ROTATIONAL DYNAMICS OF CELESTIAL DEFORMABLE BODIES

III: *Effects of Viscosity on the Dynamics of the Earth-Moon System*

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Abstract. In a previous paper of this series (Tokis, 1974b), we have discussed the solution of the Eulerian equation which governs the axial rotation, applied to the effects of viscous friction exhibited in binary systems which consist of a close pair of fluid bodies of arbitrary structure. The aim of the present paper will be to give an application of those results to the Earth-Moon system.

It is shown that synchronism between the axial rotation of the Earth and the revolution of the Moon will occur at the value of 650 h, in a time scale which depends strongly on the value of the mean viscosity of the Earth (regarded as spherical or spheroidal). In particular, the variation of rotational angular velocity of the Earth over the next ten centuries commencing from 1900 A.D., depends sensitively on the value of viscosity. On the other hand, the time for synchronism of axial rotation of the Moon is not affected by the viscosity for values between 10^{24}g cm^{-1} s^{-1} and 10^{27}g cm^{-1} s^{-1}.

1. Introduction: Equations of the Problem

In preceding papers of this series (Tokis, 1974a; 1974b; hereafter referred to as Papers I and II, respectively) the Eulerian equations have been set up explicity (cf. Equations (8.2) of Paper I) for three-dimensional rotation of self-gravitating compressible celestial bodies of arbitrary structure, for the purpose of describing the effects of tidal reformation in a close pair of such bodies. The viscosity of their material has been regarded as an arbitrary function of spatial coordinates. Moreover, the solution of one of those differential equations, namely that which governs the simple rotation about an axis perpendicular to the orbital plane of binary system, has already been discussed (see Sections 3 and 4 of Paper II).

In the present paper we shall study an important application of the previous theoretical results to the Earth-Moon system. We shall try to throw additional light on the dynamical evolution of the system; in particular we shall examine the effects of viscosity on the astronomical future of the rotation of the Earth and the Moon.

The variations of their axial rotation depend on the complex gravitational interaction of a many-body system; however, we shall concentrate on the aspects of evolution of a close planet-satellite system arising from tidal processes, as we have already discussed (Section 1 of Paper II).

In specifying our problem in the present paper, we consider that the Moon travels in an elliptic orbit around the Earth and that their axies of rotation are (almost) perpendicular to the orbital plane. In accordance with the theory of Papers I and II,

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we shall suppose that, when considering tidal effects on the rotation of one component of the system, we may treat its companion as a mass point. Thus, when the components of our system are treated as nearly spherical, the equation which governs the rate of change of angular velocity ω of axial rotation of each component is (from Paper II, Equation (21))

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} + \frac{q}{\Delta^3}\omega - \frac{q}{\Delta^3}n = 0, \tag{1}$$

where:

$$\Delta = \frac{A(1-e^2)}{1+e\cos v} \tag{2}$$

is the separation of the two components;

$$n = \omega_K\left(\frac{A}{\Delta}\right)^2\sqrt{1-e^2} = \omega_K\frac{(1+e\cos v)^2}{(1-e^2)^{3/2}} \tag{3}$$

is the instantaneous orbital angular velocity, with

$$\omega_K^2 = \frac{G(m_1+m_2)}{A^3} \tag{4}$$

denoting the Keplerian mean angular velocity of orbital revolution, A = semi-major axis, e = orbital eccentricity, v = true anomaly measured from the periastron and G = gravitational constant; and

$$q = \frac{4\pi(1+k_2)\,m_2}{5Cm_1}\left[\mu r^6\right]_0^{R_1} \tag{5}$$

is a constant (cf. Equation (22) of Paper II). In latter equations the 'apsidal motion' constant k_2, ths mass m_1, the moments of inertia

$$C = m_1 h_1^2, \tag{6}$$

where h_1 = radius of gyration, the mean radius R_1 and the viscosity μ refer to the rotating component under consideration and m_2 to the revolving companion.

The solution of the differential Equation (1) is, with a sufficient approximation (omitting terms of magnitude $\leqslant 10^{-13}$),

$$\omega(t) = \omega_K\frac{1+e^2}{(1-e^2)^{3/2}} + \left\{\omega_0 - \omega_K\frac{1+e^2}{(1-e^2)^{3/2}}\right\}\exp\left[-\frac{qt}{A^3(1-e^2)^{3/2}}\right] \tag{7}$$

for the axial rotation of the Earth, in which we treat the orbital elements A and e to be nearly constant and the terrestrial viscosity has value $\mu_\oplus \leqslant 10^{11}$ g cm^{-1} s^{-1} (cf. equation (44) of Paper II).

The solution of (1) for the axial rotation of the Moon will be found by numerical methods, as was suggested in Section 3.2 of Paper II.

In the simple case of a circular relative orbit, the solution of (1) is given by

$$\omega(t) = \omega_K + (\omega_0 - \omega_K) \exp\left[-\frac{gt}{\Delta^3}\right] \tag{8}$$

(cf. Equation (25) of Paper II).

Moreover, if the Earth is treated as a body with spheroidal form, the appropriate differential equation is

$$C \frac{\mathrm{d}\omega}{\mathrm{d}t} + \tfrac{4}{5}\pi(1 + k_2)(\omega - n) \frac{[\mu r^6]_0^{R_1} m_2}{\Delta^3 m_1} + \tfrac{4}{105}\pi(\omega - n) \frac{\omega^2 m_2}{G m_1} \times$$
$$\times \left\{114 \frac{1 + k_2}{\Delta^3} \int_0^{R_1} \mu r^8 \,\mathrm{d}r + 85 \frac{1 + k_4}{\Delta^5} \int_0^{R_1} \mu r^{10} \,\mathrm{d}r + \right.$$
$$\left. + 5 \frac{1 + k_4}{\Delta^5} [\mu r^{11}]_0^{R_1} \right\} = 0, \tag{9}$$

where the moment of inertia is given by

$$C = m_1 h_1^2 + \frac{2k_2 R_1^5 \omega^2}{9G} + \frac{k_2 m_2 R_1^5}{3\Delta^3}; \tag{10}$$

k_4 is the 'apsidal motion' constant, and the other symbols refer to the Earth, except for m_2, the mass of the Moon. The solution of this equation will be found by numerical methods, as we suggested in Section 4 of Paper II, regarding the viscosity as constant throughout the whole Earth.

2. Parameters of the Earth-Moon System

In order to solve the differential Equations (1) or (9) for the case of the Earth-Moon system, we must assign numerical values to the parameters occurring in these equations.

