations r and $\mathbf{r}^*$ are used for $|\mathbf{r}|$ and $\mathbf{N} \times \mathbf{r}$.

The circular, epicyclic and vertical frequencies of the disc are Ω, κ and μ, respectively. Their definitions (cf. Binney and Tremaine, 1987) provide the auxiliary equation

$$\partial\Omega/\partial r = -(4\Omega^2 - \kappa^2)/2r\Omega , \tag{81}$$

and hence,

$$\overline{\nabla\dot{\mathbf{R}}} \cong \nabla(\Omega\mathbf{r}^*) = -\Omega[\mathbf{r}^*\mathbf{r} + (2\Omega^2 - \kappa^2)\mathbf{r}\mathbf{r}^*/2\Omega^2]/r^2 . \tag{82}$$

An expression for $\ddot{\mathbf{R}}$ follows from the Taylor expansion of the potential gradient:

$$\begin{aligned} \ddot{\mathbf{R}} &= -\Omega^2\mathbf{r} - \mu^2 z\mathbf{N} - \mu\mu' z^2\mathbf{r}/r - \cdots , \\ \mu' &= \partial\mu/\partial r . \end{aligned} \tag{83}$$

The tensors **H** and **K** are replaced by the scalars $H = \overline{z^2}$ and $K = \overline{z\dot{z}}$ $(z, \bar{\dot{z}} = 0)$, because **H** is now 1-dimensional and the components of **K** along the equatorial plane can be assumed to vanish.

If ϵ and the high-order terms are neglected, Equations (63)–(67) and (71) give

$$\mathrm{d}\mathbf{T}/\mathrm{d}t = 2W(\dot{\mathbf{R}}\dot{\mathbf{R}}) - [\mathbf{T} + 2W(\dot{\mathbf{R}}\mathbf{R})] \cdot (\nabla\bar{\dot{\mathbf{R}}} + K\mathbf{NN}/H) - - (\nabla\bar{\dot{\mathbf{R}}} + K\mathbf{NN}/H)^{\dagger} \cdot [\mathbf{T} + 2W(\mathbf{R}\dot{\mathbf{R}})]\,, \tag{84a}$$

$$\mathrm{d}H/\mathrm{d}t = 2K\,, \tag{84b}$$

$$\mathrm{d}K/\mathrm{d}t = T_{zz} - \mu^2 H + 2W_{zz}(\mathbf{R}\dot{\mathbf{R}})\,, \tag{84c}$$

$$\mathrm{d}\mathbf{Q}/\mathrm{d}t = 2W(\mathbf{qq})\,, \tag{84d}$$

$$\mathrm{d}\bar{\mathbf{q}}/\mathrm{d}t = S(\mathbf{q})\,, \tag{84e}$$

where the term $\mathbf{K}^{\dagger} \cdot \mathbf{H}^{-1} \cdot \mathbf{K}(= K^2\mathbf{NN}/H)$ in Equations (66) is included for the differentiation $\mathrm{d}\mathbf{T}/\mathrm{d}t$ but elsewhere neglected (see Section 10C).

To construct the flux vector we assume $\bar{\dot{\mathbf{R}}} = \Omega\mathbf{r}^* + \mathbf{u}$. The vector $\mathbf{u}$, a small second-order quantity, is parallel to the equatorial plane and represents the contribution of impacts. If it is split into the local Cartesian components u_r (radial) and u_t (tangential), Equations (62), (63), (66), (81), and (82) give

$$\mathrm{d}u_r/\mathrm{d}t = 2\Omega u_t - \mu\mu' H + S_r(\dot{\mathbf{R}}) + [T_{tt} + 2W_{tt}(\mathbf{R}\dot{\mathbf{R}})]/r - - (1/nr)\partial\{nr[T_{rr} + 2W_{rr}(\mathbf{R}\dot{\mathbf{R}})]\}/\partial r \tag{85a}$$

$$\mathrm{d}u_t/\mathrm{d}t = -\kappa^2 u_r/2\Omega + S_t(\dot{\mathbf{R}}) - [T_{tr} + 2W_{tr}(\mathbf{R}\dot{\mathbf{R}})]/r - - (1/nr)\partial\{nr[T_{rt} + 2W_{rt}(\mathbf{R}\dot{\mathbf{R}})]\}/\partial r\,. \tag{85b}$$

According to Equations (61), (68), (71a), and (71d), the radial flux and $\mathbf{V}$ are nu_r and $\mathbf{u}' - \mathbf{u}$, respectively.

An alternative system of variables follows from the epicyclic coordinates $\mathbf{r}$, $\mathbf{e}$, z, and $\dot{z}$, which correspond to the Taylor expansions (Hämeen-Anttila *et al.*, 1988)

$$\begin{aligned} \mathbf{R} &= \mathbf{r} - (\mathbf{rr} + 2\Omega\mathbf{r}^*\mathbf{r}^*/\kappa) \cdot \mathbf{e}/r + z\mathbf{N} + \cdots\,, \\ \dot{\mathbf{R}} &= \Omega\mathbf{r}^* + [\Omega\mathbf{r}^*\mathbf{r} + (2\Omega^2 - \kappa^2)\mathbf{rr}^*/\kappa] \cdot \mathbf{e}/r + \dot{z}\mathbf{N} + \cdots\,. \end{aligned} \tag{86}$$

The vector $\mathbf{r}$ is not the same as above, but it represents the circular component of motion ($\dot{\mathbf{r}} = \Omega\mathbf{r}^*$), and $\mathbf{e}$ gives the superposed epicycles. We shall use these variables for the Keplerian systems only, in which case $\mathbf{r}$ represents the orbital semi-major axis and mean length of the particle while $\mathbf{e}$ is a constant of motion equal to the projection of the perihelion vector $\dot{\mathbf{R}} \times (\mathbf{R} \times \dot{\mathbf{R}})/GM - \mathbf{R}/|\mathbf{R}|$ (M stands for the central mass) onto the equatorial plane (Hämeen-Anttila, 1978). The vector $\mathbf{e}$ is assumed to be statistically independent of z and $\dot{z}$.

