Z. Wahrscheinlichkeitstheorie verw. Gebiete
59, 117–138 (1982)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

A Donsker-Varadhan Type of Invariance Principle*

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Summary. Let $X_1, X_2, \ldots$ be i.i.d. random variables in the domain of attraction of a stable law G, and denote $S_n = X_1 + \ldots + X_n$, $L_n(\omega, A) = n^{-1} \sum_{j=0}^{n-1} \chi_A(S_j/a(n))$, where the real sequence $a(n)$ satisfies $a(n)^{-1} S_n \to G$. Large deviation probability estimates of Donsker-Varadhan type are obtained for $L_n(\omega, \cdot)$, and these are then used to study the behavior of "small" values of $(S_n/a(n))$. These latter results are analogues of Strassen's results which described the behavior of "large" values of $(S_n/a(n))$ when the limit law was Gaussian. The limiting constants are seen to depend only on the limit law G and not on the distribution of X_1. The techniques used are those developed by Donsker and Varadhan in their theory of large deviations.

1. Introduction

Let $X_1, X_2, \ldots$ be real-valued independent identically distributed (i.i.d.) random variables and let

$$S_n = X_1 + \ldots + X_n; \qquad S_0 = 0. \tag{1.1}$$

F will denote the distribution function of X_1. We assume throughout this paper that F is in the domain of attraction of a stable law G of index α, $0 < \alpha \leqq 2$, which has a strictly positive density and satisfies the scaling property. The scaling property requirement rules out certain asymmetric situations, when $\alpha = 1$. Under our conditions we have

$$\frac{S_n}{a(n)} \Rightarrow G \quad \text{as} \quad n \to \infty \tag{1.2}$$

* This work was partially supported by the National Science Foundation.

0044-3719/82/0059/0117/$04.40

where $a(n)=n^{1/\alpha}l(n)$, l being a slowly varying function and "$\Rightarrow$" denotes the weak convergence of the corresponding measures, i.e. for any bounded continuous function f on R (the real line)

$$\lim_n E[f(S_n/a(n))]=\int f(y)\,dG(y).$$

We will always use this arrow symbol in this sense.

$y_\alpha(t)$, $t\geqq 0$, will denote the stable process on R of index α with $y_\alpha(1)$ having the distribution G. It should always be understood that y_α has sample paths in $D[0,\infty)$, the space of real-valued functions on $[0,\infty)$ which are right-continuous and possess left limits. For $h>0$, $D[0,h]$ will be considered as a separable metric space with the Skorohod metric. The process y_α has the scaling property: for $c>0$, the process $c^{-1/\alpha}y(ct)$, $t\geqq 0$, has the same finite dimensional distributions as $y_\alpha(t)$, $t\geqq 0$. For $\alpha=2$, the process is a constant multiple of a standard Brownian motion process, and in this case we will use the normalization that G has variance 1, so that the constant multiple is actually 1.

Donsker and Varadhan [8] use their powerful theory developed in [4]-[7] to prove a class of results for a stable process of the above type (actually, they assume symmetry which is not necessary). These results are analogues of the well-known invariance principle for Brownian motion due to Strassen [13]. Strassen's results deal with "large values" of the process and hold only in the Brownian motion case whereas the results of Donsker and Varadhan deal with "small values" of the process and hold for not just Brownian motion but all stable processes of the above type. For a discussion of this see Sect. 4 [8], p. 751, keeping in mind that the Brownian motion in [8] has variance 2 at time 1.

In [13] Strassen also obtains results corresponding to Brownian motion for sums of i.i.d. random variables which are in the domain of normal attraction of $N(0,1)$, i.e. for which

$$\frac{S_n}{n^{1/2}}\Rightarrow N(0,1). \tag{1.3}$$

For comparison purposes we state some of Strassen's results here. Under (1.3) he shows that for $a\geqq 1$

$$P\left[\limsup_n \frac{1}{n}\sum_{j=0}^{n-1}|S_j|^a/(n\log\log n)^{a/2}=\theta_a\right]=1 \tag{1.4}$$

where θ_a is evaluated explicitly in terms of a. He also shows

$$P[\limsup_n v_n=1-\exp(-4(c^{-2}-1))]=1 \tag{1.5}$$

where

$$v_n=\frac{1}{n}\sum_{j=4}^{n-1}\chi_{(c,\infty)}(S_j/(2j\log\log j)^{1/2})$$

for any c, $0 \leqq c \leqq 1$. Here $\chi_A(x) = 1(0)$ if $x \in A$ $(x \notin A)$. To obtain these results the important tool is the Skorokhod imbedding technique.

Our aim here is to obtain analogues of these results for "small values" of the partial sums under the condition (1.2). To obtain our results we rely very heavily on the techniques and results of Donsker and Varadhan [4]–[8]. However, the exposition here is reasonably self-contained. Before we describe our results one more result needs to be introduced, the Chung type law of the iterated logarithm. Let

$$A_n = \max_{1 \leqq j \leqq n} |S_j| \tag{1.6}$$

and for $t \geqq 16$

$$b(t) = [t/\log\log t] \tag{1.7}$$

where $[x]$ = greatest integer $\leqq x$.

It was shown in [10] that under (1.2) there exists $0 < c < \infty$ such that

$$\liminf_n \frac{A_n}{a(b(n))} = c, \quad \text{a.s.} \tag{1.8}$$

Although the conditions imposed on G in [10] are more general than in (1.2), the method of proof there does not tell us whether the constant c depends on F or just on G. Only under (1.3) it was later shown [11] that $c = \pi/\sqrt{8}$, same for all F. In this connection it is useful to mention the analogue of (1.8) for a stable process. Assume that G satisfies our conditions and define

$$A_\alpha(t) = \sup_{s \leqq t} |y_\alpha(s)|. \tag{1.9}$$

Then it was shown by Chung [3] that for $\alpha = 2$

$$\liminf_{t \to \infty} \frac{A_\alpha(t)}{b(t)^{1/2}} = \pi/\sqrt{8}, \quad \text{a.s.} \tag{1.10}$$

and, more generally, by Taylor [14] that for $0 < \alpha \leqq 2$

$$\liminf_{t \to \infty} \frac{A_\alpha(t)}{b(t)^{1/\alpha}} = c_G, \quad \text{a.s.} \tag{1.11}$$

where $0 < c_G < \infty$. The result in (1.11) is also derived in [8], but here the advantage is that c_G is identified in terms of the I-functional of the process (to be introduced in the next section). With this introduction we can now describe some of our results.

The sequences $a(n)$ and $b(n)$ have been introduced above. We will denote

$$c(n) = a(b(n)). \tag{1.12}$$

In the theorems below we assume that (1.2) holds and G satisfies our conditions.

Theorem 1.1. *For $c>0$ there exists a constant $k_{c,G}$ such that a.s.*

$$\limsup_n \frac{1}{n}\sum_{j=0}^{n-1} \chi_{[0,c]}(|S_j|/c(n)) = k_{c,G}, \tag{1.13}$$

and for $c \geqq c_G$ the constant $k_{c,G}=1$, where c_G is the constant in (1.11).