A. 'APSIDAL-MOTION' CONSTANTS

The 'apsidal-motion' constants depend on the internal structure (density concentration) of the rotating body, and are given by the form

$$k_j = \frac{j + 1 - \eta_j(R_1)}{j + \eta_j(R_2)}, \tag{11}$$

where η_j are the logarithmic derivatives of functions f_j which satisfy the Clairaut equation (Kopal, 1969a). The k_j rapidly become small if the density concentration of the respective configuration grows large. On the other hand, for homogeneous configurations we have

$$k_j = \frac{3}{2(j - 1)}. \tag{12}$$

For the Earth we have $\eta_2(R_1)=0.5627$ and $\eta_4(R_1)=3.246$ (James and Kopal, 1963; Melchior, 1972) and, therefore, it follows from (11) that $k_2=0.951$ and $k_4=0.242$. For the 'apsidal-motion' constant k_2 of the Moon we can take the value 1.5, from (12), if we regard the Moon to be a very nearly homogeneous body.

B. RADIUS OF GYRATION

When the value of k_2 is known it is possible to find the radius of gyration h_1 of a fluid body, which is defined by the equation

$$m_1 h_1^2 = \tfrac{8}{3}\pi \int_0^{R_1} \varrho r^4 \,\mathrm{d}r, \tag{13}$$

where R_1 denotes the mean radius of this body, while the k_2 is given approximately by

$$k_2 = \frac{32\pi}{5 m_1 R_1^5} \int_0^{R_1} \varrho r^7 \,\mathrm{d}r \tag{14}$$

(Kopal, 1972b; p. 164).

The constant h_1 could be evaluated by quadratures from (13) if we knew their individual internal structure; in the present case, in which the internal structure is unknown, it may be determined, very approximately, from an empirical relationship between h_1/R_1 and k_2 from the Equations (13) and (14). Numerical work by Motz (1952) has disclosed that

$$\log\left(\frac{h_1}{R_1}\right)^2 = 0.42 \log k_2 - 0.47, \tag{15}$$

where the value of the constant on the right-hand side has been adjusted so that the values of k_2 are twice those used by Motz. Thus, from (15), we can obtain the fractional radii of gyration h_1/R_1 of our body.

For homogeneous bodies we have, as is well known,

$$\frac{h_1^2}{R_1^2} = \tfrac{2}{5}. \tag{16}$$

In the present work we shall use the latter equation for the determination of the radius of gyration of the Moon.

For the Earth, from (15), it may be calculated that

$$\frac{h_\oplus^2}{R_\oplus^2} = 0.33, \tag{17}$$

where we have used $k_2=0.951$.

C. VISCOSITY

The constituent components of the Earth-Moon system should be regarded as consisting of compressible fluid of arbitrary structure, and the viscosity of their material is an appropriate function of its physical state.

The mean viscosity of the Moon remains still largely a matter of conjecture and different values have been suggested. Using various techniques it is possible to derive a value of average viscosity of the lunar curst of about 10^{25} or 10^{26} g cm^{-1} s^{-1} – at least down to a depth of some 100 km (Urey, 1968; Kopal, 1969b; Baldwin, 1970). On the other hand, the proposition of convection in the Moon to explain the departure of the lunar surface from hydrostatic equilibrium (Runcorn, 1967) requires that the viscosity of the lunar interior be less than 10^{24} g cm^{-1} s^{-1} (Turcotte and Oxburgh, 1969). But, recently, using data from the surface topography and gravitational potential on the Moon, the relaxation time ($\sim$5 b.y.) of the mascons may be estimated, by means of which the viscosity of the lunar interior was found to be at least 10^{25} g cm^{-1} s^{-1} (Arkani-Hamed, 1973a). Moreover, using data of the Apollo 15 mission it was calculated that a lower limit to the viscosity of the lunar interior within the last 3.3 b.y. was about 8×10^{26} g cm^{-1} s^{-1} (Arkani-Hamed, 1973b) and in the period 3.8 to 3.3 b.y. ago about 10^{25} g cm^{-1} s^{-1}.

In our work we propose to use for the mean viscosity of the lunar crust the value of

$$\mu_{☾} = 10^{26}\ \mathrm{g\ cm^{-1}\ s^{-1}}, \tag{18}$$

but we will examine the effects on our results of choosing values for the viscosity between 10^{24}–10^{27} cm^{-1} s^{-1}.

In order to estimate the mean value of the viscosity of the Earth, we consider the special case in which the Moon moves around the Earth in a circular orbit. In this case the Equation (1), governing the axial rotation of the Earth, applies as follows from a consideration of the assumptions made for the Earth-Moon system, which were needed for the derivation of this equation. The solution (8) of this equation indicates that the value of ω is bound to approach ω_K (or n) asymptotically as $t \to \infty$. The rate at which the difference $\omega - \omega_K$ decreases with time depends on the value of $q_\oplus / A^3$ in (8), and will diminish to one-half in the time

$$t_{1/2} = \frac{A^3}{q_\oplus} \ln 2 . \tag{19}$$

But, from (5) by (6) and (17) it follows that

$$\frac{q_\oplus}{A^3} = \frac{4\pi (1 + k_2)}{5 (h_\oplus / R_\oplus)^2} \left(\frac{m_{☾} R_\oplus^4}{m_\oplus^2 A^3} \right) \mu_\oplus = 8.82 \times 10^{-26}\, \mu_\oplus\ \mathrm{s}^{-1}, \tag{20}$$

where $m_{☾} = 7.35 \times 10^{25}$ g the mass of the Moon; $m_\oplus = 5.977 \times 10^{27}$ g, $R_\oplus = 6371$ km are the mass and the radius of the Earth; $A = 384400$ km the mean Earth-Moon distance; $k_2 = 0.951$ the 'apsidal motion' constant of the Earth; and $\mu_\oplus$ the viscosity of the Earth expressed in cgs units.

Moreover, if we assume that the synchronism between the rotation of the Earth and the revolution of the Moon will still be far from attained after a time lapse of the order of 10^9 yr (or 10^{16}s), we can conclude from (19) and (20) that

$$\mu_{\oplus} \sim 10^9 \text{ g cm}^{-1} \text{ s}^{-1} \tag{21}$$

(Kopal, 1972a). On the other hand Kaula (1968) suggested for numerical estimates the value of viscosity of order 10^{22} g cm^{-1} s^{-1} for the upper mantle and 10^{26} g cm^{-1} s^{-1} for the lower mantle of the Earth. Our numerical calculations will be made with the values of viscosity of the Earth in the region of (21). We shall examine the results for various values of viscosity between 10^8 g cm^{-1} s^{-1} and 10^{22} g cm^{-1} s^{-1}.

The numerical values of each of the parameters of the Earth-Moon system are presented in Table I.