The above-mentioned paper (Hämeen-Anttila *et al.*, 1988) gives the derivatives of $\mathbf{r}$ and $\mathbf{e}$ which are needed for Equations (65) and (67). Inserting $\Omega = \kappa = \mu$ and defining $\mathbf{P} = \overline{\mathbf{ee}} - \bar{\mathbf{e}}\bar{\mathbf{e}}$ and $Z = \overline{z^2} - \bar{z}^2$, we obtain

$$\mathrm{d}\mathbf{P}/\mathrm{d}t = 2[(2\mathbf{rr}^* - \mathbf{r}^*\mathbf{r}) \cdot W(\dot{\mathbf{R}}\dot{\mathbf{R}}) \cdot (2\mathbf{r}^*\mathbf{r} - \mathbf{rr}^*) + \\ + \Omega(\mathbf{rr} - \mathbf{r}^*\mathbf{r}^*) \cdot W(\mathbf{R}\dot{\mathbf{R}}) \cdot (2\mathbf{r}^*\mathbf{r} - \mathbf{rr}^*) + \\ + \Omega(2\mathbf{rr}^* - \mathbf{r}^*\mathbf{r}) \cdot W(\dot{\mathbf{R}}\mathbf{R}) \cdot (\mathbf{rr} - \mathbf{r}^*\mathbf{r}^*)]/\Omega^2 r^6 , \tag{87a}$$

$$\mathrm{d}Z/\mathrm{d}t = 2[W_{zz}(\dot{\mathbf{R}}\dot{\mathbf{R}}) - \mu^2 K] , \tag{87b}$$

$$\mathbf{T} = (\Omega/2r)^2(2\mathbf{rr}^* - \mathbf{r}^*\mathbf{r}) \cdot \mathbf{P} \cdot (2\mathbf{r}^*\mathbf{r} - \mathbf{rr}^*) + (Z - K^2/H)\mathbf{NN} . \tag{87c}$$

Equations (84b)–(84e) are still valid, but the flux vector is different: namely,

$$F^r = (2n/\Omega)[S_t(\dot{\mathbf{R}}) + \text{trace } W(\dot{\mathbf{R}}\mathbf{R})/\Omega r] - \\ - (4/\Omega^2 r^2)\partial\{nr^2[W_{tt}(\dot{\mathbf{R}}\dot{\mathbf{R}}) + 2\Omega W_{rt}(\mathbf{R}\dot{\mathbf{R}})]\}/\partial r . \tag{88}$$

If the system consists of a mixture of particles, $S(\dot{\mathbf{R}})$ introduces the vector $\mathbf{V} = \mathbf{u}' - \mathbf{u}$ which is calculated from Equations (85). The new variables then increase the number of differential equations. If, however, the particles are identical, $S(\dot{\mathbf{R}})$ vanishes and Equations (85) become unnecessary. In this case the computational work is reduced. Equation (88) is also useful for studying of local density fluctuations, since the gradient term then dominates.

A comparison of Equation (88) with the hydrodynamical treatment of Keplerian discs (cf. Stewart *et al.*, 1984) indicates that $4[W_{tt}(\dot{\mathbf{R}}\dot{\mathbf{R}}) + 2\Omega W_{rt}(\mathbf{R}\dot{\mathbf{R}})]/3\Omega^2$ represents the coefficient of kinematic viscosity. If this is calculated for the equilibrium solution of Equations (84) (the state in which H, K and the components of $\mathbf{T}$, $\mathbf{Q}$ and $\bar{\mathbf{q}}$ are constants), the resulting expression agrees with the sum of local and non-local viscosity as given bv Araki and Tremaine (1986). However, Equation (88) is more generally valid, since it can also be used for the non-equilibrium states.

The remaining terms of F^r correspond to the loss of energy in partially elastic collisions.

12. Comparison with Computer Simulations

A. Simulation method

All the numerical simulations were performed with the local simulation code (Salo, 1991; see also Wisdom and Tremaine, 1988). Assuming that the system is axially symmetric, the calculations are confined to a small volume which follows the mean orbital motion of the particles. Each time one of them leaves the cell, another enters it with an appropriately modified position and velocity. By adjusting the volume, one can study various optical thicknesses up to values $\gg 1$ with a reasonable number of particles. Numerical tests (Wisdom and Tremaine, 1988; Salo, 1991) indicate that the method is valid as long as the mean free path between the impacts remains below the dimensions of the cell. Compared with Wisdom and Tremaine's (1988) simulations of identical frictionless particles, the present paper includes the size distribution and axial rotation. The number of particles is also essentially larger.

Since the orbital calculations are carried out in a non-inertial frame, the simulation velocities contain an extra term arising from the rotation of coordinate axes:

$$\dot{\mathbf{R}}_\Omega = \dot{\mathbf{R}}_I - \Omega \mathbf{N} \times \dot{\mathbf{R}}_I . \tag{89}$$

The subscripts Ω and I distinguish between the rotating coordinates and inertial frame aligned with the instantaneous direction of the simulation system, $\mathbf{R}_\Omega = \mathbf{R}_I$. Equations (23)–(29) indicate that the expressions for the collisional changes of velocity and spin vectors are not modified by the use of $\dot{\mathbf{R}}_\Omega$ instead of $\dot{\mathbf{R}}_I$, but $\mathbf{q} + \mathbf{q}'$ in Equations (26) and (29) must be replaced by $\mathbf{q} + \mathbf{q}' - (\sigma + \sigma')\Omega\mathbf{N}$. This is caused by the different inertial velocities of the particle centres due to the rotating coordinate system. Also, before applying Equations (26) and (29), the spin-axes referring to the fixed initial coordinates must be rotated to the instantaneous system. After the impact they are rotated back to the initial one.

The simulation system consists of 200 to 2000 particles and corresponds to Saturn's rings at $r = 10^5$ km. The program did not include γ, which is therefore zero for all the comparisons. The coefficient β was constant, but for α we also used the expression

$$\alpha = \mathrm{Min}[1, (|\mathbf{c} \cdot \mathbf{v}|/v_c)^{-0.234}] , \tag{90}$$

which is an empirical result for ice (Bridges *et al.*, 1984). Since the original constant $v_c = 0.01 \text{ cm s}^{-1}$ does not distinguish sufficiently between $\alpha = \alpha(|\mathbf{c} \cdot \mathbf{v}|)$ and $\alpha =$ const., it was replaced by $v_c = 0.2 \text{ cm s}^{-1}$. Some tests were also carried out for $\alpha = 1/(1 + |\mathbf{c} \cdot \mathbf{v}|/v_c)$.