Theorem 1.2. *Let A_n be defined by* (1.6), *then*

$$\liminf_n \frac{A_n}{c(n)} = c_G, \quad a.s. \tag{1.14}$$

where c_G is the constant in (1.11).

This theorem tells us, in particular, that if instead of (1.3) we have

$$\frac{S_n}{n^{1/2}\, l(n)} \Rightarrow N(0,1), \tag{1.3'}$$

i.e. F is in the domain of attraction of $N(0,1)$ and could have infinite variance ((1.3) implies that F has variance 1) then

$$\liminf_n \frac{A_n}{b(n)^{1/2}\, l(b(n))} = \pi/\sqrt{8}, \quad \text{a.s.} \tag{1.8'}$$

This, of course, includes the result of [11] but the methods of that paper do not appear to be adequate for the identification of the constant in (1.8').

Theorem 1.3. *For $a>0$ there exists a constant $A_{a,G}$ such that a.s.*

$$\liminf_n \frac{1}{n}\sum_{j=0}^{n-1} (|S_j|/c(n))^a = A_{a,G}, \tag{1.15}$$

$A_{a,G} \leqq (c_G)^a$ in view of Theorem 1.2.

It is shown in [8] that $A_{2,G}=1/4$, if $G=N(0,1)$. The constant θ_a in (1.4) corresponding to $a=2$ equals $8/\pi^2$.

It should be noted that the results of Sects. 3–5 are more basic and the above theorems are only their applications obtained in Sect. 6. Theorem 1.3 is a special case of Example 6.2 in Sect. 6. The notation introduced here will be used throughout; more notation and some preliminaries are given in Sect. 2. Sections 3 and 4 contain the basic results leading to theorems in Sect. 5.

2. Preliminaries and Notation

We will denote

$\mathscr{U}$ = Class of real-valued Borel measurable functions u on R such that $0<a\leqq u \leqq b<\infty$, where a, b depend on u;

$\mathscr{U}_1$ = continuous functions in $\mathscr{U}$;

$\mathscr{U}_2$ = members of $\mathscr{U}_1$ for which $\lim_{|x|\to\infty} u(x)$ exists;
$\mathscr{U}_3$ = members of $\mathscr{U}_2$ such that u = constant outside of a compact set;
$\mathscr{U}_4$ = infinitely differentiable members of $\mathscr{U}_3$.

$\mathscr{B}$ will denote Borel subsets of the real line R and for $x \in R$, $A \in B$, $\pi(x, A)$ will mean a transition probability function, i.e. $\pi(\cdot, A)$ is a Borel measurable function for each $A \in \mathscr{B}$, and $\pi(x, \cdot)$ is a probability measure on $\mathscr{B}$ for each $x \in R$. π will be called a *Feller transition probability function* if $\pi f(x) = \int f(y)\pi(x, dy)$ is a bounded continuous function whenever f is a bounded continuous function.

If v is a measure on $\mathscr{B}$ and f is a Borel measurable extended real-valued function we will write

$$v(f) = \int f(y)\, v(dy) \tag{2.1}$$

whenever the integral makes sense. Thus we will also write $\pi(x, f)$ in place of $\pi f(x)$.

Following the notation in [8] we write

$$M = \text{Subprobability measures on } (R, \mathscr{B}),$$

$$\mathscr{M} = \text{Probability measures on } (R, \mathscr{B}),$$

and for π a transition probability function and $\mu \in M$ we denote by $I(\mu)$ the I-functional corresponding to π given by

$$I(\mu) = -\inf_{u \in \mathscr{U}} \int \log\left(\frac{\pi u}{u}\right)(x)\, \mu(dx). \tag{2.2}$$

As noted in [8], $\mathscr{U}$ can be replaced by $\mathscr{U}_1$ in (2.2).

Lemma 2.2 shows that if π is Feller then $\mathscr{U}$ can be replaced by $\mathscr{U}_3$ in (2.2) and Lemma 2.1 is proved to prove Lemma 2.2.

Lemma 2.1. *Suppose π is Feller. Let a be a fixed constant. Then given $\varepsilon > 0$, there exists K compact such that $\sup_{|x| \leq a} \pi(x, K^c) < \varepsilon$.*

We always write A^c to be the complement of the set A.

Proof. Let $K_n \uparrow R$, K_n compact, and let f_n be a (pointwise) decreasing sequence of continuous functions such that $0 \leq f_n \leq 1$, $f_n = 0$ on K_n, $f_n = 1$ on O_n^c, where O_n is a bounded open neighborhood of K_n. Then $\pi f_n \downarrow 0$ pointwise and since πf_n is continuous, the convergence is uniform on compacts. Therefore $\sup_{|x| \leq a} \pi f_{n_0}(x) < \varepsilon$ and so $\sup_{|x| \leq a} \pi(x, \bar{O}_{n_0}^c) < \varepsilon$ for some n_0.

Lemma 2.2. *If π is Feller, then for $\mu \in M$*

$$I(\mu) = -\inf_{u \in \mathscr{U}_3} \int \log\left(\frac{\pi u}{u}\right)(x)\, \mu(dx). \tag{2.3}$$

Proof. Let $I'(\mu)$ denote the right side in (2.3). Clearly $I(\mu) \geq I'(\mu)$. To see the reverse inequality, let $c > 0$ and suppose that $I(\mu) > c$. We will show that then

$I'(\mu)>c$. By definition of $I(\mu)$ there exists $u_0 \in \mathscr{U}_1$ such that

$$-\int \log \left(\frac{\pi u_0}{u_0}\right)(x)\,\mu(dx)>c. \tag{2.4}$$

Hence there exists $a>0$ such that

$$-\int_{|x|\leqq a} \log \left(\frac{\pi u_0}{u_0}\right)(x)\,\mu(dx)>c. \tag{2.5}$$

If $0<a_1 \leqq u_0 \leqq b_1<\infty$, we can pick a larger so that

$$-\int_{|x|\leqq a} \log \left(\frac{\pi u_0}{u_0}\right)(x)\,\mu(dx)-\left(\log \frac{b_1}{a_1}\right)\mu[x:|x|>a]>c. \tag{2.6}$$

Now define u_1 such that $u_1 \in \mathscr{U}_3$, $u_1(x)=u_0(x)$ for $|x|\leqq \gamma$, where $\gamma>a$ will be picked suitably, and $a_1 \leqq u_1 \leqq b_1$. Then

$$|\pi u_0(x)-\pi u_1(x)|\leqq 2b_1 \int_{|y|>\gamma} \pi(x,dy). \tag{2.7}$$

For $|x|\leqq a$, by Lemma 2.1, we can pick γ so large that the right side in (2.7) is small enough so (note that $u_1(y)=u_0(y)$ for $|y|\leqq a$) that by (2.6)

$$-\int_{|y|\leqq a} \log \left(\frac{\pi u_1}{u_1}\right)(y)\,\mu(dy)>c+\log \frac{b_1}{a_1}\mu[|y|>a]$$

which shows that

$$-\int \log \left(\frac{\pi u_1}{u_1}\right)(y)\,\mu(dy)>c.$$

We also define the I-functional for the stable process $y_\alpha(t)$ with infinitesimal generator L as in [8]. This will be denoted by I_G. For $\mu \in M$

$$I_G(\mu)=-\inf_{u\in \mathscr{U}_4}\int \left(\frac{Lu}{u}\right)(x)\,\mu(dx). \tag{2.8}$$

If $h>0$, let

$$\begin{aligned}\pi^{(h)}(x,A)&=P[x+y_\alpha(h)\in A]\\ &=P^x[y_\alpha(h)\in A].\end{aligned} \tag{2.9}$$

Then $\pi^{(h)}$ is a Feller probability transition and $I_h(\mu)$ will denote the I-functional given by (2.3) corresponding to $\pi^{(h)}$. The dependence on G will be clear from the context and will be suppressed.

