GEOIDAL POTENTIAL FREE OF ZERO-FREQUENCY TIDAL DISTORTION

MILAN BURŠA
Astronomical Institute, Czech Acad. Sci., Prague

(Received 2 March 1995)

Abstract. It has been proved that the geoidal value W_0 is not dependent on the system used for defining the geoid surface. It is the same for the zero-frequency tidal system, mean system and tide-free system. It has been suggested, W_0 be adopted as primary constant defining the length dimensions of celestial bodies.

1. Introduction

There are the direct, as well as indirect tidal Earth's distortions due to the Moon and the Sun (Zadro and Marussi, 1973) which affect the undulations of the equipotential surfaces, in general. These effects make the semimajor axis and the flattening of the Earth's ellipsoid larger compared to the tide-free parameters. However, the geopotential value W_0 on the geoid remains unchanged. The aim of the paper is to prove this significant fact which makes W_0 advantageous as the primary fundamental constant defining the length dimensions of the planets.

2. Geoidal Potential

If no tidal forming bodies, the geoidal potential W_0 at any point (ρ, ϕ, Λ) situated on the geoid $W = W_0$ can be expressed as

$$W_0 = GM\frac{U}{\rho}, \tag{1}$$

$$\begin{aligned} U \; = \; & 1 + \sum_{n=2}^{\infty}\sum_{k=0}^{n}\left(\frac{a_0}{\rho}\right)^n (J_n^{(n)} \cos k\Lambda + S_n^{(k)} \sin k\Lambda) \times \\ & \times P_n^{(k)}(\sin\phi) + \frac{1}{3} q \left(\frac{a_0}{\rho}\right)^{-3} [1 - P_2^{(0)}(\sin\phi)]; \end{aligned}$$

$$q = \frac{\omega^2 a_0^3}{GM}. \tag{2}$$

Notations: GM is the geocentric gravitational constant; $J_n^{(k)}$ and $S_n^{(k)}$ the Stokes parameters (geopotential coefficients) at the Legendre associated function $P_n^{(k)}$, degree n, order k; a_0 the scaling parameter associated with $J_n^{(k)}$ and $S_n^{(k)}$ to be dimensionless; ω the angular velocity of the Earth's rotation; ρ, ϕ, Λ the geocentric radius vector, latitude and longitude, respectively.

If tidal forming bodies, the Moon and the Sun, the tidal perturbations occur, direct and indirect. However, only the zonal zero-frequency tidal terms are of interest from the point of view of the problem posed. The direct zero-frequency tidal zonal term due to the Moon reads (Zadro and Marussi, 1973),

$$V(\rho,\phi,\Lambda) = \frac{GM_{☾}}{\Delta_{\oplus☾}} \sum_n \left(\frac{\rho}{\Delta_{\oplus☾}}\right)^n V_n^{(0)} P_n^{(0)}(\sin\phi), \tag{3}$$

$$V_2^{(0)} = \frac{3}{4}\sin^2\epsilon - \frac{1}{2} + \Delta V_2^{(0)}, \tag{4}$$

$$V_4^{(0)} = \frac{105}{64}\sin^4\epsilon - \frac{15}{8}\sin^2\epsilon + \frac{3}{8} + \Delta_4^{(0)}, \tag{5}$$

$$V_6^{(0)} = \frac{1155}{256}\sin^6\epsilon - \frac{945}{128}\sin^4\epsilon + \frac{105}{32}\sin^2\epsilon - \frac{5}{16} + \Delta_6^{(0)}. \tag{6}$$

Notations: $GM_{☾} = 4902.799 \times 10^9$ m^3 s^{-2} is the selenocentric gravitational constant, $\Delta_{\oplus☾}$ = 384 400 km the mean Earth–Moon distance, $\epsilon = 23°26'21.4''$ the obliquity of the ecliptic. Terms $\Delta_n^{(0)}$ represent corrections to the main terms, functions of the orbital elements e, i of the Moon's orbit.

The indirect zero-frequency tidal effect δV in the geopotential due to the mass transfer is

$$\delta V(\rho,\phi,\Lambda) = \frac{GM_{☾}}{\Delta_{\oplus☾}} \sum_n \left(\frac{\rho}{\Delta_{\oplus☾}}\right)^n \delta V_n^{(0)} P_n^{(0)}(\sin\phi), \tag{7}$$

$$\delta V_n^{(0)} = k_{s,n} V_n^{(0)}; \tag{8}$$

$k_{s,n}$ are the secular Love numbers responsible for mass deformations.

