THE USE OF THE SAROS IN LUNAR DYNAMICAL STUDIES*

A. E. ROY
Dept. of Astronomy, Glasgow University, Scotland

(Received 12 July, 1972)

Abstract. It is shown that the near-periodicity in the Earth-Moon-Sun system demonstrated by the possibility of using the Saros to predict eclipses, suggests that the Saros can also be used in a fast and accurate method of special perturbations which can be applied for long term study of the evolution of the Moon's orbit.

1. Introduction

The Moon in its geocentric orbit is subject to many disturbing forces, among which are the gravitational pulls of Sun and planets, the departure of the Moon's and Earth's gravitational fields from those due to point-masses, tidal effects and so on. The main disturber is of course the Sun. A break-down of the secular rates of motion of the lunar orbit's apse and line of nodes, taken from Brown's lunar theory, is instructive in this respect (see Table 1).

TABLE I

Mean annual motion of	Perigee	Node
Principal solar action	+ 146 426.92″	− 69 672.04″
Mass of the Earth	− 0.68	+ 0.19
Direct planetary action	+ 2.69	− 1.42
Indirect planetary action	− 0.16	+ 0.05
Figure of the Earth	+ 6.41	− 6.00
Figure of the Moon	+ 0.03	− 0.14
	+ 146 435.21	− 69 679.36

The main lunar problem therefore deals with the Earth-Moon-Sun system, taken as point masses. Everything else may be added later in the form of small but very numerous corrections. From the time of Newton, many people have attempted to produce an analytical lunar theory capable of predicting the Moon's position to at least the accuracy to which observations could be made, for ephemeris purposes, for the study of the evolution of the lunar orbit, for geophysical study and towards a complete understanding of how close to reality Newton's law of gravitation was.

Newton, Euler, Clairaut, Hansen, Delaunay, Hill, Brown and Deprit have been a few who have produced such theories, Deprit's solution of the main lunar problem being achieved by programming an electronic computer to obtain an analytical lunar ephemeris.

* Paper dedicated to Professor Harold C. Urey on the occasion of his 80th birthday on 29 April 1973.

The Moon 7 (1973) 6–13. *All Rights Reserved*

All of these theories have two common features – the large number of terms they contain and the need for selection of a zero order intermediate orbit. For example Brown's lunar theory contains 1500 separate terms. The number of terms required is dictated not only by the required accuracy but also by the choice of intermediate orbit and method of development. Most theories began with the equations of motion expressed in terms of polar coordinates or functions of the orbital elements though Euler's theory of 1772 used rectangular coordinates, the x- and y-axes rotating with the Moon's mean angular motion. Similarly, Hill's theory utilised rotating rectangular coordinates but with the x-axis restrained to point at the Sun's mean position. A fixed keplerian ellipse, a rotating ellipse of fixed shape, a periodic orbit more complicated than either, have all been used at various times as intermediate orbits.

In this paper we are interested in Hill's approach as the example of an intermediate orbit most similar to the one suggested below. Hill chose a periodic orbit which was a particular solution of two second-order differential equations in u and s, where

$$u = X + iY\,, \quad s = X - iY\,, \quad i = \sqrt{-1}\,;$$

X and Y being the Moon's geocentric ecliptic coordinates, the X-axis always pointing to the Sun's mean geocentric direction. The independent variable ζ was defined by

$$\zeta = \exp\left\{i\,(n - n')\,(t - t_0)\right\},$$

where n' is the mean motion of the Sun about the Earth, t is time and t_0 and n are undetermined constants at that stage.

Hill obtained these equations by neglecting the solar eccentricity e', the solar parallax $1/a'$, the Moon's latitude Z and the lunar eccentricity e. The solution used by Hill as his intermediate orbit was expressed in Fourier series of $(n - n')\,t$. The deviations of the real lunar orbit from this intermediate orbit were then developed analytically by Hill and Brown to give the lunar theory still used in preparing the lunar ephemeris.

It is our suggestion in this paper that a far better intermediate orbit can be found by considering the existence of the Saros. A method of special perturbations can then be developed that may well be superior to any standard numerical method that can be applied to the lunar problem in step-size, minimisation of error and computing time.