Finally, on M we will use the topology of vague convergence under which $\lambda_n \to \lambda$ means $\int f\,d\lambda_n \to \int f\,d\lambda$ for each continuous function f with limits 0 at $\pm\infty$, and on $\mathscr{M}$ the topology of weak convergence under which $\lambda_n \to \lambda$ means $\int f\,d\lambda_n \to \int f\,d\lambda$ for each bounded continuous function f. Vague (weak) neighborhood of a member of $M(\mathscr{M})$ will have the obvious meaning.

3. An Asymptotic Upper Estimate

The main result of this section is Theorem 3.2. Theorem 3.1 may be of some independent interest; it facilitates the proof of Theorem 3.2.

Theorem 3.1. *Let (k_n) be a strictly increasing sequence of integers and for $n \geqq 1$ let $y_0^{(n)}, y_1^{(n)}, \ldots, y_{k_n-1}^{(n)}$ be a Markov process with state space R and stationary transition probability function π_n. Assume*

$$\pi_n \text{ is Feller, } n \geqq 1; \tag{3.1}$$

$$\pi_n(x, \cdot) \Rightarrow \pi(x, \cdot),\ x \in R, \text{ where } \pi \text{ is again Feller;} \tag{3.2}$$

$$\text{if } u \in \mathscr{U}_3,\ \pi_n u(x) \to \pi u(x) \text{ uniformly in } x; \text{ and} \tag{3.3}$$

$$\lim_{|x| \to \infty} (\pi u/u)(x) = 1 \text{ for } u \in \mathscr{U}_3. \tag{3.4}$$

For $A \in \mathscr{B}$, define

$$L_n(\omega, A) = \frac{1}{k_n} \sum_{j=0}^{k_n-1} \chi_A(y_j^{(n)}(\omega)) \tag{3.5}$$

and for B a Borel subset of M (vague topology)

$$Q_{n,x}(B) = P^x[L_n(\omega, \cdot) \in B]. \tag{3.6}$$

Then, if C is a vaguely closed subset of M, we have

$$\limsup_n \frac{1}{k_n} \log \sup_x Q_{n,x}(C) \leqq -\inf_{\lambda \in C} I(\lambda), \tag{3.7}$$

where I denotes the I-functional of π defined by (2.2).

Proof. Let $u \in \mathscr{U}_3$ and let

$$V_n = \pi_n u, \qquad \exp(-W_n) = u V_n^{-1}.$$

W_n is a real bounded continuous function on R. Using induction and the definitions we have

$$E^x[V_n(Y_{k_n-1}) \exp\{-(W_n(Y_0^{(n)}) + \ldots + W_n(Y_{k_n-1}^{(n)}))\}] = u(x).$$

From this we conclude that for all $x \in R$

$$E^x\left[\exp\left\{-\sum_{j=0}^{k_n-1} W_n(Y_j^{(n)})\right\}\right] \leqq c < \infty \tag{3.8}$$

where c depends only on the bounds of u. This inequality is the same as (recall (2.1))

$$E^x[\exp\{-k_n L_n(\omega, W_n)\}] \leqq c.$$

Therefore

$$E^x[(\exp\{-k_n L_n(\omega, W_n)\}) \chi_{[L_n(\omega, \cdot) \in C]}] \leqq c$$

and we get

$$Q_{n,x}(C)[\exp\{-k_n \sup_{\lambda\in C} \lambda(W_n)\}] \leqq c.$$

This last inequality is the same as

$$Q_{n,x}(C) \leqq c \exp\{k_n \sup_{\lambda\in C} \lambda(W_n)\}. \tag{3.9}$$

Since the right side in (3.9) does not depend on x and $W_n = \log(\pi_n u/u)$, we get

$$\frac{1}{k_n} \log \sup_x Q_{n,x}(C) \leqq \frac{\log c}{k_n} + \sup_{\lambda\in C} \int \log(\pi_n u/u)(y)\,\lambda(dy). \tag{3.10}$$

Now using (3.3) we get

$$\limsup_n \frac{1}{k_n} \log \sup_x Q_{n,x}(C) \leqq \sup_{\lambda\in C} \int \log(\pi u/u)(y)\,\lambda(dy). \tag{3.11}$$

Since the left side in (3.11) does not depend on u, we can take the inf in $u\in\mathscr{U}_3$ on the right side. We thus obtain for a vaguely measurable subset C of M

$$\limsup_n \frac{1}{k_n} \log \sup_x Q_{n,x}(C) \leqq \inf_{u\in\mathscr{U}_3} \sup_{\lambda\in C} \int \log(\pi u/u)(y)\,\lambda(dy). \tag{3.12}$$

If $C \subset \bigcup_{j=1}^{k} C_j$, then the left-side in (3.12) is dominated by

$$\max_{1\leqq j\leqq k} \limsup_n \frac{1}{k_n} \log \sup_x Q_{n,x}(C_j)$$

and therefore the right-side in (3.12) can be replaced by the expression

$$\inf \max_{1\leqq j\leqq k} \inf_{u\in\mathscr{U}_3} \sup_{\lambda\in C_j} \int \log(\pi u/u)(y)\,\lambda(dy), \tag{3.13}$$

where the first inf is over all measurable $C_1, \ldots, C_k$ whose union contains C. Denote the integral in (3.13) by $H(u, \lambda)$. For $u\in\mathscr{U}_3$, $\log(\pi u/u)$ is a continuous function and by assumption (3.4) it tends to 0 as $|x|\to\infty$, therefore for a fixed $u\in\mathscr{U}_3$ the function $H(u,\lambda)$ is continuous in λ. If C is compact, then we will show that the expression in (3.13) is dominated by

$$\sup_{\lambda\in C} \inf_{u\in\mathscr{U}_3} H(u,\lambda) \tag{3.14}$$

and this will prove the theorem via Lemma 2.2. Let the quantity in (3.14) equal $l<\infty$ (otherwise there is nothing to prove). Let $\varepsilon>0$, then by the continuity of H in λ, given λ there is a neighborhood N_λ of λ and a u_λ such that $H(u_\lambda, v) < l + \varepsilon$ for $v\in N_\lambda$. Using compactness of C we conclude that there exist $N_{\lambda_1}, \ldots, N_{\lambda_k}$, open subsets of M, such that $\bigcup_{j=1}^{k} N_{\lambda_j} \supset C$ and

$$\max_{1 \leqq j \leqq k} \inf_{u \in \mathcal{U}_3} \sup_{\lambda \in N_{\lambda_j}} H(u, \lambda) \leqq l + \varepsilon,$$

which is what we had to show.