TABLE I

Numerical values of parameters

Parameter	Dimensions	c.g.s. units	Earth planetary units	Moon planetary units
Gravitation const.	$M^{-1}L^3T^{-2}$	6.67×10^{-8}	1.0	1.0
Earth				
Mass	M	5.977×10^{27}	1.0	81.3
Mean radius	L	6.371×10^8	1.0	3.67
Angular velocity (1900.0)	T^{-1}	7.2921×10^{-5}	0.0588	
Apsidal-motion const. k_2		0.951		
Apsidal-motion const. k_4		0.242		
Moon				
Mass	M	7.35×10^{25}	0.0123	1.0
Mean radius	L	1.738×10^8	0.2725	1.0
Angular velocity (1900.0)	T^{-1}	$2.661\,699\,489 \times 10^{-6}$		0.002758
Apsidal-motion const. k_2		1.5		
Present Orbit				
Semi-major axis	M	3.844×10^{10}	60.3	221.7
Eccentricity		0.0549		

(Kaula, 1968)

3. Synchronism of Rotation and Revolution of the Moon

In the present section we consider the following situation: the Earth moves around the Moon in a Keplerian orbit so that, at all times, the equatorial plane of the rotating Moon is identified with the orbital plane of the Earth. In addition, the Moon rotates about an axis perpendicular to the Earth's orbital plane.

A. CIRCULAR ORBIT

When the Earth moves in circular orbit about the Moon the rate of change of the rotation of the Moon is governed by the differential Equation (1), the solution of which is given by (8). For a homogeneous Moon the parameters of (8) were given in the previous section; then, from (5) it follows that

$$\frac{q_☾}{A^3} = 5\pi \frac{m_\oplus R_☾^4}{m_☾^2 A^3} \mu_☾ = 2.791\,755 \times 10^4 \text{ s}^{-1}, \tag{22}$$

where we have used the relation (6) and Table I. With this value of $q_☾/A^3$ the Equation (8) makes it evident that the angular velocity is approximately equalized with the orbital angular velocity n (or ω_K) in a time scale of the order of seconds. Table II

TABLE II

Variation of angular velocity of axial rotation of the Moon

Time (sec)	Angular velocity (10^{-6} rad s^{-1})	
	Circular orbit	Elliptic orbit
0.	2.661 699 489	2.661 699 489
0.004	2.664 119 061	2.882 316 759
0.007	2.665 010 062	2.949 877 541
0.011	2.665 338 171	2.970 567 033
0.014	2.665 458 996	2.976 902 885
0.018	2.665 503 490	2.978 843 146
0.022	2.665 519 874	2.979 437 322
0.025	2.665 525 908	2.979 619 280
0.029	2.665 528 130	2.979 675 002
0.032	2.665 528 948	2.979 692 066
0.038	2.665 529 249	2.979 697 291
0.043	2.665 529 401	2.979 699 381
0.053	2.665 529 424	2.979 699 592
0.064	2.665 529 425	2.979 699 598
0.072	2.665 529 425	2.979 699 598

displays the solutions of (1) from (8) for various values of the time t starting from $t=0$ until the values of ω tend to a constant (equal to $\omega_K = 2.665\,529\,425 \times 10^{-6}$ rad s^{-1}); thus, from this table, the time scale of synchronism is of order 10^{-2}s.

B. ELLIPTIC ORBIT

If the Earth moves in an elliptic orbit around the Moon, the solution of the differential Equation (1) can be found by numerical integration of the system of Equations (54) and (50) (of Paper II). The results of this integration are again presented in Table II. The time scale, in which the ω becomes constant, is again of order 10^{-2} s, as in the circular case, but the value of ω tends asymptotically to the value

$$n_p = \omega_K \frac{(1+e)^2}{(1-e^2)^{3/2}} \simeq 2.979\,699\,598 \times 10^{-6} \text{ rad s}^{-1}, \tag{23}$$

which is the mean angular valocity n_p at the pericentre passage which may be found from (3) setting the true anomaly $v=0$.

In the two cases, namely those of circular and elliptical orbits, the value of the mean viscosity $\mu_☾$ of lunar crust was chosen to be 10^{26} g cm^{-1} s^{-1}. For different values of $\mu_☾$ the results are unchanged: that is, we find the values given in the Table II for values of $\mu_☾$ lying between 10^{24} g cm^{-1} s^{-1} and 10^{27} g cm^{-1} s^{-1}. Figure 1 illustrates the results of Table II.

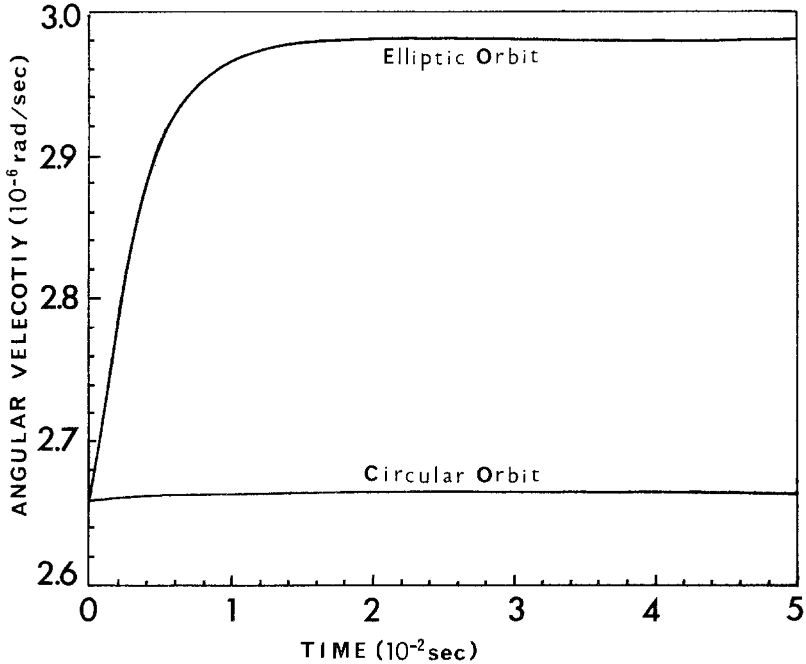

Fig. 1. Variation of angular velocity of axial rotation of the Moon.

Finally, for viscosities of the crustal order of magnitude (10^{24}–10^{27} g cm^{-1} s^{-1}) the tidal control by the Earth of the lunar rotation should be virtually instantaneous, and see to it that the angular velocity should be virtually equal to the mean angular velocity n at all times.

4. Viscosity and Axial Rotation of the Earth

In the present section we assume that the Moon moves in a Keplerian orbit around the Earth, which itself rotates about an axis perpendicular to the plane of the lunar orbit, so that the equatorial plane of the Earth always coincides with the orbital plane of the Moon.