B. Evolution of Keplerian Discs

Orbital motion in Keplerian systems is characterized by the mean-square eccentricity and inclination, which are obtained with a high degree of accuracy from the expressions

$$\begin{aligned} e_m^2 &= \overline{\mathbf{e}^2} \cong \operatorname{trace} \mathbf{P} , \\ i_m^2 &= \overline{i^2} = (H + Z/\mu^2)/r^2 . \end{aligned} \tag{91}$$

Since $\bar{\mathbf{q}}$ turns out to be parallel to $\mathbf{N}$, the axial rotation of particles, $\boldsymbol{\omega} = \mathbf{q}/\sigma$, can be discussed in terms of the quantities.

$$\begin{aligned} [\Delta(\omega)]^2 &= \overline{\boldsymbol{\omega}^2} - \overline{\boldsymbol{\omega}}^2 = \operatorname{trace} \mathbf{Q}/\sigma^2 , \\ \omega_m &= \overline{q_z}/\sigma . \end{aligned} \tag{92}$$

Figure 2 shows the evolution of e_m, i_m, ω_m and $\Delta(\omega)$ for identical particles having α, $\beta = 0.5$, $\sigma = 1$ m and $\tau = n\pi\sigma^2 = 1$.

Since trace $\mathbf{D}$ now vanishes, the true frequency of impacts, $\nu + \Delta\nu$, does not differ from the "apparent" one, ν, as is seen from Equation (77). Its behaviour

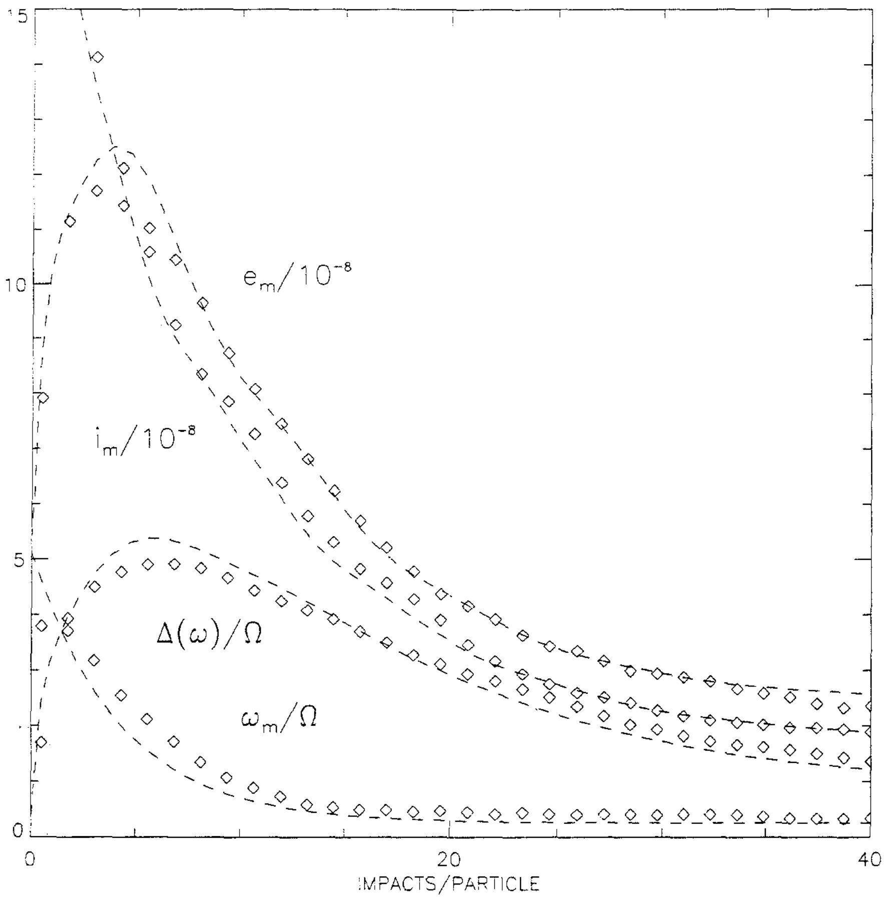

Fig. 2. Evolution of a Keplerian disc having α, $\beta = 0.5$, $\sigma = 1$ m, and $\tau = 1$. The dashed lines are theoretical curves and the squares represent the results of simulations. The notations e_m, i_m, ω_m and $\Delta(\omega)$ are defined by Equations (91) and (92).

is illustrated in Figure 3. The form of the theoretical curve, including the wavy appearance of the simulated evolution, is correct but gives a slightly too low ν. The error is probably caused by g or approximations in the differential rotation terms.

The results for a velocity-dependent α are similar. The accuracy obviously varies with the functional form of this quantity if $\sqrt{\text{trace}(\mathbf{T} + \mathbf{T}')}$ is substituted for $|\mathbf{c} \cdot \mathbf{v}|$ as explained in Section 10D.

The evolution of the radial dispersion of identical particles in a ring-shaped system can be calculated from Equations (85) or (88), both agreeing with the simulations. Some problems arise with the initial values of u_r and u_t if Equations (85) are used, since a minor error in $\mathbf{u}$ produces periodic oscillations in the radial expansion. In this respect Equation (88) is superior for identical particles.