In the following $d(n) \nearrow \infty$ and $k(n) \nearrow \infty$ denote positive integer sequences and

$$r(n) = d(n)\, k(n). \tag{3.15}$$

One should think of $d(n)$ as the block size and $k(n)$ as the number of blocks. For $A \in \mathcal{B}$, we define

$$\hat{L}_n^x(\omega, A) = \frac{1}{r(n)} \sum_{j=0}^{r(n)-1} \chi_{A-x}(S_j(\omega)/a(d(n))), \tag{3.16}$$

where $a(n) = n^{1/\alpha} l(n)$, as before. For $x = 0$ we simply write $\hat{L}_n(\omega, A)$.

Theorem 3.2. *Let $X_1, X_2, \ldots$ be i.i.d. random variables satisfying* (1.2). *Then for a vaguely closed subset C of M we have*

$$\limsup_n \frac{1}{k(n)} \log \sup_x P[\hat{L}_n^x(\omega, \cdot) \in C] \leqq -\inf_{\lambda \in C} I_G(\lambda) \tag{3.17}$$

where $I_G(\lambda)$ is given by (2.8).

To prove the theorem we will need the following lemma.

Lemma 3.3. *If $u \in \mathcal{U}_3$ and $\mu_n \Rightarrow \mu$, μ_n, μ in $\mathcal{M}$, then*

$$\int u(x+y)\, \mu_n(dy) \to \int u(x+y)\, \mu(dy)$$

uniformly in x.

Proof of the Lemma. Since u is bounded and $\mu_n \Rightarrow \mu$, it is enough to show that for $a > 0$

$$\Big| \int_{|y| \leqq a} u(x+y)\, \mu_n(dy) - \int_{|y| \leqq a} u(x+y)\, \mu(dy) \Big| \to 0 \tag{3.18}$$

uniformly in x, and a can be picked so $\pm a$ are continuity points of μ. Let $c > a$ be so large that for $|y| \leqq a$, $u(x+y)$ is constant for $|x| > c$ $(u \in \mathcal{U}_3)$. It is thus enough to consider $|x| \leqq c$. Let $\varepsilon > 0$ be given. Since u is uniformly continuous, $[-c, c]$ can be partitioned $-c = x_0 < x_1 < \ldots < x_N = c$, N depending on ε, so $|u(x'+y) - u(x''+y)| < \varepsilon$ for x', x'' in any closed subinterval of the partition and all y. If $x \in [x_i, x_{i+1}]$, then writing $I = [y: |y| \leqq a]$

$$\Big|\int_I u(x+y)\, \mu_n(dy) - \int_I u(x+y)\, \mu(dy)\Big| \leqq \Big|\int_I (u(x+y) - u(x_i+y))\, \mu_n(dy)\Big|$$

$$+ \Big|\int_I u(x_i+y)\, \mu_n(dy) - \int_I u(x_i+y)\, \mu(dy)\Big| + \Big|\int_I (u(x_i+y) - u(x+y))\, \mu(dy)\Big|.$$

Each of the first and the last terms on the right is $\leqq \varepsilon$, and the middle term tends to 0 as $n \to \infty$. Therefore there exists n_0 depending on N (hence on ε only) such that for $n \geqq n_0$ the right side $\leqq 3\varepsilon$. This finishes the proof.

Proof of Theorem 3.2. Let $h > 0$, and define

$$d'(n) = [h\, d(n)], \qquad k'(n) = [r(n)/d'(n)] + 1$$

and

$$J_k(n) = [j: (k-1)\, d'(n) \leqq j < k\, d'(n)].$$

Also, let

$$Y_k^{(n)}(i) = \sum_{j \in J_k(n)} X_{i+j}/a(d(n)), \qquad 1 \leqq k \leqq k'(n) - 1, \; i \geqq 0, \tag{3.19}$$

and for $n \geqq 1$, let

$$\pi_n(x, A) = P[x + Y_1^{(n)}(0) \in A]. \tag{3.20}$$

For $x \in R$ we have

$$\pi_n(x, \cdot) \Rightarrow \pi^{(h)}(x, \cdot) \tag{3.21}$$

where

$$\pi^{(h)}(x, A) = P^x[y_\alpha(h) \in A].$$

The transitions π_n and $\pi^{(h)}$ are easily seen to satisfy conditions (3.1), (3.2) and (3.4) where $\pi^{(h)}$ plays the role of π. Lemma 3.3 shows that condition (3.3) is also satisfied. Now let $u \in \mathscr{U}_3$ and let $\exp(-W_n) = u/\pi_n u$. As in the proof of Theorem 3.1, for $i \geqq 0$, $n \geqq 1$, we have $(Y_0^{(n)}(i) = 0)$

$$E^x\left[\exp\left\{-\left(\sum_{r=0}^{k'(n)-1} W_n(Y_0^{(n)}(i) + \ldots + Y_r^{(n)}(i))\right)\right\}\right] \leqq c$$

where c depends only on the bounds of u. Since this holds for all x, we can replace x by $x + (S_i/a(d(n))$ to conclude that

$$E^x\left[\exp\left\{-\sum_{r=0}^{k'(n)-1} W_n(Z_r^{(n)}(i))\right\}\right] \leqq c$$

where we write

$$Z_r^{(n)}(i) = S_{r d'(n) + i}/a(d(n)).$$

By Jensen's inequality

$$E^x\left[\exp\left\{-\frac{1}{d'(n)} \sum_{i=0}^{d'(n)-1} \left(\sum_{r=0}^{k'(n)-1} W_n(Z_r^{(n)}(i))\right)\right\}\right]$$
$$\leqq E^x\left[\frac{1}{d'(n)} \sum_{i=0}^{d'(n)-1} \exp\left\{-\sum_{r=0}^{k'(n)-1} W_n(Z_r^{(n)}(i))\right\}\right] \leqq c.$$

Since W_n is a bounded function and the incomplete block consists of no more than $d'(n)$ terms, we have

$$E^x\left[\exp\left\{-\frac{1}{d'(n)}(W_n(S_0/a(d(n)) + \ldots + W_n(S_{r(n)-1}/a(d(n)))\right\}\right] \leqq c_1 \tag{3.22}$$

where c_1 again depends only on the bounds of u. Let $\varphi(n)=r(n)/d'(n)$. Then (3.22) is the same as

$$E[\exp\{-\varphi(n)\hat{L}_n^x(\omega, W_n)\}] \leqq c_1. \tag{3.23}$$

Now following the proof of Theorem 3.1 we get

$$h \limsup_n \frac{1}{k(n)} \log \sup_x P[\hat{L}_n^x(\omega, \cdot) \in C] \leqq -\inf_{\lambda \in C} I_h(\lambda), \tag{3.24}$$

where I_h is given by

$$I_h(\lambda) = -\inf_{u \in \mathscr{U}_3} \int \log\left(\frac{\pi^{(h)} u(x)}{u(x)}\right) \lambda(dx). \tag{3.25}$$