Because of additional deforming potential, Equation (7), the distortions $\delta J_n^{(0)}$ in the zonal Stokes parameters $J_n^{(0)}$ occur. They result from the solution of the first (Dirichlet's) boundary-value problem for the sphere, radius $\bar{R}$ = 6 371 km, as

$$\delta J_n^{(0)} = k_{s,n} \frac{GM_{☾}}{GM} \left(\frac{\bar{R}}{\Delta_{\oplus☾}}\right)^{n+1} \left(\frac{\bar{R}}{a_0}\right)^n V_n^{(0)}. \tag{9}$$

At any fixed point the zero-frequency direct plus indirect tidal distortion makes the actual geopotential different from the ideal tide-free potential. As the Bruns theorem suggests, the tidally distorted equipotential surface is shifted from that tide-free, however, the same geopotential value is appropriate to both surfaces. It means, geoidal potential W_0 should remain unchanged for both, tide-free geoid, as well as, tidally distorted one. It makes W_0 advantageous primary constant of common relevance of astronomy, geodesy, and geodynamics. However, we wish to prove the above fact without applying the Bruns theorem.

3. Zero-frequency Tidal Distortions of the Geoid Surface

We wish to introduce the fictitions variation δW_0 due to the direct plus indirect zero-frequency zonal tidal terms, as follows:

$$\begin{aligned} W_0 + \delta W_0 &= GM\frac{U}{\rho} + V(\rho,\phi,\Lambda) - \frac{GM}{\rho^2}(\Delta\rho + \delta\rho) + \\ &+\frac{GM}{\rho}\sum_n \left(\frac{a_0}{\rho}\right)^n \delta J_n^{(0)} P_n^{(0)}(\sin\phi); \end{aligned} \tag{10}$$

$\Delta\rho$ is the direct zero-frequency tidal distortion in ρ,

$$\Delta\rho = \frac{V(\rho,\phi,\Lambda)}{g}, \tag{11}$$

$\delta\rho$ the indirect one

$$\delta\rho = \frac{\delta V(\rho,\phi,\Lambda)}{g}; \tag{12}$$

g is gravity at $P(\rho,\phi,\Lambda)$. Considering the frequently used spherical tidal distortion model,

$$\Delta\rho = \bar{R}\frac{GM_☾}{GM}\sum_n \left(\frac{\bar{R}}{\Delta_{\oplus☾}}\right)^{n+1} V_n^{(0)} P_n^{(0)}(\sin\phi), \tag{13}$$

$$\delta\rho = \bar{R}\frac{GM_☾}{GM}\sum_n k_{s,n}\left(\frac{\bar{R}}{\Delta_{\oplus☾}}\right)^{n+1} V_n^{(0)} P_n^{(0)}(\sin\phi). \tag{14}$$

Considering Equations (1), (3), (13) and (14), the fictitions variation in the geoidal value can be expressed as follows:

$$\begin{aligned} \delta W_0 &= \frac{GM_☾}{\Delta_{\oplus☾}}\sum_n \left(\frac{\bar{R}}{\Delta_{\oplus☾}}\right)^n V_n^{(0)} P_n^{(0)}(\sin\phi) - \\ &-\frac{GM_☾}{\bar{R}}\sum_n (1+k_{s,n})\left(\frac{\bar{R}}{\Delta_{\oplus☾}}\right)^{n+1} V_n^{(0)} P_n^{(0)}(\sin\phi) + \\ &+\frac{GM}{\bar{R}}\sum_n \left(\frac{a_0}{\bar{R}}\right)^n \delta J_n^{(0)} P_n^{(0)}(\sin\phi). \end{aligned}$$

After inserting $\delta J_n^{(0)}$ by Equation (9) we get

$$\delta W_0 = 0.$$

4. Effect of the Long-term Variation in the Second Zonal Stokes Parameter

The fact, W_0 does not depend on the zero-frequency tidal distortions, makes it advantageous for solving global geodynamic problems. It is the same for the tide-free, mean and zero-frequency tidal systems. Moreover, it does not depend, e.g., on the long-term variation in the second zonal Stokes parameter detected by LAGEOS orbit dynamics (Nerem *et al.*, 1994). To prove it, the rotational second-degree spheroid defined by four parameters GM, $\omega(q)$, W_0, $J_2^{(0)}$ will be used. Its radius is

$$\begin{aligned}\rho_{sph} = \frac{GM}{W_0}\Bigg\{1 + \frac{1}{3}\left(\frac{R_0}{a_0}\right)^3 q + \\ + \left[\left(\frac{a_0}{R_0}\right)^2 J_2^{(0)} - \frac{1}{3}\left(\frac{R_0}{a_0}\right)^3 q\right] P_2^{(0)}(\sin\phi)\Bigg\},\end{aligned} \tag{15}$$

and, if $\rho_{sph}(\phi = 0) = a$,

$$W_0 = \frac{GM}{a}\left\{1 + \frac{1}{3}\left(\frac{R_0}{a_0}\right)^3 q - \frac{1}{2}\left[\left(\frac{a_0}{R_0}\right)^2 J_2^{(0)} - \frac{1}{3}\left(\frac{R_0}{a_0}\right)^3 q\right]\right\}; \tag{16}$$