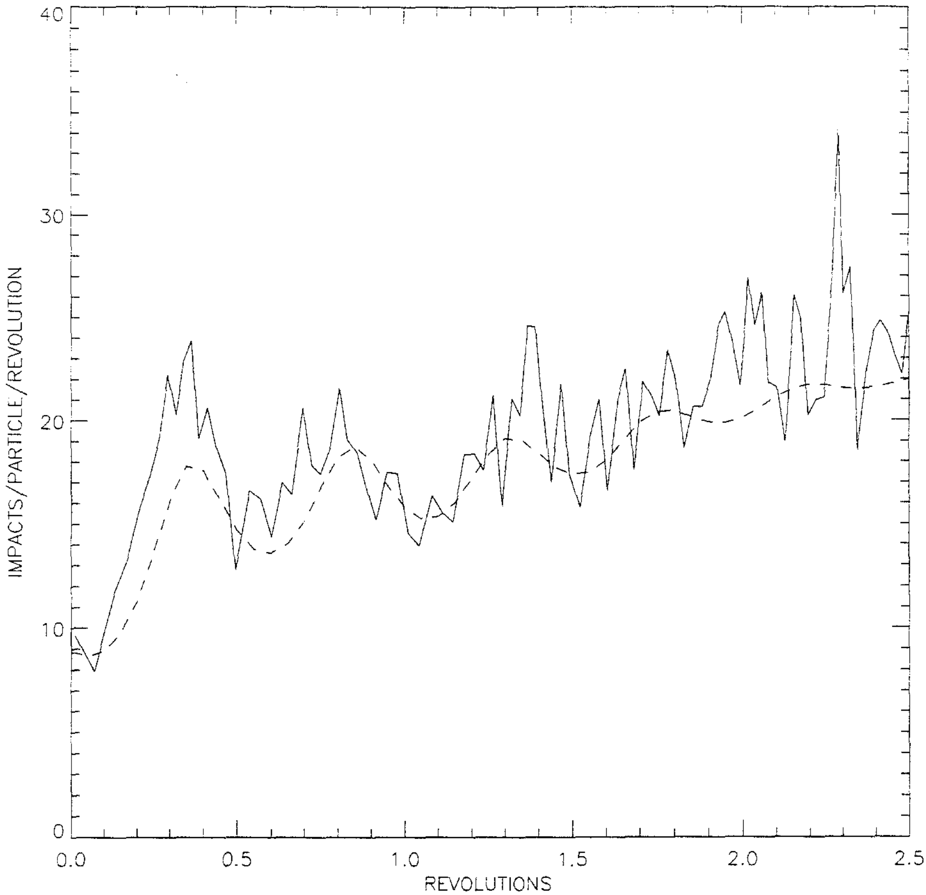

Fig. 3. Impact frequency as a function of time according to the theory (dashed line) and simulation (solid line). The system is the same as in Figure 2.

Figure 4 represents the radial evolution for a mixture of two particle types according to Equations (85). Since the above-mentioned problem is complicated by the rapid initial variations of **T**, the simulation particles were first confined to a co-moving cell for a period of 5 orbital revolutions, after which the radial boundaries were removed and comparison with the theory began. The smaller particles are spread more rapidly than the larger ones, but they were not observed to accumulate at the edges of the ring as was found by Brophy *et al.* (1990) in their work.

C. Equilibrium states

Figure 5 gives the equilibrium values for some of the relevant quantities for α satisfying Equation (90) and $\beta = 0.5$. The effective thickness h is defined by the relation

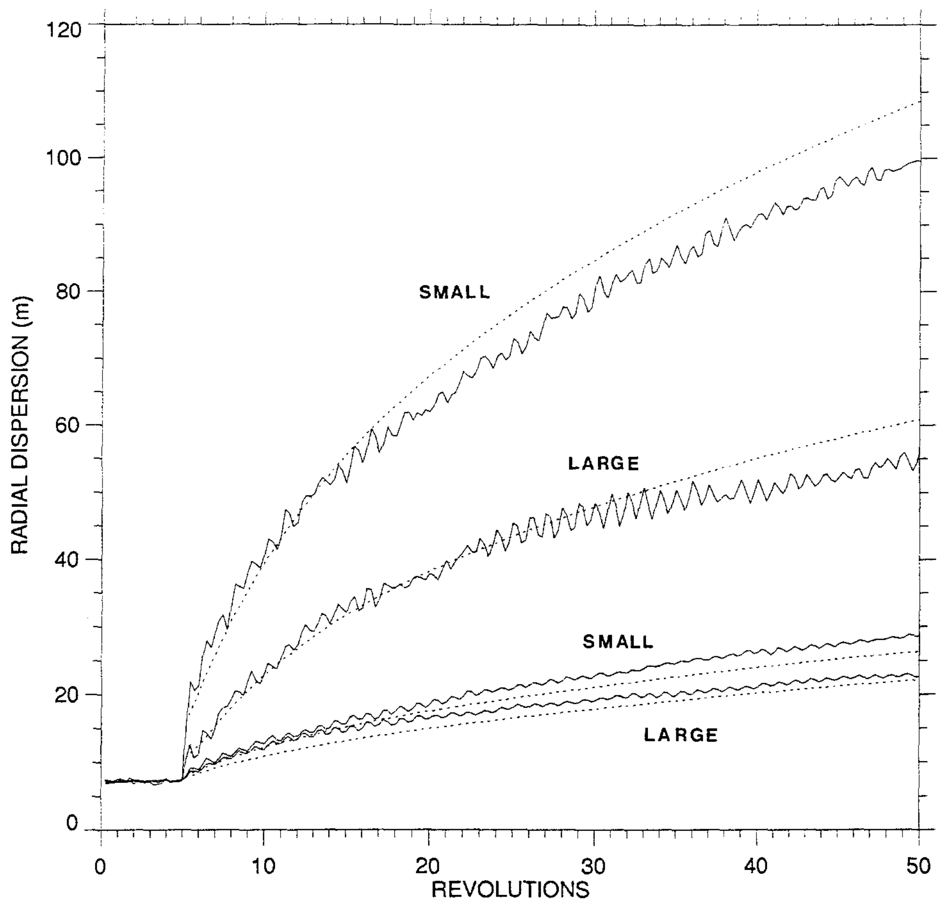

Fig. 4. Radial expansion in a mixture of two particle types having $\sigma_2 = 1$ m and $(\sigma_2/\sigma_1)^3 = 16$. The upper curves correspond to the restitution coefficient (90) and the lower ones to $\alpha = 0.5$. In both cases $\beta = 0.5$.

$$h = \sqrt{12H} \tag{93}$$

and represents the geometric thickness of a homogeneous layer having the same vertical dispersion $\overline{z^2} - \overline{z}^2$ as the disc. The error in trace $\mathbf{T}$ is probably caused by the substitution of $\sqrt{\text{trace}(\mathbf{T} + \mathbf{T}')}$ for $|\mathbf{c} \cdot \mathbf{v}|$ in α. A similar effect in h is explained by Equations (84c) and (93), since $W_{zz}(\mathbf{R}\dot{\mathbf{R}})$ is insignificant if $h \gg \sigma$.

If α, β = const. and $\tau < 1$ the accuracy of the theoretical predictions is almost as good as in Figure 5, but for $\tau > 1$ the finite size of the particles produces considerable errors in some quantities. If $\tau = 3$, $\sqrt{\text{trace}\,\mathbf{Q}}$ and ν are twice as large as in the simulations, while the axial ratios of the tensor ellipsoids for $\mathbf{T}$ and $\mathbf{Q}$ contain errors of up to 30%. The orientation of the latter is correct to a high degree of precision, but the former shows a deviation of 10%. The other quantities agree with the simulations as far as these were carried out ($\tau < 5$).