We have thus shown that for C a closed subset of M,

$$\limsup_n \frac{1}{k(n)} \log \sup_x P[\hat{L}_n^x(\omega, \cdot) \in C] \leqq -\limsup_{h \to 0} \inf_{\lambda \in C} (I_h(\lambda)/h). \tag{3.26}$$

If $u \in \mathscr{U}_4$ and L is the infinitesimal generator of the process $y_\alpha(t)$, then

$$\frac{\pi^{(h)} u}{u} = 1 + h \frac{Lu}{u} + o(h)$$

where $o(h)$ is uniform in x. From this we get

$$-\frac{1}{h} \int \log\left(\frac{\pi^{(h)} u(x)}{u(x)}\right) \lambda(dx) = -\int \left(\frac{Lu}{u}\right)(x)\, \lambda(dx) + o(1)$$

where $o(1)$ is uniform in λ as $h \to 0$. Taking the sup over $u \in \mathscr{U}_3$ on the left side we therefore get

$$I_h(\lambda)/h \geqq -\int \left(\frac{Lu}{u}\right)(x)\, \lambda(dx) + \varepsilon_h(u)$$

for every $u \in \mathscr{U}_4$, where $\varepsilon_h(u)$ depends on u and tends to 0 uniformly in λ as $h \to 0$ (u fixed). Therefore,

$$\limsup_{h \to 0} \inf_{\lambda \in C} (I_h(\lambda)/h) \geqq -\inf_{\lambda \in C} \int \left(\frac{Lu}{u}\right)(x)\, \lambda(dx)$$

for all $u \in \mathscr{U}_4$. Since the left side does not depend on u, this implies

$$\limsup_{h \to 0} \inf_{\lambda \in C} (I_h(\lambda)/h) \geqq -\inf_{u \in \mathscr{U}_4} \sup_{\lambda \in C} \int \left(\frac{Lu}{u}\right)(x)\, \lambda(dx). \tag{3.27}$$

This and (3.26) give

$$\limsup_n \frac{1}{k(n)} \log \sup_x P[\hat{L}_n^x(\omega, \cdot) \in C] \leqq \inf_{u \in \mathscr{U}_4} \sup_{\lambda \in C} \int \left(\frac{Lu}{u}\right)(x)\, \lambda(dx). \tag{3.28}$$

If $C \subset \bigcup_{j=1}^{k} C_j$, C_j closed in M, then the left-side in (3.28) is dominated by

$$\max_{1 \leqq j \leqq k} \limsup_n \frac{1}{k(n)} \log \sup_x P[\hat{L}_n^x(\omega, \cdot) \in C_j]$$

and then by (3.28) we get

$$\limsup_n \frac{1}{k(n)} \log \sup_x P[\hat{L}_n^x(\omega, \cdot) \in C]$$
$$\leqq \inf_{C \subset \bigcup_{j=1}^{k} C_j} \max_{1 \leqq j \leqq k} \inf_{u \in \mathcal{U}_4} \sup_{\lambda \in C_j} \int \left(\frac{Lu}{u}\right)(x)\, \lambda(dx). \tag{3.29}$$

Let $H(u, \lambda)$ denote this last integral, which is continuous in λ for each $u \in \mathcal{U}_4$ because $(Lu/u)(x) \to 0$ as $|x| \to \infty$. The rest of the argument is the same as in the proof of Theorem 3.1 except that we should take N_λ's to be closed neighborhoods to show that

$$\inf_{\substack{C \subset \bigcup_{j=1}^{k} C_j \\ C_j \text{ closed}}} \max_{1 \leqq j \leqq k} \inf_{u \in \mathcal{U}_4} \sup_{\lambda \in C_j} \int \left(\frac{Lu}{u}\right)(x)\, \lambda(dx) \leqq \sup_{\lambda \in C} \inf_{u \in \mathcal{U}_4} \int \left(\frac{Lu}{u}\right)(x)\, \lambda(dx),$$

and since the right side equals $-\inf_{\lambda \in C} I_G(\lambda)$, this together with (3.29) implies (3.17).

4. An Asymptotic Lower Estimate

Let $d(n)$ and $k(n)$ be nondecreasing positive integer sequences tending to ∞, and let $r(n) = k(n)\, d(n)$ as in (3.15). The main result of this section is Theorem 4.1 which is an analogue of Lemma 2.12 of [8].

Theorem 4.1. *If β is a probability measure on R such that $\beta\{x: |x| \leqq a\} = 1$ and V is a weak neighborhood of β in $\mathcal{M}$, then for $a' > a > 0$ we have*

$$\liminf_n \frac{1}{k(n)} \log \inf_{|x| \leqq a} P[\hat{L}_n^x(\omega, \cdot) \in V;\, |x + (S_j/a(d(n)))| \leqq a',$$
$$0 \leqq j < r(n) - 1] \geqq -I_G(\beta), \tag{4.1}$$

where $\hat{L}_n^x$ is defined by (3.16) and I_G by (2.8).

The proof of this theorem depends on the following theorem which is a restatement of Lemma 2.12 [8]. First we need a notation.

$$L_t(\omega, A) = \frac{1}{t} \int_0^t \chi_A(y_\alpha(s, \omega))\, ds, \qquad A \in \mathcal{B}. \tag{4.2}$$

Theorem 4.2. *If β is a probability measure on R such that $\beta\{x:|x|\leqq a\}=1$ and V is a weak neighborhood of β in $\mathcal{M}$, then for $a'>a>0$ we have*

$$\liminf_{t\to\infty}\frac{1}{t}\log\inf_{|x|\leqq a} P^x[L_t(\omega,\cdot)\in V; |y_\alpha(s)|\leqq a', 0\leqq s\leqq t]\geqq -I_G(\beta). \tag{4.3}$$

What we actually need is the following corollary of Theorem 4.2.

Corollary 4.3. *For β and V as above and $0<a<a'$ we have*

$$\liminf_{t\to\infty}\frac{1}{t}\log\inf_{|x|\leqq a} P^x[L_t(\omega,\cdot)\in V; |y_\alpha(s)|\leqq a', 0\leqq s<t;$$
$$|y_\alpha(t)|\leqq a]\geqq -I_G(\beta). \tag{4.4}$$

Proof of Corollary. Without any loss of generality we may take

$$V=\{v\in\mathcal{M}: |\int f_i\,dv-\int f_i\,d\beta|<\varepsilon, 1\leqq i\leqq k\} \tag{4.5}$$

where f_i, $1\leqq i\leqq k$, are uniformly continuous bounded functions on R. Then

$$|L_{t-1}(\omega,f_i)-L_t(\omega,f_i)|\leqq\frac{2\|f_i\|_\infty}{t-1} \tag{4.6}$$

and the upper bound tends to 0 uniformly in ω as $t\to\infty$; it follows that there exists t_0 (independent of ω) such that $L_{t-1}(\omega,\cdot)\in V_1$ implies $L_t(\omega,\cdot)\in V$, whenever $t\geqq t_0$, where

$$V_1=\{v\in\mathcal{M}: |\int f_i\,dv-\int f_i\,d\beta|<\varepsilon/2, 1\leqq i\leqq k\}. \tag{4.7}$$

Therefore, if $t\geqq t_0$, then by the Markov property the probability in (4.4) dominates $(a<a''<a')$

$$P^x[L_{t-1}(\omega,\cdot)\in V_1; |y_\alpha(s)|<a'', 0\leqq s\leqq t-1].$$
$$\inf_{|z|\leqq a''} P^z[|y_\alpha(s)|<a', 0\leqq s<1, |y_\alpha(1)|\leqq a]. \tag{4.8}$$

The second quantity in (4.8) is positive since G has strictly positive density. The corollary now follows from (4.8) by applying (4.3).