The calculations were performed for a range of values of the mean viscosity of the Earth and for various time scales; these are given in the tables and are illustrated in the figures. However, in the text, reference will be made to the time scale fixed by the value of the viscosity 10^9 g cm^{-1} s^{-1}, because this value gives results comparable with those of some previous investigators. The Adams-Moulton method for numerical integration of the first-order differential equations is used.

A. CIRCULAR ORBIT

The variation of the angular velocity of axial rotation of the Earth with the Moon travelling in a circular orbit (whose elements are regarded as constant) is given by Equation (8). Using this equation we obtain values given in Table III which presents

TABLE III

Variation of angular velocity of axial rotation of the Earth (spherical model; e, A = constant)

Time (10^9 yr)			Angular velocity (10^{-5} rad s^{-1})		
$\mu = 10^8$ cgs	$\mu = 10^9$ cgs	$\mu = 10^{10}$ cgs	Circular orbit	Elliptic orbit	
				Numerical integration	Series expansion
0.	0.	0.	7.2921	7.2921	7.2921
0.1788	0.0179	0.0018	6.9495	6.9481	6.9480
0.4470	0.0447	0.0045	6.4666	6.4633	6.4633
1.3410	0.1341	0.0134	5.0951	5.0877	5.0876
2.2351	0.2235	0.0224	4.0271	4.0175	4.0173
3.1291	0.3129	0.0313	3.1952	3.1850	3.1848
3.5761	0.3576	0.0358	2.8511	2.8409	2.8407
7.1522	0.7152	0.0715	1.2174	1.2109	1.2105
10.7282	1.0728	0.1073	0.6163	0.6139	0.6135
14.3043	1.4304	0.1430	0.3952	0.3953	0.3949
17.8803	1.7880	0.1788	0.3139	0.3152	0.3148
21.4564	2.1456	0.2146	0.2840	0.2859	0.2855
25.0325	2.5032	0.2503	0.2730	0.2752	0.2748
28.6086	2.8609	0.2861	0.2689	0.2712	0.2708
32.1847	3.2185	0.3218	0.2674	0.2698	0.2694
35.7608	3.5761	0.3576	0.2669	0.2693	0.2689
39.3369	3.9337	0.3934	0.2667	0.2691	0.2687
42.9130	4.2913	0.4291	0.2666	0.2690	0.2686
46.4891	4.6489	0.4649	0.2666	0.2690	0.2686

the variation of ω with different values of viscosity of the Earth. Thus, the angular velocity ω becomes constant and equal to

$$\omega_s = 2.666 \times 10^{-6} \text{ rad s}^{-1} \simeq \omega_K \tag{24}$$

after a time of about 4.3 b.y. (in fact this time scale depends on the chosen value for the viscosity). Figure 2 illustrates the variation of angular velocity for various time scales corresponding to different values of viscosity.

Hence, the period of the Earth is at present increasing and, from (24), will become

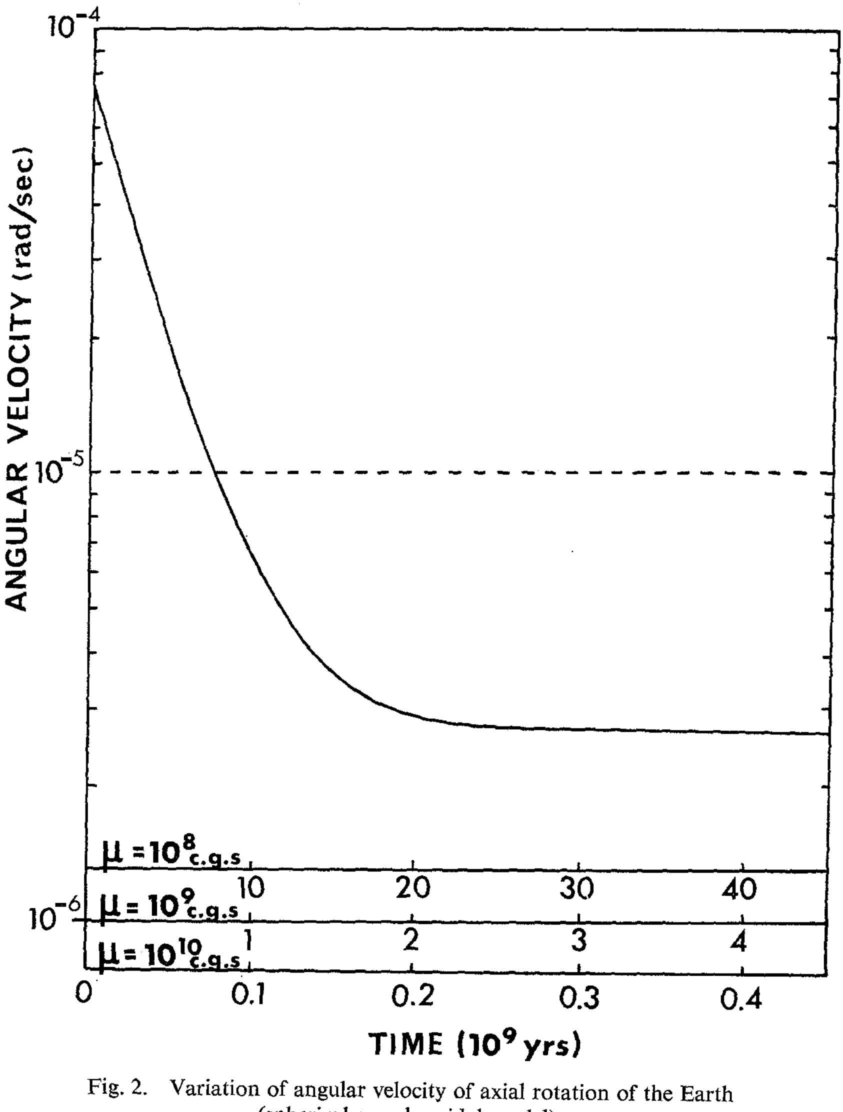

Fig. 2. Variation of angular velocity of axial rotation of the Earth (spherical or spheroidal model).

655 hr long about 4.3 b.y. from now. The length of the terrestrial day and the sidereal month will be approximately in synchronism after 4.3 b.y.

B. ELLIPTIC ORBIT

When the Moon travels in an elliptic orbit (with constant elements) we must integrate the differential Equation (1) as mentioned in Section 3.2 of Paper II. Table III contains the results of these integrations for different values of viscosity. At the same time these results have been checked by comparison with the approximate solution (7) of (1); as we can see from Table III an accuracy of order 10^{-8} is attained for results with viscosities

$$\mu_\oplus \leqslant 10^{10} \text{ g cm}^{-1} \text{ s}^{-1}. \tag{25}$$

From this table we conclude that the angular velocity becomes constant and equal to

$$\omega_s = 2.686 \times 10^{-6} \text{ rad s}^{-1} \simeq \omega_K \frac{1+e^2}{(1-e^2)^{3/2}} \tag{26}$$

after 4.3 b.y. from now, as in the circular case.