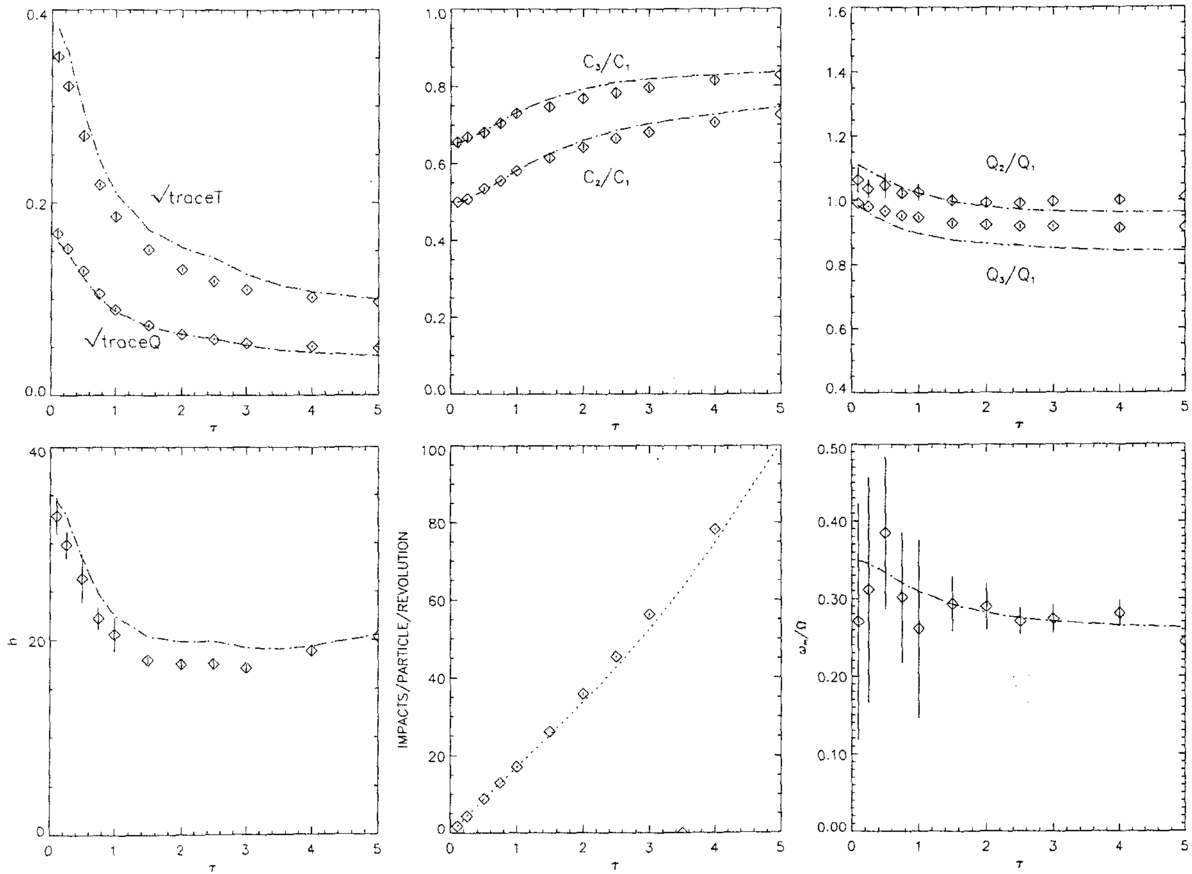

Fig. 5. Equilibrium values of some relevant quantities for particles having $\sigma = 1$ m and $\beta = 0.5$. The restitution coefficient is given by Equation (90). The axes of the velocity ellipsoid are c_1, c_2 and c_3, while Q_1, Q_2 and Q_3 represent the corresponding quantities for **Q**. The effective thickness h is in metres. Squares and dashed lines as in Figure 2.

The error in $\sqrt{\text{trace}\,\mathbf{Q}}$ can be expected, because Equation (71g) for $W(\mathbf{qq})$ is uncorrected for the differential rotation. Since $\Gamma_6, \Gamma_7 < 0.02$, $\mathbf{Q}$ does not contribute greatly to the other quantities. Better results for it would imply second-order corrections for differential rotation. Those and the uncertainty of g also influence ν. A detailed study of simulations indicated that the large number of impacts for $\tau > 1$ follows from an excess of grazing collisions. These can contribute to the rotational terms but their effects on $\mathbf{T}$, H, and K are small, $\sim(\mathbf{c}\cdot\mathbf{v})^2$. The errors in the eigenvalues and orientation of $\mathbf{T}$ probably depend on the isotropic distribution of $\mathbf{c}$, which was used in some cases to simplify the corrections for differential rotation.

The generally good agreement between the theory and the simulations is also seen in Figures 6 and 7 for a mixture of two particle types having α, $\beta = 0.5$. One can observe the above-mentioned behaviour of $\mathbf{Q}$ in Figure 7, but a curious phenomenon also appears in h, since if $\tau < 2.5$, the large particles are concentrated near to the equatorial plane, but if $\tau > 2.5$, the small ones behave in this manner. The latter case is also seen in Figure 8 for the vertical structure of the system. This effect ultimately follows from the fact that δ_{max} in Equation (22) is a function of particle type. This is not taken into account by Equation (79).

D. Gravitating particles

Gravitational interactions produce a number of effects: the vertical field of the disc, pre-collisional acceleration and post-collisional deceleration of impacting particles, growth of their collisional cross-sections and gravitational encounters.

The vertical field (disc + central body) follows from Poisson's equation in which the density of matter is replaced by $\overline{mn_R}$. This quantity is a function of particle types, as can be expected, since the amplitudes of vertical oscillations depend on m and imply a different construction of mean density for each type of particle. Combining Equation (78) with Poisson's equation and the properties of Ω and κ, one obtains

$$\mu^2 = 2\Omega^2 - \kappa^2 + 4\pi G \int \frac{m'n'}{\sqrt{2\pi(H+H')}}\,\mathrm{d}\mathbf{p}' \tag{94}$$

if $\bar{z} = \bar{z}' = 0$. Since the horizontal effect is neglected, we have $\Omega = \kappa = \sqrt{GM/r^3}$.

The increased collisional cross-sections and mutual gravitational attraction of impacting particles are given by Equations (76f) and (80), while the encounters correspond to the gravitational terms in Equations (70).

Figure 9 represents a comparison with the simulations for $\alpha = 0.5$, $\beta = 0$, $\sigma =$ 5 cm, and $\tau = 0.1$. The particles are identical and have the internal density 0.0 gr cm^{-3} (curve A), 0.9 gr cm^{-3} (curve B) and 1.8 gr cm^{-3} (curve C). The agreement between the simulations (squares) and theory (solid lines) is good for curves A and B, but curve C shows a rather large error. This is probably caused by the particle groups which are seen to grow and decay during the simulation.