Proof of Theorem 4.1. Let V_1 be given by (4.7). For $\delta>0$ we pick a large *positive integer h* so by Cor. 4.3, with $a<a''<a'$

$$\inf_{|x|\leqq a} P^x[L_h(\omega,\cdot)\in V_1; |y_\alpha(s)|\leqq a'', 0\leqq s<h,$$
$$|y_\alpha(h)|\leqq a]\geqq\exp\{-h(I_G(\beta)+\delta)\}. \tag{4.9}$$

Let

$$d'(n)=h\,d(n),\qquad k'(n)=[r(n)/d'(n)]+1. \tag{4.10}$$

The idea is to work with blocks of length $d'(n)$, so there are $k'(n)$ blocks, the last one being possibly incomplete. The event

$$\bigcap_{r=0}^{k'(n)-1}\left[\left|\frac{1}{d'(n)}\sum_{j=0}^{d'(n)-1} f_i\left(x+\frac{S_{rd'(n)+j}}{a(d(n))}\right)-\int f_i\,d\beta\right|<\varepsilon, 1\leqq i\leqq k;\right.$$
$$\left.\left|x+\frac{S_{rd'(n)+m}}{a(d(n))}\right|\leqq a', 0<m<d'(n)-1, \left|x+\frac{S_{(r+1)d'(n)}}{a(d(n))}\right|\leqq a\right]$$

is contained in the event Λ_n^x which occurs in (4.1). Therefore, by using the Markov property we see that $\inf_{|x|\leqq a} P(\Lambda_n^x)$ dominates

$$\left\{\inf_{|x|\leqq a} P\left[\left|\frac{1}{d'(n)}\sum_{j=0}^{d'(n)-1} f_i\left(x+\frac{S_j}{a(d(n))}\right)-\int f_i\,d\beta\right|<\varepsilon, 1\leqq i\leqq k;\right.\right.$$
$$\left.\left.\left|x+\frac{S_m}{a(d(n))}\right|\leqq a', 0\leqq m<d'(n); \left|x+\frac{S_{d'(n)}}{a(d(n))}\right|\leqq a\right]\right\}^{k'(n)}. \tag{4.11}$$

We now claim that the quantity within curly brackets in (4.11) has lim inf larger than the quantity on the left side in (4.9). Assume this claim for the moment. Then

$$\liminf_n \frac{1}{k'(n)}\log\inf_{|x|\leqq a} P[\Lambda_n^x]\geqq -h(I_G(\beta)+\delta).$$

Since $hk'(n)\sim k(n)$ as $n\to\infty$, we get

$$\liminf_n \frac{1}{k(n)}\log\inf_{|x|\leqq a} P[\Lambda_n^x]\geqq -(I_G(\beta)+\delta)$$

and $\delta>0$ being arbitrary, the result follows. We now establish the claim made after (4.11).

Define $Y_n(t)$, $0\leqq t\leqq h$, $n\geqq 1$, as follows.

$$Y_n(t)=\frac{S_j}{a(d(n))}, \quad \frac{j}{d(n)}\leqq t<\frac{j+1}{d(n)}, \quad 0\leqq j<hd(n),$$

$$Y_n(h)=\frac{S_{d'(n)}}{a(d(n))}.$$

The sample paths of Y_n lie in $D[0,h]$ and each Y_n is a process with independent increments. By Theorem 5 [9], p. 435, it readily follows that

$$Y_n\Rightarrow y_\alpha \tag{4.12}$$

in the sense of weak convergence of measures in $D[0,h]$. By a theorem of Skorohod [12] we can construct a probability space on which $\tilde{Y}_n$, $n\geqq 1$, and $\tilde{y}_\alpha$, $D[0,h]$-valued random variables, are defined with the distribution of $\tilde{Y}_n(\tilde{y}_\alpha)$ the same as that of $Y^{(n)}(y_\alpha)$ and such that as $n\to\infty$

$$\tilde{Y}_n\to\tilde{y}_\alpha, \quad \text{a.s.} \tag{4.13}$$

The convergence in (4.13) is in the Skorohod metric in $D[0,h]$. Since $\tilde{y}_\alpha$ is a stable process without fixed discontinuities, we assume that $\tilde{y}_\alpha$ is a.s. continuous (from the left) at h. It then follows from (4.13) that

$$\tilde{Y}_n(h)\to\tilde{y}_\alpha(h), \quad \text{a.s.} \tag{4.14}$$

The following lemma will now be useful.

Lemma 4.4. *With $\tilde{Y}_n$ and $\tilde{y}_\alpha$ as above, f a bounded uniformly continuous function on R, as $n\to\infty$*

$$\int_0^h f(x+\tilde{Y}_n(t))\,dt\to\int_0^h f(x+\tilde{y}_\alpha(t))\,dt, \quad a.s., \tag{4.15}$$

and

$$\sup_{0\leqq t\leqq h}|x+\tilde{Y}_n(t)|\to\sup_{0\leqq t\leqq h}|x+\tilde{y}_\alpha(t)|, \quad a.s., \tag{4.16}$$

where the convergence is uniform in x in (4.15) *and* (4.16).

Proof of the Lemma. Let Γ be the set of all continuous, increasing, functions λ on $[0,1]$ such that $\lambda(0)=0$, $\lambda(1)=1$. Then

$$\begin{aligned}&\left|\sup_t|x+\tilde{Y}_n(t)|-\sup_t|x+\tilde{y}_\alpha(t)|\right|=\left|\sup_t|x+\tilde{Y}_n(\lambda(t))|-\sup_t|x+\tilde{y}_\alpha(t)|\right|\\&\quad\leqq\inf_{\lambda\in\Gamma}\sup_t|(x+\tilde{Y}_n(\lambda(t)))-(x+\tilde{y}_\alpha(t))|\to 0\end{aligned}$$

as $n\to\infty$, uniformly in x, which proves (4.16). By (4.13) we have a.s., $\lambda_n(\omega)\in\Gamma$ such that

$$\sup_t|\tilde{Y}_n(t,\omega)-\tilde{y}_\alpha(\lambda_n(t),\omega)|\to 0 \tag{4.17}$$

and

$$\sup_t|\lambda_n(t)-t|\to 0. \tag{4.18}$$

Therefore

$$\begin{aligned}&\left|\int_0^h f(x+\tilde{Y}_n(t))\,dt-\int_0^h f(x+\tilde{y}_\alpha(t))\,dt\right|\\&\quad\leqq\int_0^h|f(x+\tilde{Y}_n(t))-f(x+\tilde{y}_\alpha(\lambda_n(t)))|\,dt+\int_0^h|f(x+\tilde{y}_\alpha(\lambda_n(t)))-f(x+\tilde{y}_\alpha(t))|\,dt,\end{aligned}$$

since f is bounded and uniformly continuous, each integrand on the right tends to 0 a.e. (t) boundedly (note that $\tilde{y}_\alpha$ is continuous a.e.), so each term tends to zero; the uniformity of convergence in x is clear.