Figure 2 shows the behaviour of the angular velocity as a function of the time, while Figure 3 shows the difference between the circular and elliptic cases. Figure 3 indicates that the angular velocity in the elliptic case decreases more rapidly than for the circular case until about 1.3 b.y., when this phenomenon reverses and the values become constant and equal to 2.686×10^{-6} rad s^{-1} for the former, and to 2.666×10^{-6} rad s^{-1} for the latter, at about the same time (4.3 b.y.).

Thus, the period of the rotation of the Earth in the elliptic case tends to a constant value of 650 hr after 4.3 b.y., while in the circular case the asymptotic value is about 655 hr.

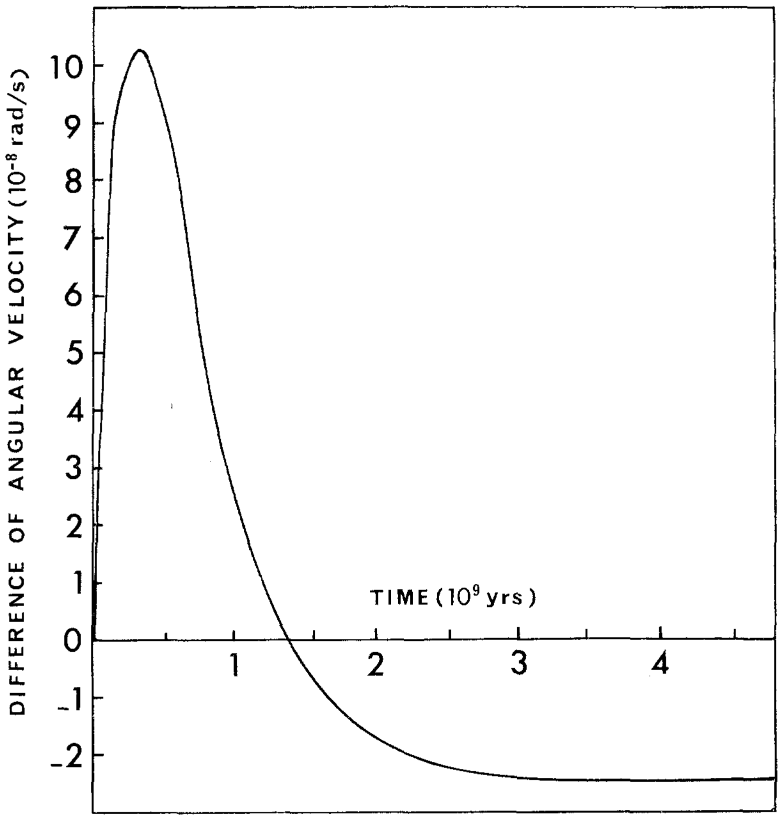

Fig. 3. Difference of angular velocity ($\omega_C - \omega_E$) between the circular and elliptic cases.

In the case in which the orbital elements (A and e) are regarded as variable with respect to the time, instead of Equation (1), we integrate the differential Equation (69) coupled with the Equations (66)–(68), as was described in Section 3.2 of Paper II (where we took for the Sun $e' = 0.017$ and $\omega_{0K}/\omega'_K = 13.368$). The variation of difference of angular velocities between the present case and the previous case, in which the semi-major axis and the eccentricity are constant, is displayed in Figure 4. Thus, as can be seen from the latter figure, the difference between these two cases becomes approximately zero after about 2.2 b.y. and, consequently, the period becomes constant at about 650 hr after 4.3 b.y. from the present time for both cases.

The shape of the Earth in the foregoing paragraphs has been regarded as approximately spherical, in accordance with the scheme of approximation of Section 3 of Paper II. If the Earth is regarded as spheroidal, instead of differential Equation (9),

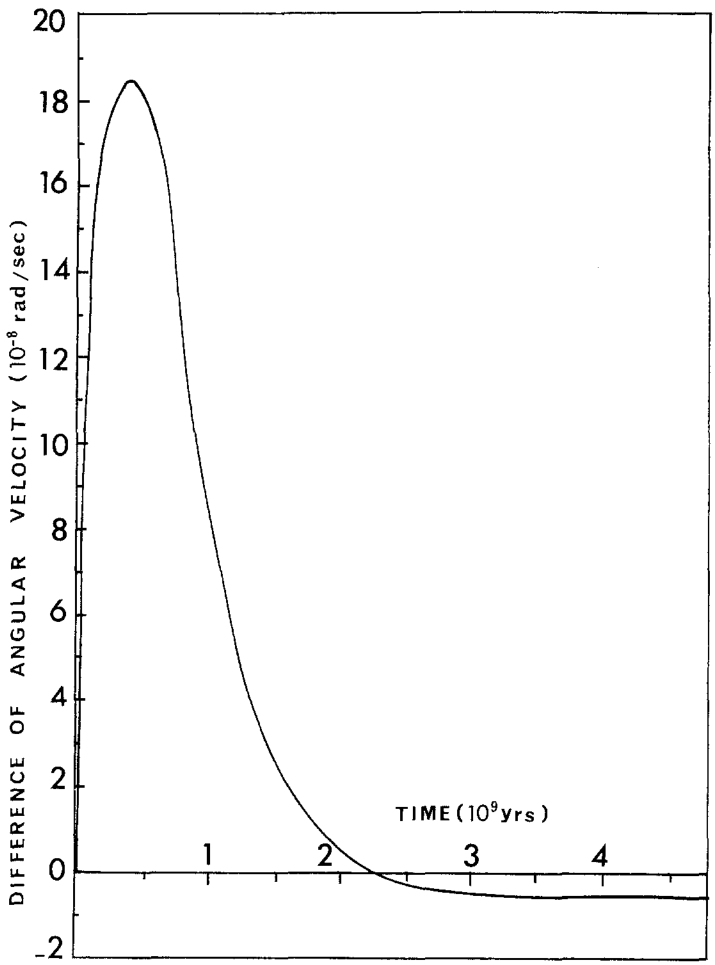

Fig. 4. Difference of angular velocities ($\omega_V - \omega_G$) between the cases of variable and constant A and e (spherical and spheroidal model).

we must integrate the Equations (90) of Paper II for A and e constant, and (94) of same paper for A and e variable. Table IV contains results of this integrations for various values of viscosity. The variation of the difference of angular velocity between these two cases is illustrated in Figure 4, since the results are the same as for the spherical model of the Earth.

In fact, in the elliptic case the results for the angular velocity agree up to the 8th decimal place for constant orbital elements as well as for variable orbital elements.