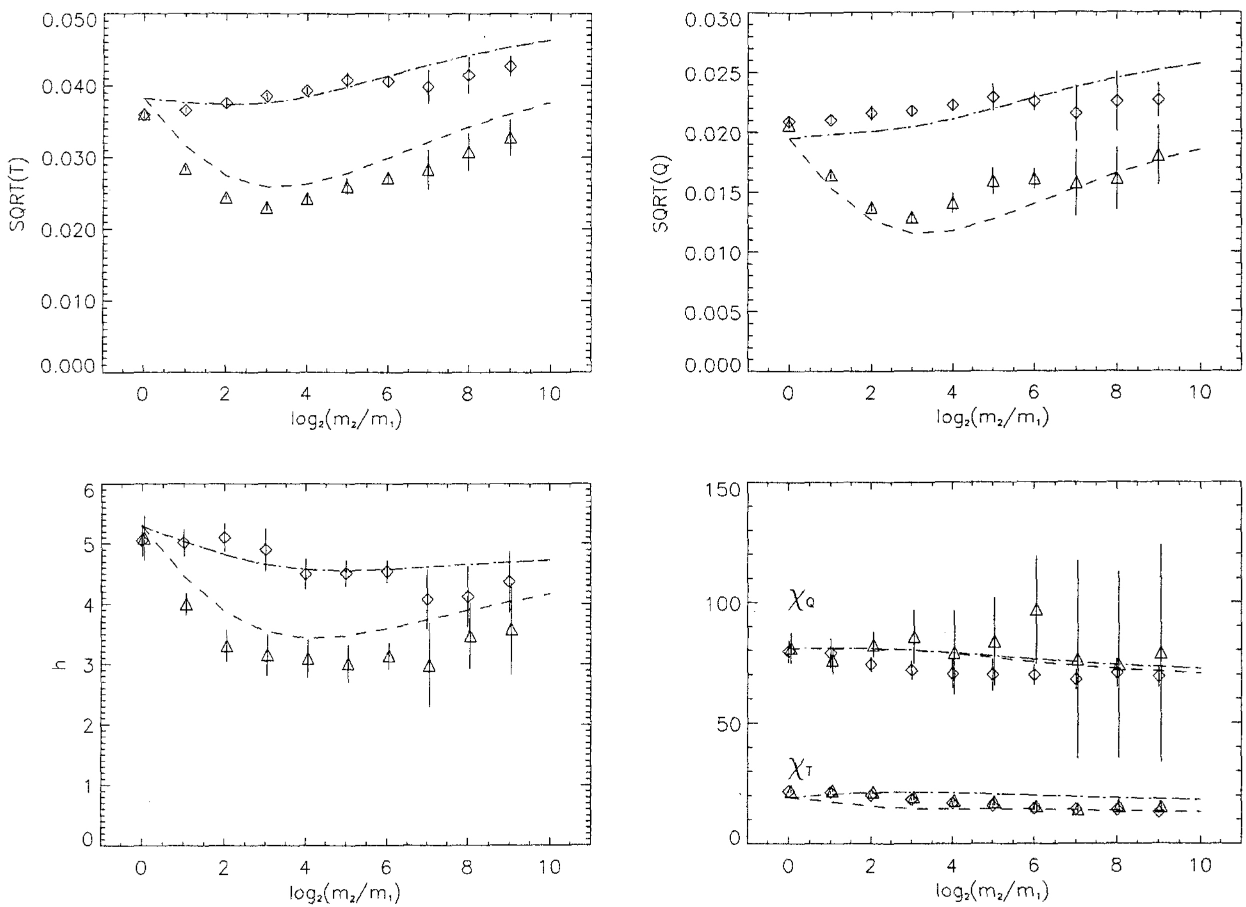

Fig. 6. Equilibrium values of some quantities for a mixture of two particle types having α, $\beta = 0.5$. The optical thickness of the disc is $\tau = 1$. The squares and triangles are simulation values for smaller and larger particles respectively. The dashed lines are as in Figure 2. The quantity χ_T is the angle between the radial direction and major axis of the velocity ellipsoid, and χ_Q is the corresponding quantity for **Q**. Both angles are given in degrees.