Now let

$$\tilde{L}^x_{h,n}(\omega,A)=h^{-1}\int_0^h\chi_A(x+\tilde{Y}_n(t))\,dt \quad\text{and}\quad \tilde{L}_h(\omega,A)=h^{-1}\int_0^h\chi_A(\tilde{y}_\alpha(t))\,dt.$$

By Lemma 4.4, (4.14), and the fact that $\tilde{y}_\alpha(h)$ has a continuous distribution, we conclude that if $a<a''<a'$ and V_1 is given by (4.7), then

$$\begin{aligned}&\liminf_n\inf_{|x|\leqq a}P[\tilde{L}^x_{h,n}(w,\cdot)\in V;\,|x+\tilde{Y}_n(s)|\leqq a',0\leqq s\leqq h,\,|x+\tilde{Y}_n(h)|\leqq a]\\&\quad\geqq\inf_{|x|\leqq a}P^x[\tilde{L}_h(\omega,\cdot)\in V_1;\,|\tilde{y}_\alpha(s)|\leqq a'',0\leqq s\leqq h,\,|\tilde{y}_\alpha(h)|\leqq a].\end{aligned}$$

This is clearly the claim we made, because

$$\tilde{L}^x_{h,n}(\omega,f)=\frac{1}{h}\int_0^h f(x+\tilde{Y}_n(t))\,dt$$

$$=\frac{1}{h}\sum_{j=0}^{hd(n)-1} d(n)^{-1} f\left(x+\frac{S_j}{a(d(n))}\right).$$

This establishes Theorem 4.1.

We now use Theorem 4.1 to prove the following analogue of Theorem 2.15 [8].

Theorem 4.5. *Let β be a probability measure on R such that $\beta\{x:|x|\leqq a\}=1$ and $I_G(\beta)<1$. Let V be a weak neighborhood of β in $\mathscr{M}$. Then for $a'>a$*

$$P[\bar{L}_n(\omega,\cdot)\in V;\ \max_{0\leqq j\leqq n}|S_j/c(n)|\leqq a';\ \text{i.o.}]=1, \tag{4.19}$$

where

$$\bar{L}_n(\omega,A)=\frac{1}{n}\sum_{j=0}^{n-1}\chi_A(S_j/c(n)), \tag{4.20}$$

and $c(n)$ is given by (1.12).

We need the following lemma for the proof of this theorem. Let $\gamma>1$ be such that $\gamma I_G(\beta)<1$ and let

$$j_n=[\exp(n^\gamma)], \tag{4.21}$$

and

$$\varphi(n)=b(j_n)\,[j_n/b(j_n)]. \tag{4.22}$$

Lemma 4.6. *We have*

$$\lim_n\,(\max_{0\leqq j\leqq \varphi(n-1)}|S_j|/c(j_n))=0, \qquad a.s. \tag{4.23}$$

Proof. By a lemma of Skorohod (Lemma 3.2 [2], p. 45) we have for $\varepsilon>0$

$$P\left[\max_{0\leqq j\leqq\varphi(n-1)}|S_j|>2\varepsilon c(j_n)\right]\leqq\frac{1}{1-c_n}P[|S_{\varphi(n-1)}|>\varepsilon c(j_n)], \tag{4.24}$$

where

$$c_n=\max_{0\leqq j\leqq\varphi(n-1)}P[|S_j|>\varepsilon c(j_n)]. \tag{4.25}$$

If $0<\alpha'<\alpha$ (the stable index is α), then by Theorem 6.1 [1]

$$\lim_n E\,|S_n/a(n)|^{\alpha'}=E\,|y_\alpha(1)|^{\alpha'}<\infty. \tag{4.26}$$

Since $\varphi(n)\sim j_n$, $a(\varphi(n-1))/c(j_n)\to 0$, and (4.26) easily implies that $c_n\to 0$. Therefore it is enough to check that $\sum P[|S_{\varphi(n-1)}|>\varepsilon c(j_n)]<\infty$. By (4.26) we have by Čebyšev's inequality

$$P[|S_{\varphi(n-1)}|>\varepsilon c(j_n)]\leqq\text{const.}\left\{\frac{a(\varphi(n-1))}{\varepsilon c(j_n)}\right\}^{\alpha'}.$$

This upper estimate is easily seen to be summable in n. The Borel-Cantelli lemma then implies (4.23).

Proof of Theorem 4.5. We take V as in (4.5), and let

$$B_n = [\bar{L}_n(\omega, \cdot) \in V;\ \max_{0 \leq j < n} |S_j/c(n)| \leq a'].$$

If j_n is given by (4.21), we will show

$$P[B_{j_n} \text{ i.o.}] = 1.$$

To make the events independent, let

$$\tilde{L}_{j_n}(\omega, A) = \frac{1}{\varphi(n)} \sum_{j=\varphi(n-1)}^{\varphi(n)-1} \chi_A((S_j - S_{\varphi(n-1)})/c(j_n)),$$

where $\varphi(n)$ is given by (4.22), and define

$$\tilde{B}_{j_n} = [\tilde{L}_{j_n}(\omega, \cdot) \in V_1;\ \max_{\varphi(n-1) \leq j < \varphi(n)} |(S_j - S_{\varphi(n-1)})/c(j_n)| \leq a''],$$

where V_1 is given by (4.7) and $a < a'' < a'$. Since $\varphi(n) \sim j_n$ and $(\varphi(n-1)/\varphi(n)) \to 0$, by Lemma 4.6 we conclude that for almost all ω, n sufficiently large, if $\tilde{B}_{j_n}$ occurs, then B_{j_n} occurs. Therefore, it suffices to prove that

$$P[\tilde{B}_{j_n} \text{ i.o.}] = 1. \tag{4.27}$$

The events $\tilde{B}_{j_n}$ are independent, so by Borel-Cantelli it suffices to prove that

$$\sum_n P(\tilde{B}_{j_n}) = \infty. \tag{4.28}$$

Now

$$P(\tilde{B}_{j_n}) = P\left[\frac{1}{\varphi(n)} \sum_{j=0}^{\varphi(n)-\varphi(n-1)-1} \chi_{(\cdot)}(S_j/c(j_n)) \in V_1;\ \max_{0 \leq j < \varphi(n)-\varphi(n-1)} |S_j/c(j_n)| \leq a''\right].$$

Let

$$V_2 = \{\nu \in \mathscr{M}\colon |\textstyle\int f_i\, d\nu - \int f_i\, d\beta| < \varepsilon/4,\ i = 1, \ldots, k\}.$$