2. The Saros

The Saros, known to the ancient Chaldeans, is a period of time of approximately 18 years 10 or 11 days (depending upon the number of leap years in the interval). At the end of a Saros, the geometry in the Earth-Moon-Sun system is repeated to a close enough extent that solar and lunar eclipses can be predicted from the occurrence of past eclipses at the Saros' beginning. Table II shows, for example, the values of the semidiameters of Moon and Sun during four eclipses, each set of four occurring in the years 1898, 1916, 1934, 1952, 1970. The eclipses were:

(1) Partial eclipse of the Moon   Date in 1970: February 21
(2) Total eclipse of the Sun   Date in 1970: March 7

(3) Partial eclipse of the Moon Date in 1970: August 17
(4) Annular eclipse of the Moon Date in 1970: August 31–Sept. 1.

TABLE II
Semidiameter of Sun and Moon during eclipse

Year	1898	1916	1934	1952	1970
Date	Jan. 7	Jan. 19	Jan. 30	Feb. 10–11	Feb. 21
Moon	14′52″.00	14′49″.8	14′48″.5	14′47″.3	14′46″.8
Sun	16′15″.87	16′15″.3	16′14″.1	16′12″.4	16′10″.3
Date	Jan. 21	Feb. 3	Feb. 13–14	Feb. 25	Mar. 7
Moon	16′24″.30	16′25″.4	16′27″.3	16′29″.2	16′31″.6
Sun	16′14″.83	16′13″.5	16′11″.6	16′09″.4	16′06″.8
Date	Jul. 3	Jul. 14	Jul. 26	Aug. 5	Aug. 17
Moon	16′43″.32	16′42″.9	16′43″.1	16′43″.2	16′43″.9
Sun	15′43″.86	15′44″.1	15′44″.9	15′46″.2	15′47″.9
Date	Jul. 18	Jul. 29	Aug. 10	Aug. 20	Aug. 31–Sep. 1
Moon	14′45″.87	14′44″.0	14′43″.2	14′42″.5	14′42″.6
Sun	15′44″.36	15′45″.3	15′46″.8	15′48″.6	15′50″.8

All four's characteristics were unchanged in the five years in which they occurred. In comparing the values of the lunar semidiameter (and therefore its geocentric distance) from Saros to Saros it is seen how little it varies. The same is true of the Sun's semidiameter even though the ranges within which both lunar and solar semidiameters can vary are large (Sun: 15′45″–16′18″; Moon: 14′42″–16′44″). If we also take additional eclipse data from the respective *Nautical Almanacs* and the 1970 *Astronomical Ephemeris* concerning solar and lunar ecliptic longitudes (λ) and latitudes (β), and also the rates of change of these quantities, we find that their values at the beginning of a Saros are very nearly repeated at the end of the Saros. Thus in Table III, data for the partial lunar eclipses of 1952 February 10–11 and 1970 February 21 are compared. In the Table the differences between the Sun and Moon's geocentric ecliptic coordinates during eclipse are tabulated for each eclipse.

Suffixes M and S refer to Moon and Sun respectively, the dots denote daily rates of change and σ stands for semidiameter.

One more example, not at an eclipse but taken at random in the lunar ephemeris is illustrated in Table IV. Again it is seen how closely the relative positions and velocities of Sun and Moon are repeated after one Saros. The reason, of course, is the interesting set of near commensurabilities existing among the Moon's synodic period, its anomalistic period and its nodical period. From the *Astronomical Ephemeris* (1970) their mean values are:

Synodic (S)	$29^{\mathrm{d}}.530\,589$
Anomalistic (L)	27.554 551
Nodical (D)	27.212 220

TABLE III

Data	$\lambda_S - \lambda_M$	$\beta_S - \beta_M$	σ_M	σ_S	$\dot{\lambda}_M$	$\dot{\beta}_M$	$\dot{\sigma}_M$	$\dot{\lambda}_S$	$\dot{\beta}_S$	$\dot{\sigma}_S$
1952 Feb. $11^{d}.02729$	179°54′.9	−0° 50′.84	14′47″.3	16′12″.4	11°.955	−65′.345	3″.84	1°.0114	−0″.15	−0″.18
1970 Feb. $21^{d}.35467$	179°54′.7	−0° 51′.69	14′46″.8	16′10″.3	11°.922	−65′.305	3″.49	1°.0069	−0″.28	−0″.22