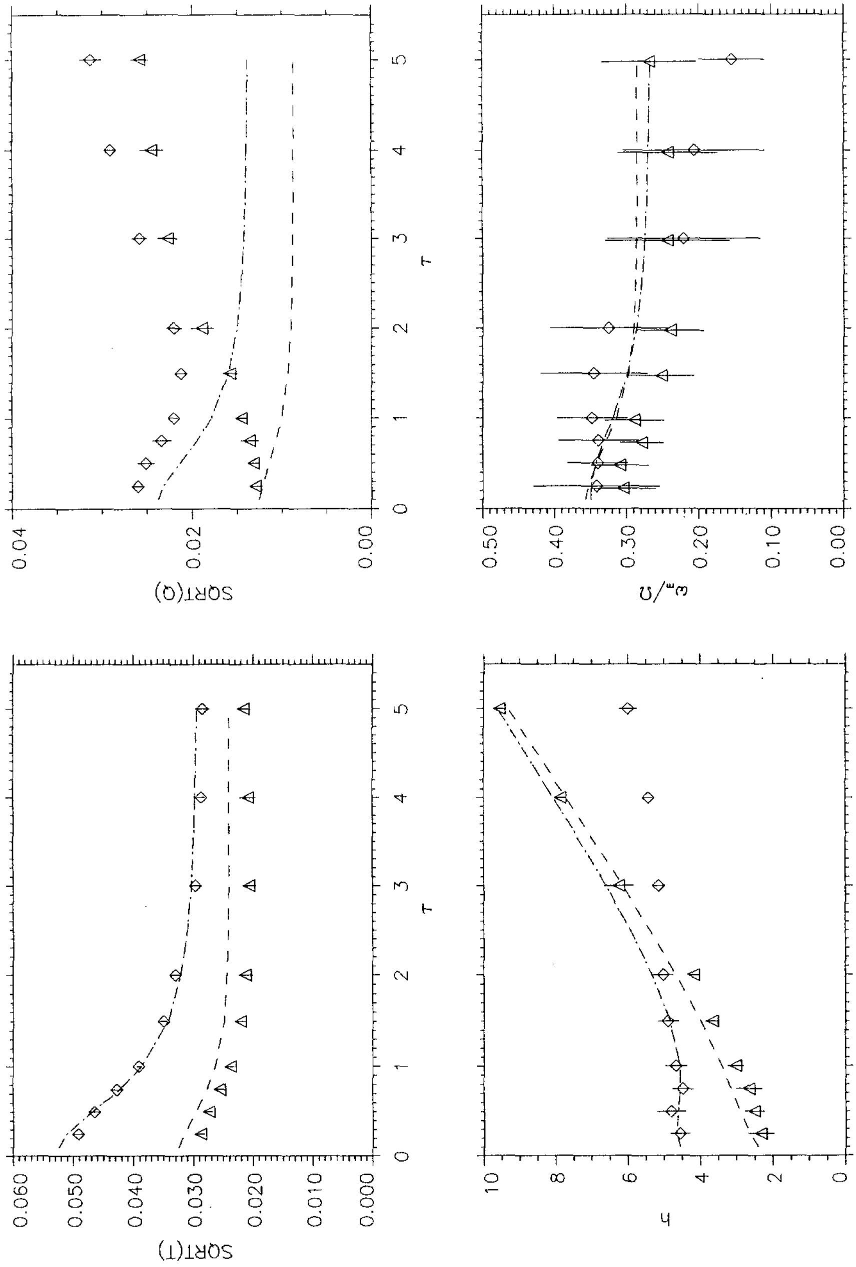

Fig. 7. Dependence on τ for some relevant quantities. The mass ratio of particles is fixed to 16. The notations are as in Figure 6.

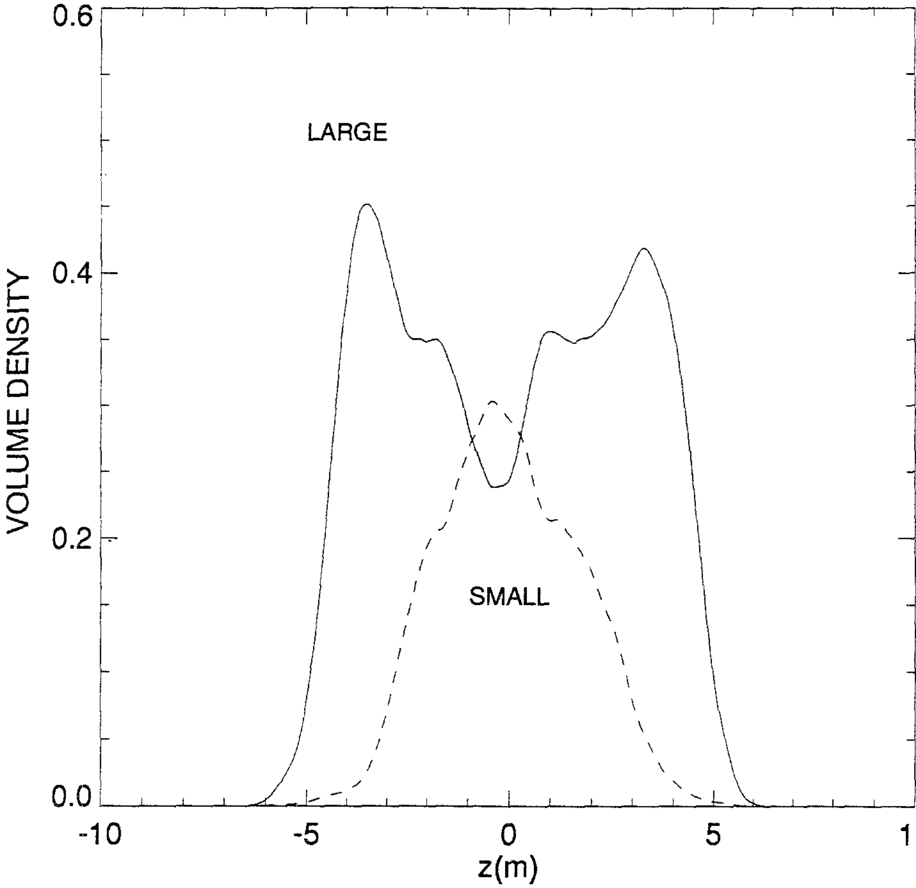

Fig. 8. Vertical equilibrium distribution of particles for the same mixture as in Figure 7. The optical thickness is $\tau = 5$. Solid lines correspond to large particles and dashed lines to small ones.

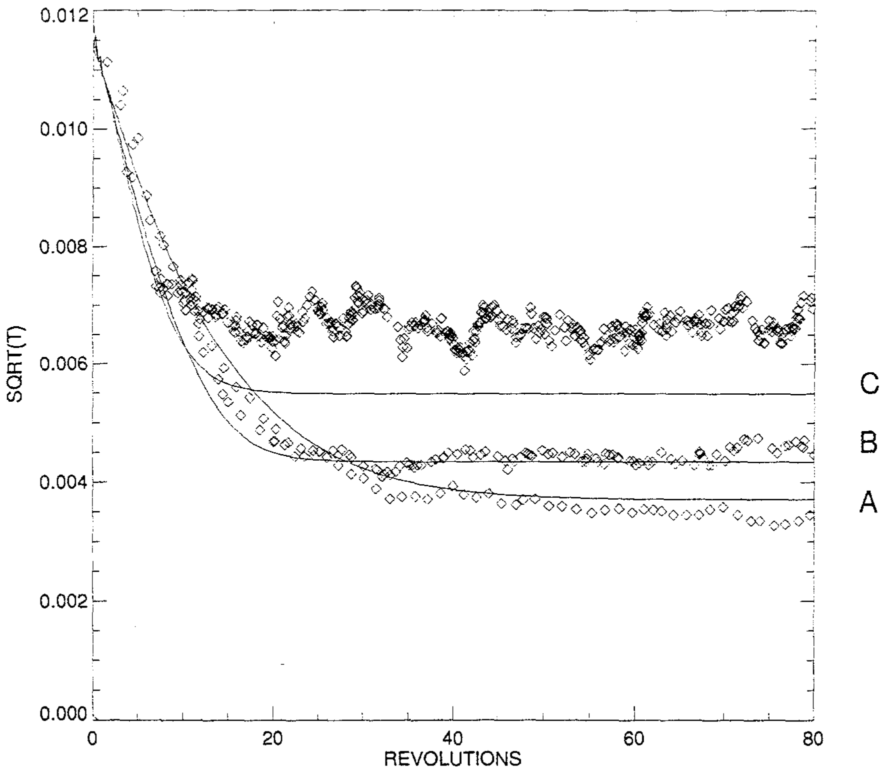

Fig. 9. Evolution of $\sqrt{\text{trace}\,\mathbf{T}}$ for gravitating particles having $\alpha = 0.5$, $\beta = 0$, $\sigma = 5$ cm and $\tau = 0.1$. The internal density is 0.0 gr cm^{-3} (curve A), 0.9 gr cm^{-3} (curve B) and 1.8 gr cm^{-3} (curve C).