TABLE IV

Variation of angular velocity of axial rotation of the Earth (spheroidal model; e, A = constant)

Time (10^9 yr)			Angular velocity (10^{-5} rad s^{-1})
$\mu = 10^8$ g cm^{-1} s^{-1}	$\mu = 10^9$ g cm^{-1} s^{-1}	$\mu = 10^{10}$ g cm^{-1} s^{-1}	
0.	0.	0.	7.2921
0.1355	0.0136	0.0014	7.0299
0.2709	0.0271	0.0027	6.7775
0.5418	0.0542	0.0054	6.3006
0.6773	0.0677	0.0068	6.0754
1.3546	0.1355	0.0136	5.0696
2.7092	0.2709	0.0271	3.5503
4.0637	0.4064	0.0406	2.5118
5.4183	0.5418	0.0542	1.8020
6.7729	0.6773	0.0677	1.3168
10.8366	1.0837	0.1084	0.6036
14.9004	1.4900	0.1490	0.3758
17.6096	1.7610	0.1761	0.3189
21.6732	2.1673	0.2167	0.2849
27.0915	2.7092	0.2709	0.2724
32.5097	3.2510	0.3251	0.2697
43.3463	4.3346	0.4335	0.2690
44.7009	4.4701	0.4470	0.2690

The variation of angular velocity is shown in Figure 2 for all of these cases, but the time scale depends strongly on the mean value of viscosity of the Earth; this variation is also shown on Figure 5.

Furthermore, we can obtain details of the effects of viscosity on the present rate of rotation of the Earth. We regard the orbital elements as constant on time scale of order 10^3 yr; then we integrate Equation (56) of Paper II, the Earth being regarded as spherical, or Equations (92) of the same paper for a spheroidal Earth. The results of (56) are checked by (7) for corresponding times; these results coincide up to the ninth decimal place (see Table V). The results are listed in Table V. Figure 6 displays the variation of rotation of the Earth for about ten centuries after 1900.0 A.D., for various values of viscosity. In Table VI, results are shown for the effect of different viscosities on the lengthening of the day over an interval of one century.

Thus, from Figure 3 we conclude that the period of axial rotation of the Earth in the elliptic case is larger than for the circular case until about 1.3 b.y. Then, this phenomenon reverses and the period in the elliptic case will become constant and equal to 650 hr at about 4.3 b.y., while in the circular case it will become constant at about 655 hr at approximately the same time. This time scale for the value of viscosity 10^9 g cm^{-1} s^{-1} is comparable with the time scale 5.3 b.y. of MacDonald of syn-

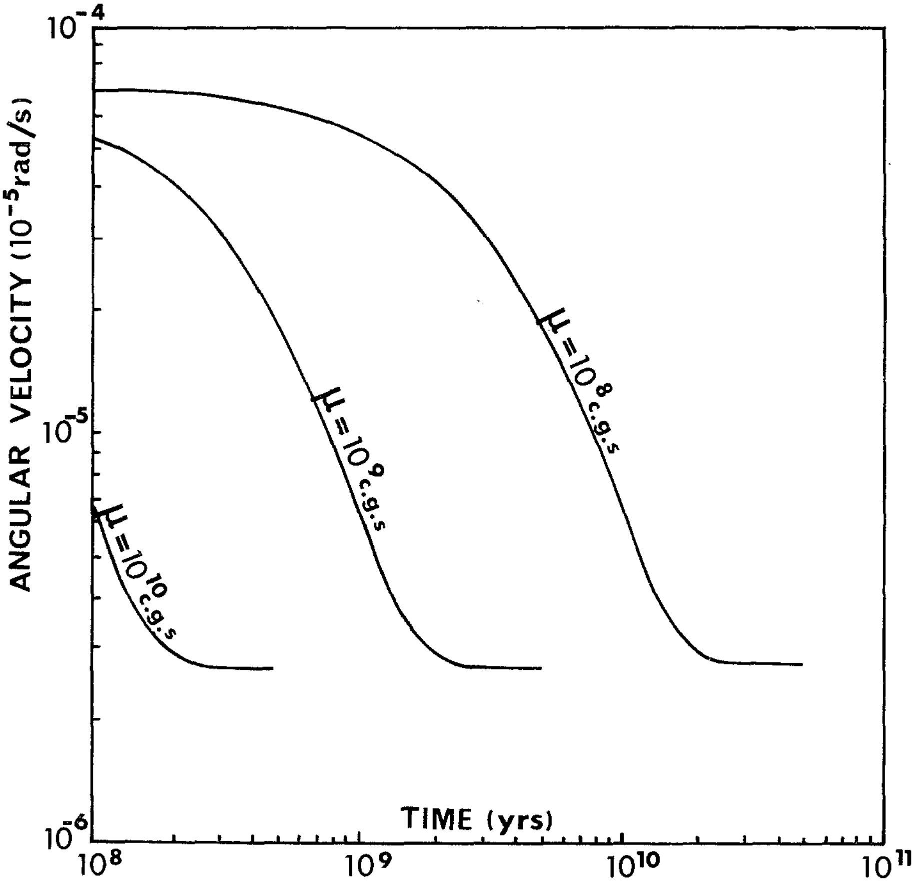

Fig. 5. Variation of angular velccity of the Earth (spherical or spheroidal model).

chronization between rotation of the Earth and revolution of the Moon (MacDonald, 1964).

5. Viscosity and Future Evolution of the Earth-Moon System

In the present section we propose to study the future evolution of the Earth-Moon system, especially the effects of viscosity on the rotation of the Earth and on its synchronism with revolution of the Moon, using the results of the previous numerical calculations.

TABLE V

Variation of angular velocity of axial rotation of the Earth (spherical and spheroidal model; e, A = constant) for 10 centuries from 1900 A.D.