TABLE IV

Data	$\lambda_S - \lambda_M$	$\beta_S - \beta_M$	σ_M	σ_S	$\dot{\lambda}_M$	$\dot{\beta}_M$	$\dot{\sigma}_M$	$\dot{\lambda}_S$	$\dot{\beta}_S$	$\dot{\sigma}_S$
1952 Mar. 18.0	104°21′.1	5° 06′.2	15′54″.7	16′05″.6	13°.801	16′.567	8″.01	0°.9945	0″.01	−0″.28
1970 Mar. 29.347	104°21′.9	5° 05′.8	15′52″.9	16′02″.6	13°.984	16′.006	8″.52	0°.9883	−0″.14	−0″.28

Then, as is well-known

$$223\ \mathrm{S} = 6585\overset{\mathrm{d}}{.}3213,$$
$$239\ \mathrm{L} = 6585.5377,$$
$$242\ \mathrm{D} = 6585.3572.$$

The close agreement ensures that the geometry of the Earth-Moon-Sun system at any epoch is almost exactly repeated one Saros later. When the Moon's elongation is repeated at the end of the Saros its argument of perigee and true anomaly also have very nearly the same values as before. In addition, because the Saros length is only 10 days longer than 18 yr, the Sun is almost back to its original true anomaly and length of radius vector. Thus the closeness of the fit is not only in position but in velocities as well.

It should also be noted that within any Saros, the perturbations of the Sun on the Earth-Moon system almost completely cancel themselves out, in particular the large disturbances in semimajor axis, eccentricity and inclination.

It is perhaps easiest to see this if we take the situation at the beginning of a Saros to be such that full Moon happens when the Moon and the Sun are at perigee, the Moon's latitude being zero. Then the velocity vectors of the Sun and Moon are perpendicular to both the radius vectors. This is a mirror condition and by the mirror theorem, (Roy and Ovenden, 1955) the history of the system after that time is a mirror image of its history prior to that time.

But 9 years and approximately 5 days later, a new mirror condition occurs very nearly – a new Moon, Sun within 6° of perigee, Moon at apogee, Moon's latitude zero. The velocity vectors of Sun and Moon are very nearly perpendicular to both the radius vectors. If this second mirror configuration were exact, the Moon's orbit would be exactly periodic, returning at the end of the Saros to a repeat of the first mirror configuration so that the perturbations built up in the first half of the Saros would have been cancelled completely in the second, the only result being that the sidereal position of the line of nodes of the Moon's orbital plane would have regressed approximately 11°.

As it is, the Moon's orbit under solar perturbation is very nearly periodic in a period of length one Saros, the close repetition of the geometrical properties of solar and lunar eclipses being the outward manifestation of how closely the system Earth-Moon-Sun approximates to a purely periodic motion. All other perturbations (planetary, tidal, figures of Earth and Moon) are very small indeed.

According to the present viewpoint, therefore, by far the best reference orbit for dynamical study of the Earth-Moon system would be the set of geocentric positions of Moon and Sun from the previous Saros.

3. Procedure

Let the Moon's geocentric equation of motion be

$$\ddot{\mathbf{r}} = -\mu \mathbf{r} u + \mathbf{F}, \tag{1}$$

where $\mathbf{r}$ is the Moon's geocentric radius vector, $\mu = G(m_{\oplus} + m_{\ominus})$, $u = r^{-3}$. G is the constant of gravitation, $m_{\oplus}$ and $m_{\ominus}$ are the masses of Earth and Moon respectively and $\mathbf{F}$ is the acceleration due to the perturbing effects of Sun, planets, shape of Earth and Moon, tides, and so on.

Let $\mathbf{r}$ be almost periodic in a time interval T (one Saros) and let suffix n refer to a value of $\mathbf{r}$ or any other variable anywhere within the nth time interval, i.e. nth Saros.

Define $\mathbf{r}_n'$ by

$$\mathbf{r}_n = \mathbf{r}_{n-1} + \mathbf{r}_n'.$$

Introduce the general difference formula

$$\mathbf{r}_n^{(i)} = \mathbf{r}_{n-1}^{(i)} + \mathbf{r}_n^{(i+1)}, \quad i = 0, 1, \ldots, n-1$$

where (i) denotes the i-th difference.