Their gravitational fields heat the system and probably cause the discrepancy. They also produce the fluctuations in curve C. The system seems to be on the verge of gravitational instability, and an increase of τ actually led to the formation of semi-permanent particle clouds.

13. Discussion

The advantages of the theory discussed in the preceding sections are its great generality and the avoidance of multidimensional numerical integrations without any appreciable loss of accuracy. Better results could be achieved if the corrections for differential velocity were calculated more precisely, but since the approximations chiefly affect less important quantities in special circumstances ($\tau > 1$, $h \sim \sigma$), the compromises between simplicity and accuracy are justified. There is also an ultimate limit beyond which the analytical methods become uncertain, this limit being set by the filling factor g and the maximum packing density δ_{max}, which must be replaced by empirical functions if high accuracy is required for dense systems. These quantities can occasionally produce drastic effects, as seen in the concentration of the smallest particles near to the equatorial plane in discs having $\tau > 2.5$ (Figures 7 and 8).

The treatment of gravitational interactions between mutually colliding particles is of a preliminary nature, since the results derived from the two-body approximation can only be valid for rarefied matter. The model also ought to be generalized for β, $\gamma \neq 0$. An important phenomenon which was neglected in the theory of encounters is the formation of particle groups (see Section 12D). They can occasionally produce larger gravitational effects than the single particles.

The simple, crude method of substituting $\sqrt{\mathrm{trace}(\mathbf{T} + \mathbf{T}')}$ for $|\mathbf{c} \cdot \mathbf{v}|$ in the relevant functions $f(\alpha, \beta, \gamma)$ turns out to be better than could be expected. A relatively simple alternative would be to calculate $\langle f(\alpha, \beta, \gamma)(\mathbf{c} \cdot \mathbf{v})^2 \rangle / \langle (\mathbf{c} \cdot \mathbf{v})^2 \rangle$ for an isotropic velocity field and to fit an empirical function of $\mathrm{trace}(\mathbf{T} + \mathbf{T}')$ to the data.

The ϵ terms in Equations (61)–(65) were not tested at all. One important application would be the gravitational coagulation of particles and the destruction of the resulting loose configurations in subsequent impacts, but this would have extended the theoretical calculations more than was desirable. The tests were restricted to discs for the same reason, although the basic relations are more generally valid.

References

Anderson, Ian: 1974, *A First Course in Combinatorial Mathematics*, Clarendon Press, Oxford, p. 100.
Araki, S.: 1988, *Icarus* **76**, 182–198.
Araki, S.: 1991, *Icarus* **90**, 139–171.
Araki, S. and Tremaine, S.: 1986, *Icarus* **65**, 83–109.
Barge, P. and Pellat, R.: 1990, *Icarus* **85**, 481–498.
Barge, P. and Pellat, R.: 1991, *Icarus* **93**, 270–287.
Binney, J. and Tremaine, S.: 1987, *Galactic Dynamics*, Princeton University Press, Princeton, p. 121.
Boll, M.: 1957, *Tables numériques universelles*, 2. ed. Dunod, Paris, p. 580.
Borderies, N., Goldreich, P. and Tremaine, S.: 1982, *Nature* **299**, 209–211.
Borderies, N., Goldreich, P. and Tremaine, S.: 1983, *Icarus* **55**, 124–132.
Brahic, A.: 1977, *Astron. Astrophys.* **54**, 895–907.
Bridges, F. G., Hatzes, A. and Lin, D. N. C.: 1984, *Nature* **309**, 333–335.
Brophy, T. G., Stewart, G. R. and Esposito, L. W.: 1990, *Icarus* **83**, 133–155.
Chandrasekhar, S.: 1960, *Principles of Stellar Dynamics*, Dover Publications Inc., New York, p. 251.
Enskog, D.: 1922, Kungliga Svenska Vetenskapsakademiens Handlingar 63, No. 4.
Goldreich, P. and Tremaine, S.: 1978, *Icarus* **34**, 227–239.
Hämeen-Anttila, K. A.: 1978, *Astrophys. Space Sci.* **58**, 477–519.
Hämeen-Anttila, K. A.: 1983, *Moon and Planets* **28**, 267–303.
Hämeen-Anttila, K. A.: 1984, *Earth, Moon, and Planets* **31**, 271–299.
Hämeen-Anttila, K. A., Hänninen, J. and Verronen, M.: 1988, *Earth, Moon, and Planets* **43**, 61–86.
Hämeen-Anttila, K. A. and Lukkari, J.: 1980, *Astrophys. Space Sci.* **71**, 475–497.
Hornung, P., Pellat, R. and Barge, P.: 1985, *Icarus* **64**, 295–307.
Lukkari, J.: 1978, *Astrophys. Space Sci.* **58**, 113–123.
Salo, H.: 1987a, *Earth, Moon, and Planets* **38**, 149–181.
Salo, H.: 1987b, *Icarus* **70**, 37–51.
Salo, H.: 1991, *Icarus* **90**, 254–270.
Shu, F. and Stewart, G.: 1985, *Icarus* **62**, 360–383.

Stewart, G. R., Lin, D. N. C. and Bodenheimer, P.: 1984, in R. Greenberg and A. Brahic (eds.), *Planetary Rings*, Univ. of Arizona Press, Tucson, pp. 447–512.
Trulsen, J.: 1972a, *Astrophys. Space Sci.* **17**, 241–262.
Trulsen, J.: 1972b, *Astrophys. Space Sci.* **18**, 3–20.
Trulsen, J. and Wikan, A.: 1980, *Astron. Astrophys.* **91**, 155–160.
Verronen, M.: 1989, *Earth, Moon, and Planets* **47**, 51–59.
Wetherill, G. W. and Cox, L. P.: 1984, *Icarus* **60**, 40–55.
Wetherill, G. W. and Cox. L. P.: 1985, *Icarus* **63**, 290–303.
Wisdom, J. and Tremaine, S.: 1988, *Astron. J.* **95**, 925–940.