Time (yr)	Angular velocity (10^{-5} rad s^{-1})					
	$\mu = 10^8$ g cm^{-1} s^{-1}		$\mu = 10^9$ g cm^{-1} s^{-1}		$\mu = 10^{10}$ g cm^{-1} s^{-1}	
	Numer. integrat.	From Eq. (25); (44)	Numer. integrat.	From Eq. (25); (44)	Numer. integrat.	From Eq. (25); (44)
0.	7.2921000	7.2921000	7.292100	7.292100	7.292100	7.292100
49.96	7.2920999	7.2920999	7.292099	7.292099	7.292090	7.292090
99.92	7.2920998	7.2920998	7.292098	7.292098	7.292080	7.292080
149.88	7.2920997	7.2920997	7.292097	7.292097	7.292070	7.292070
199.84	7.2920996	7.2920996	7.292096	7.292096	7.292060	7.292060
249.80	7.2920995	7.2920995	7.292095	7.292095	7.292051	7.292051
299.76	7.2920994	7.2920994	7.292094	7.292094	7.292041	7.292041
349.71	7.2920992	7.2920993	7.292093	7.292093	7.292031	7.292031
399.67	7.2920991	7.2920992	7.292092	7.292092	7.292021	7.292021
449.63	7.2920990	7.2920991	7.292091	7.292091	7.292011	7.292011
499.59	7.2920989	7.2920990	7.292090	7.292090	7.292001	7.292001
549.55	7.2920988	7.2920989	7.292089	7.292089	7.291992	7.291992
599.51	7.2920987	7.2920988	7.292088	7.292088	7.291982	7.291982
649.47	7.2920986	7.2920987	7.292087	7.292087	7.291972	7.291972
700.62	7.2920985	7.2920986	7.292086	7.292086	7.291962	7.291962
750.58	7.2920984	7.2920985	7.292085	7.292085	7.291952	7.291952
800.58	7.2920983	7.2920984	7.292084	7.292084	7.291942	7.291942
850.50	7.2920982	7.2920983	7.292083	7.292083	7.291933	7.291933
900.46	7.2920981	7.2920982	7.292082	7.292082	7.291923	7.291923
950.42	7.2920980	7.2920981	7.292081	7.292081	7.291913	7.291913
1000.21	7.2920979	7.2920980	7.292080	7.292080	7.291903	7.291903

The results of Section 3 indicate that the effects of viscosity on the rate of synchronization between the angular velocity of axial rotation of the Moon and its orbital revolution are not important, because this synchronism is obtained on a time scale of the order of 10^{-2} s for values of viscosity between 10^{24} g cm^{-1} s^{-1} and 10^{27} g cm^{-1} s^{-1}. This synchronism will not be maintained for all time, because the angular velocity of the Moon will vary with the Moon's position in its orbit (even for a circular orbit, if it is inclined to the equator) and will be synchronized only with the mean angular velocity n, which is given by (3).

On the other hand, the angular velocity of axial rotation of the Earth is slowed down by bodily tides raised on the Earth by the Moon (see Figures 2, 5 and 6). It is well known that the rotation of the Earth will eventually be synchronized with the revolution of the Moon. When this has occurred the angular velocity ω_s of synchronism will lie between the lower limit ω_K (when the orbit of the Moon is regarded very near to circular) and the upper limit $\omega_K(1+e)^2/(1-e^2)^{3/2}$ (which is the value of n at perigee passage) – e.g.,

$$\omega_K \leqslant \omega_s \leqslant \omega_K \frac{(1+e)^2}{(1-e^2)^{3/2}}. \tag{27}$$

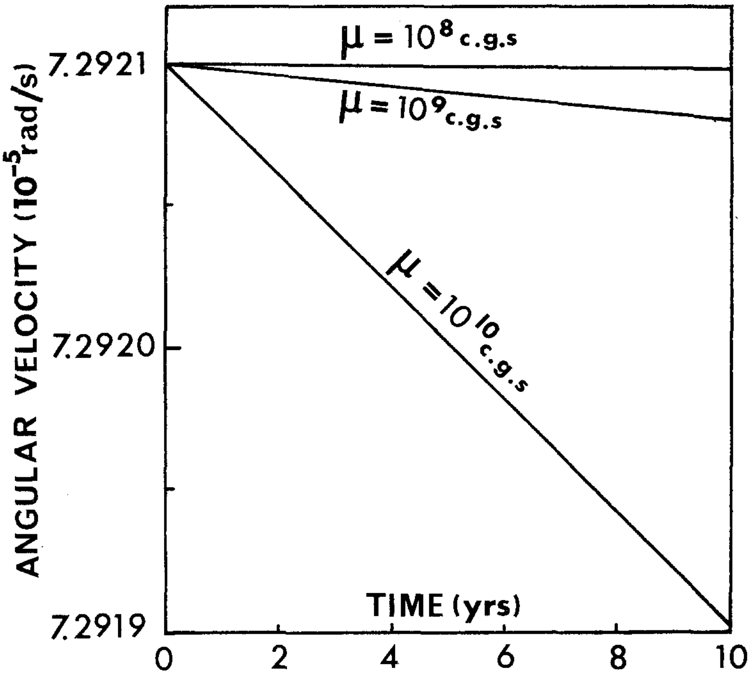

Fig. 6. Variation of angular velocity of axial rotation of Earth for 10 centuries from 1900 A.D.

TABLE VI

Increase of the length of the day for the Earth (for 10 centuries from 1900 A.D.)

Viscosity ($g\ cm^{-1}\ s^{-1}$)	Increment (sec per century)
10^8	2.16×10^{-3}
10^9	2.17×10^{-2}
10^{10}	2.18×10^{-1}

In particular, the angular velocity of the Earth will become constant and equal to ω_K (see Equation (24)) in the case of a circular orbit. Thus the synchronism will be exact and, therefore, the terrestrial day will become precisely equal to the sidereal month. But the orbit of the Moon is elliptic, and so this synchronism will not be exact; instead the angular velocity of the Earth will become constant at the value

$$\omega_s \simeq \omega_K \frac{1 + e^2}{(1 - e^2)^{3/2}} < \omega_K \frac{(1 + e)^2}{(1 - e^2)^{3/2}}, \tag{28}$$

from (26), whether the orbital elements are regarded as constant or variable (as given by Equations (58)–(60) of Paper II).

The time scale for any of these types of 'synchronization' between the rotation of the Earth and the revolution of the Moon is dependent on the mean value of the viscosity of the Earth (see Figures 2 and 5). A wide range exists between values of the viscosity attributed to different parts of the Earth. The astronomers suggest a value of the mean viscosity of the terrestrial globe as a whole to be of the order of 10^9 g cm^{-1} s^{-1} or smaller (cf., e.g., Kopal, 1972a); while the geophysicists arrived at a value of the order of 10^{22} g cm^{-1} s^{-1} or larger for the Eaıth's crust (e.g., Kaula, 1968).

Supposing now that the actual value of the viscosity is between 10^8 g cm^{-1} s^{-1} and 10^{22} g cm^{-1} s^{-1}, we can repeat the numerical integration of Equations (1) or (9) for these values of viscosity for all cases. The variation of the time t_s of synchronization (the time when $\omega \simeq \omega_s$) with respect to the mean value of viscosity is shown in Figure 7. Thus, the time t_s decreases as the viscosity increases. It is obvious that we cannot employ the value of viscosity which is given by geophysicists, because it would mean that the synchronism occurs on a time scale of a few hundred years, which is impossible. Probably the values of viscosity in the region of the order of 10^{22} g cm^{-1} s^{-1} refer only to the viscosity of the outermost solid crust of the Earth. The value of the order 10^9 g cm^{-1} s^{-1}, which we estimated in Section 2 and used in our calculations, is more compatible with the mean rigidity of the Earth, as a whole which is of the order of 10^{12} dyn cm^{-2}.