$R_0 = GM/W_0 = 6\ 363\ 672.4$ m is the geopotential scale factor. The fictitions variation in W_0 due to $\mathrm{d}J_2^{(0)}/\mathrm{d}t$ expressed using Equation (16) (Burša, 1995)

$$\frac{\mathrm{d}W_0}{\mathrm{d}t} = -\frac{GM}{a^2}\frac{\mathrm{d}a}{\mathrm{d}t} - \frac{1}{2}\frac{GM}{a}\left(\frac{a_0}{R_0}\right)^2 \frac{\mathrm{d}J_2^{(0)}}{\mathrm{d}t}. \tag{17}$$

Variation in semimajor axis follows from

$$\begin{aligned}\frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}t} &= \frac{1}{\rho g}\frac{\mathrm{d}W}{\mathrm{d}t} P_2^{(0)}(\sin\phi) \\ &= \frac{GM}{\rho^2 g}\left(\frac{a_0}{R_0}\right)^2 \frac{\mathrm{d}J_2^{(0)}}{\mathrm{d}t} P_2^{(0)}(\sin\phi)\end{aligned} \tag{18}$$

after specifying $\phi = 0$, as

$$\frac{\mathrm{d}a}{\mathrm{d}t} = -\frac{1}{2} a \left(\frac{a_0}{R_0}\right)^2 \frac{\mathrm{d}J_2^{(0)}}{\mathrm{d}t}. \tag{19}$$

TABLE I

Geopotential value on the geoid computed in the tide-free, the zero frequency and the mean systems; adopted: $GM = 398\,600.4415 \times 10^9$ m^3 s^{-2}, $\omega = 7\,292\,115 \times 10^{-5}$ rad s^{-1}, Love number 0.3

System	a (m)	$1/\alpha$	W_0 (m^2 s^{-2})	$R_0 = GM/W_0$ (m)
Mean	6 378 136.59	298.25232	62 636 856.83	6 363 672.463
Zero	6 378 136.49	298.25643	62 636 856.85	6 363 672.461
Tide-free	6 378 136.46	298.25766	62 636 856.85	6 363 672.461

After inserting into Equation (17) we get

$$\frac{dW_0}{dt} = 0$$

which was to prove.

Again, surface $W = W_0$ has been perturbed by $dJ_2^{(0)}/dt$, however, W_0 does not vary. That is why, W_0 is significant for the global geodynamics, e.g., for the low-frequency geodynamics.

At the end, we wish to demonstrate once again numerically, that W_0 is independent on zero-frequency tidal distortions (Burša, 1995). For this we shall use the most recent solution for semimajor axis (a) and flattening (α) by Rapp (1994) in the mean and the zero tidal systems, as well as, in the tide-free system (Table I). Geoidal value W_0 was computed by the well known formula appropriate to the rotational level ellipsoid defined by four parameters GM, q, a, α, given e.g., in (Moritz, 1984). The differencies in W_0 are due to the rounding errors in the input parameters.

5. Conclusions

Geoidal potential value W_0 as free of tidal system can be selected as primary constant of common relevance of astronomy, geodesy, and geodynamics. It is appropriate to the zero-frequency tidal system, mean system, as well as, tide-free system.

The major advantage of W_0 is, its determination does not need the global coverage by the data used for its determination. Theoretically, one exact geocentric position on the geoidal surface is sufficient, even if situated in a geoidal area very anomalous. That is why, the area for determining W_0 can be selected to be free of the outer masses, i.e., it can represent the oceans and seas only. On the contrary, the continental areas should be avoided, because surface $W = W_0$ intersects the

Earth's mass there and the regularization problem arises. Geoidal value W_0 is appropriate to any point of surface $W_i = W_0$. Limited coverage, e.g., by satellite altimeter data, makes the estimation for semimajor axis on dm-level difficult as regards the actual accuracy However, any estimate of W_0 is free of it.

If adopted as primary, the length dimension of the celestial body is defined. The semimajor axis of the "best fitting" ellipsoid can be derived on the basis of W_0, however, a condition is needed, to be posed *a priori*.

References

Burša, M.: 1995, 'Primary and Derived Parameters of Common Relevance of Astronomy, Geodesy, and Geodynamics', *Earth, Moon, and Planets* **65**, (in print).

Moritz, H.: 1984, 'Geodetic Reference System 1980', in C. C. Tscherning (ed.), *Bull. Géodesique AIG*, Vol. 58, No. 3, pp. 388–398.

Nerem, R. S., Chao, B. F., Au, A. Y., Chan, J. C., Klosko, S. M., Pavlis, N. K., and Williamson, R. G.: 1993, 'Time Variations of the Earth's Gravitational Field from Satellite Laser Ranging to LAGEOS', *Geophys. Res. Lett.* **20**(7), 595–598.

Rapp, R. H., Yi, Y., and Wang, Y. M.: 1994, 'Mean Sea Surface and Geoid Gradient Comparisons with Topex Altimeter Data', *J. Geophys. Res.* **99**, C12, 24, 657-24, 667.

Zadro, M. B. and Marussi, A.: 1973, 'On the Static Effect of Moon and Sun on the Shape of the Earth', in *ATTI Proc. V Simp. sulla Geodesia Matematica*, Firenze, 25–26 Ottobre 1972, pp. 249–267.