Then it is easily seen that

$$\mathbf{r}_n = \sum_{i=0}^{n-2} \mathbf{r}_{n-1}^{(i)} + \mathbf{r}_n^{(n-1)}. \tag{2}$$

Also

$$\mathbf{r}_n = \mathbf{r}_1 + \sum_{i=2}^{n} \mathbf{r}_i'$$

or

$$\mathbf{r}_n = \sum_{i=0}^{n-1} {}^{n-1}C_i \, \mathbf{r}_{i+1}^{(i)}. \tag{3}$$

Now by Equation (1)

$$\ddot{\mathbf{r}}_2 = \mathbf{F}_2 - \mu \mathbf{r}_2 u_2 \, ; \quad u_2 = r_2^{-3}, \tag{4}$$

$$\ddot{\mathbf{r}}_1 = \mathbf{F}_1 - \mu \mathbf{r}_1 u_1 \, ; \quad u_1 = r_1^{-3}. \tag{5}$$

Let

$$\left.\begin{aligned} \mathbf{F}_n^{(i)} &= \mathbf{F}_{n-1}^{(i)} + \mathbf{F}_n^{(i+1)} \\ u_n^{(i)} &= u_{n-1}^{(i)} + u_n^{(i+1)} \end{aligned}\right\} \quad i = 0, 1, 2, \ldots, n-1. \tag{6}$$

Then, substracting (5) from (4),

$$\ddot{\mathbf{r}}_2' = \mathbf{F}_2' - \mu \left(\mathbf{r}_1 u_2' + \mathbf{r}_2' u_2\right).$$

Similarly,

$$\ddot{\mathbf{r}}_3' = \mathbf{F}_3' - \mu \left(\mathbf{r}_2 u_3' + \mathbf{r}_3' u_3\right),$$

giving, on substraction

$$\ddot{\mathbf{r}}_3'' = \mathbf{F}_3'' - \mu \left(\mathbf{r}_1 u_3'' + 2\mathbf{r}_2' u_3' + \mathbf{r}_3'' u_3\right).$$

Hence, in general,

$$\ddot{\mathbf{r}}_n^{(n-1)} = \mathbf{F}_n^{(n-1)} - \mu \sum_{i=0}^{n-1} {}^{n-1}C_i \, \mathbf{r}_{n-i}^{(n-1-i)} \, u_n^{(i)} . \tag{7}$$

Rewriting (2) and (3), we have

$$\mathbf{r}_n = \sum_{i=0}^{n-2} \mathbf{r}_{n-1}^{(i)} + \mathbf{r}_n^{(n-1)} \tag{8}$$

and

$$\mathbf{r}_n = \sum_{i=0}^{n-1} {}^{n-1}C_i \, \mathbf{r}_{i+1}^{(i)} . \tag{9}$$

Equations (7), (8) and (9) form the set of equations for the problem.

The procedure may be sketched as follows.

During the first Saros cycle, Equation (7) with $n = 1$ is integrated numerically by some standard procedure. The step lengths will be of order a few hours at most. The positions and velocities are stored.

During the succeeding Saros cycle, Equation (7), with $n = 2$ is integrated numerically, use being made of the positions and velocities of Saros 1. Equation (9) is also used. Because of the almost periodic nature of the Earth-Moon-Sun system's geometry in a Saros period, it is to be expected that in this cycle of numerical integration, the step-lengths will be much larger. It may be remarked that in this cycle the method is essentially an Encke-type procedure except that the reference orbit is not a fixed keplerian orbit osculating with the real lunar orbit at some epoch and subsequently computed from the usual two-body formulae but is much more efficient since it includes to a high degree of accuracy solar perturbations.

For the third Saros cycle, Equation (7), with $n = 3$, is integrated numerically, use being made of the positions and velocities stored from the first two Saros periods. The step-length may be increased still further. Indeed by proceeding to this and higher differences, we may expect the real power of the method to be realised.

It is hoped to publish elsewhere in detail numerical studies of the above procedure in which consideration is given to such questions as to what order in n it is practicable to take the differencing to, at what order loss of accuracy sets in appreciably, and so on. It seems likely, indeed, that the law of diminishing returns must set in when the value of $\mathbf{r}_n^{(n-1)}$ becomes so small that it is of the order of the non-solar perturbations. But as seen from Table I, these are very small so that there are good grounds for hoping that by a procedure of this kind, the main lunar problem may be accurately 'subtracted out' allowing really long timescale study to be made of such questions as the evolution of the lunar orbit under tidal effects.

Acknowledgement

In conclusion, my thanks are due to my colleague W. Black for much helpful advice.

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