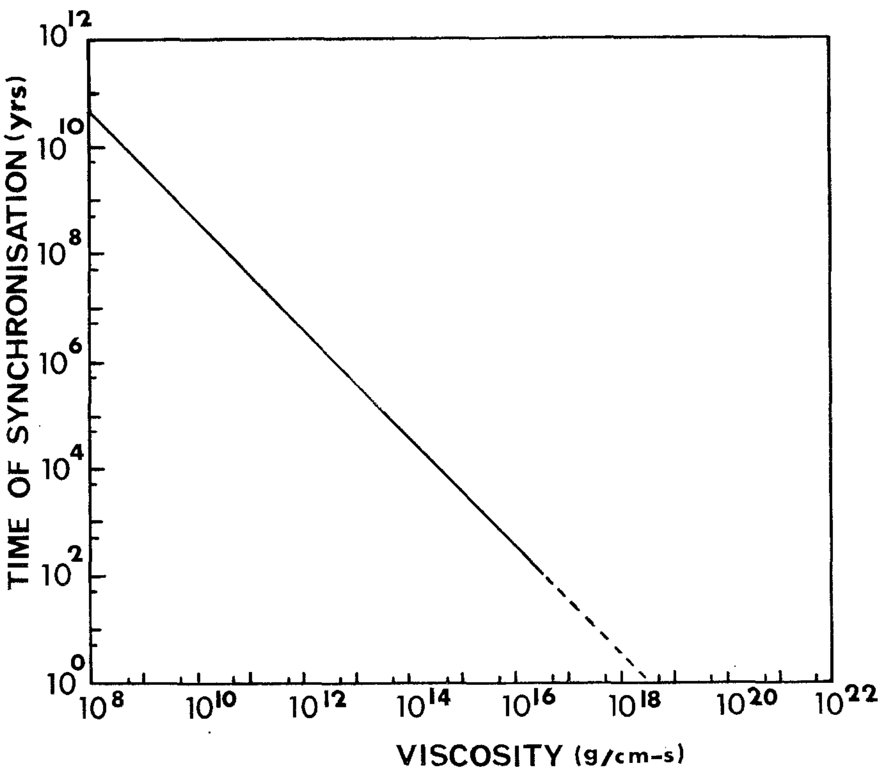

Fig. 7. Variation of the time of synchronization with the viscosity.

In conclusion, we can say that the terrestrial day will become constant and equal to about 650 hr, which is smaller by about 5 hr than the present sidereal month, in a time scale depending on the mean value of the viscosity of the Earth, when the Moon travels in an elliptic orbit.

6. Discussion

The aim of the present paper has been to study the effects of viscosity on the future evolution of the Earth-Moon system. The matter of the viscosity in this problem is a complicated one, because the variation of viscosity of the material neither of the Earth nor of the Moon is known. Supposing that the viscosity is a function of the radius of each material point, we have constructed the differential equation for the angular velocity of each component of such a close system (Tokis, 1974a).

Regarding the viscosity to be constant at the surface (or, for 'cold' celestial bodies, at the outer crust) of spherical bodies, or throughout the entire body in the case of spheroidal bodies, we then discussed the solution of one of these equations (Tokis, 1974b).

The application of this theory to the Earth-Moon system showed that the results of the spherical model of the Earth are the same as those for the spheroidal model (Figure 2). The times required for synchronism between rotation of the Earth and revolution of the Moon, and the variation of angular velocity in the next ten centuries from 1900 A. D. are again the same for the two models. These results were found for the values of viscosity of the Earth in the region of 10^9 g cm^{-1} s^{-1} and they vary with the viscosity (Figures 2, 6). On the other hand, the Moon is not affected by the variation of viscosity (for values between 10^{24} g cm^{-1} s^{-1} and 10^{27} g cm^{-1} s^{-1}).

Moreover, the results for the Earth-Moon system remain approximately the same (Figure 2) when we solve numerically the previous equations with variable orbital elements (with the aid of the approximate Equations (58)–(60) of Paper II).

Thus, the synchronism of the period of the Earth's rotation and the revolution of the Moon occurs at the value 650 hr, which is 5 hr less than for the case of a circular orbit. The time scale for this synchronism strongly depends on the value of the viscosity. There is a big difference between the results (Figure 7) for the values of viscosity, which are suggested for the Earth by astronomers and those suggested by geophysicists.

The results of the present work indicate that the mean value of viscosity of the Earth should be of the order of 10^8 g cm^{-1} s^{-1}, because this value gives rise to an increase of the length of the day per century (Table VI) compatible with the value of 1.8×10^{-3} seconds per century which is obtained from astronomical observations over the past two centuries, as well as from palaeontological evidence.

The general conclusion drawn from the present application to the Earth-Moon system at the theory of Paper I and II is that the value of the viscosity of the material of the Earth is a very important factor affecting phenomena arising from tidal friction and the rotation of the Earth; and, together with the separation of the Earth and Moon, the most important characteristics which determine the rate of tidal dissipative processes in this system.

References

Arkani-Hamed, J.: 1973a, *The Moon* **6**, 100.
Arkani-Hamed, J.: 1973b, *The Moon* **6**, 112.
Baldwin, R. B.: 1970, *Icarus* **13**, 215.
James, R. A. and Kopal, Z.: 1963, *Icarus* **1**, 442.
Kaula, W. M.: 1968, *An Introduction to Planetary Physics*, J. Wiley, London, New York, p. 110 and p. 186.
Kopal, Z.: 1969a, *Astrophys. Space Sci.* **4**, 330.
Kopal, Z.: 1969b, *The Moon* (2nd ed.), D. Reidel Publ. Co., Dordrecht, Holland, pp. 203–204.
Kopal, Z.: 1972a, *Astrophys. Space Sci.* **16**, 3.
Kopal, Z.: 1972b, *Astrophys. Space Sci.* **17**, 161.
MacDonald, G. J. F.: 1964, *Rev. Geophys.* **2**, 467.
Melchior, P.: 1972, *Physique et Dynamique Planétaires*, Vol. 3, Vander, p. 30.
Motz, L.: 1952, *Astrophys. J.* **115**, 562.
Runcorn, S. K.: 1967, *Roy. Soc. London Proc.* **A296**, 270.
Tokis, J. N.: 1973, Ph.D. Thesis Univ. of Manchester, (unpublished).
Tokis, J. N.: 1974a, *Astrophys. Space Sci.* **26**, 447.
Tokis, J. N.: 1974b, *Astrophys. Space Sci.* **26**, 477.
Turcotte, D. L. and Oxburgh, E. R.: 1969, *Nature* **223**, 250.
Urey, H. C.: 1968, *Science* **162**, 1408.