If $k(n)$ is chosen to be $[n/b(n)]$, then $\varphi(n) - \varphi(n-1) - 1 \sim \varphi(n) \sim k(j_n)\, b(j_n) \sim j_n$. It follows that for n sufficiently large, writing $r(n) = k(n)\, b(n)$, we have

$$P(\tilde{B}_{j_n}) \geq P\left[\frac{1}{r(j_n)} \sum_{j=0}^{r(j_n)-1} \chi_{(\cdot)}(S_j/c(j_n)) \in V_2;\ \max_{0 \leq j < r(j_n)} |S_j/c(j_n)| \leq a''\right]. \tag{4.29}$$

We now use the estimate of Theorem 4.1 given in (4.1), with $b(n)$ playing the role of $d(n)$ in this estimate, to conclude that if $\delta > 0$ is picked so $\gamma(I_G(\beta) + \delta) < 1$, then, for all n sufficiently large, the right side in (4.29) dominates $\exp\{-k(j_n)(I_G(\beta) + \delta)\} \sim n^{-\gamma(I_G(\beta)+\delta)}$. This completes the proof of the theorem.

5. Important Corollaries

We use the topology of vague convergence in M. Closures in M will be with respect to this topology.

Theorem 5.1. *Let* $C_G=\{\beta\in M\colon I_G(\beta)\leqq 1\}$. *Let* $\bar{L}_n(\omega,\cdot)$ *be defined by* (4.20). *Then for almost all* ω

$$\bigcap_{m=1}^{\infty}\overline{\bigcup_{n\geqq m}\{\bar{L}_n(\omega,\cdot)\}}=C_G. \tag{5.1}$$

Proof. With $d(n)=b(n)$, $k(n)=[n/b(n)]$ and $x=0$ we define $\hat{L}_n$ as in (3.16). Since

$$\sup_{\substack{\omega\in\Omega\\ A\in\mathscr{B}}}|\hat{L}_n(\omega,A)-\bar{L}_n(\omega,A)|\leqq\frac{2(n-r(n))}{r(n)}\to 0$$

(note that $r(n)=k(n)\,d(n)\sim n$), the set of limit points of $\{\hat{L}_n(\omega,\cdot)\}$ is the same as that of $\{\bar{L}_n(\omega,\cdot)\}$ for each ω. Therefore, it suffices to prove (5.1) for $\hat{L}_n$.

We first prove that the left side in (5.1) is contained in the right side. Let N_1 be an open neighborhood of C_G. Since I_G is lower semicontinuous on M, we have $\inf\limits_{\lambda\in N_1^c}I_G(\lambda)=\theta>1$. Let $0<\gamma<1$ be such that $\theta\gamma>1$ and let $j_n=[\exp(n^\gamma)]$. Let $\varepsilon>0$ be such that $\gamma(\theta-\varepsilon)>1$. By Theorem 3.2

$$P[\hat{L}_{j_n}(\omega,\cdot)\in N_1^c]\leqq\exp\{-k(j_n)(\theta-\varepsilon)\} \tag{5.2}$$

for all n sufficiently large. The right side of (5.2) summed on n converges. Therefore, by the Borel-Cantelli lemma

$$P[\hat{L}_{j_n}(\omega,\cdot)\in N_1^c \text{ i.o.}]=0. \tag{5.3}$$

This means

$$P[\bigcap_m\overline{\bigcup_{n\geqq m}\{\hat{L}_{j_n}(\omega,\cdot)\}}\subset\bar{N}_1]=1. \tag{5.4}$$

Now, if $j_{n-1}\leqq p_n<j_n$, then $c(p_n)/c(j_n)\to 1$, consequently for any continuous f with compact support and $\omega\in\Omega$

$$\lim_n|\hat{L}_{p_n}(\omega,f)-\hat{L}_{j_n}(\omega,f)|=0.$$

Therefore $\{\hat{L}_{p_n}(\omega,\cdot)\}$ and $\{\hat{L}_{j_n}(\omega,\cdot)\}$ have the same vague limit points and

$$P[\bigcap_m\overline{\bigcup_{n\geqq m}\{\hat{L}_n(\omega,\cdot)\}}\subset\bar{N}_1]=1. \tag{5.5}$$

Since we can pick $N_j\supset\bar{N}_{j+1}$, N_j open, $j\geqq 1$, such that $\bigcap_{j=1}^{\infty}N_j=C_G$, (5.5) implies that the left side in (5.1) is contained in the right side.

To see the converse, by Lemma 2.16 [8], the set D of probability measures β with compact supports and satisfying $I_G(\beta)<1$ is dense in C_G, so there exists $D_1=\{\beta_1,\beta_2,\ldots\}\subset D$ which is dense in C_G. By Theorem 4.5 the left side in (5.1) contains D_1 a.s. and since the left side is a closed set it must contain $\bar{D}_1$, hence C_G.

The following theorem is an immediate corollary of Theorem 5.1.

Theorem 5.2. *If Φ is a functional on M which is lower (upper) semicontinuous on M in the vague topology, then*

$$\limsup_n \Phi(\bar{L}_n(\omega, \cdot)) \underset{(\leqq)}{\geqq} \sup_{\beta \in C_G} \Phi(\beta), \qquad \text{a.s.,} \tag{5.6}$$

where $\bar{L}_n$ is defined by (4.20) *and* $C_G = \{\beta \in M : I_G(\beta) \leqq 1\}$.

This has the following corollary.

Corollary 5.3. *If Φ is a continuous functional on M in the vague topology, then*

$$\limsup_n \Phi(\bar{L}_n(\omega, \cdot)) = \sup_{\beta \in C_G} \Phi(\beta), \qquad \text{a.s.} \tag{5.7}$$

6. Applications

We now turn to applications similar to those in Sect. 4 [8]. As before

$$C_G = \{\beta \in M : I_G(\beta) \leqq 1\}.$$

Recall that $c(n) = a(b(n))$.

Example 6.1. Let V be a continuous function on R such that $V(x) \to 0$ as $|x| \to \infty$, then

$$\limsup_n \frac{1}{n} \sum_{j=0}^{n-1} V(S_j/c(n)) = \sup_{\beta \in C_G} \int V(x)\, d\beta(x). \tag{6.1}$$

Proof. For $\beta \in M$, $\Phi(\beta) = \int V(x)\, d\beta(x)$ defines a continuous functional on M and (6.1) follows from Cor. 5.3.

Example 6.2. Let V be a continuous function on R such that $V(x) \to \infty$ as $|x| \to \infty$. For $\beta \in \mathcal{M}$, let

$$\begin{aligned} \Phi(\beta) &= \int V(x)\, d\beta(x), && \text{if the integral is finite} \\ &= \infty, && \text{otherwise.} \end{aligned}$$

Define $\Phi(\beta) = \infty$ on $M - \mathcal{M}$. Then

$$\liminf_n \frac{1}{n} \sum_{j=0}^{n-1} V(S_j/c(n)) = \inf_{\beta \in C_G \cap \mathcal{M}} \int V(x)\, d\beta(x). \tag{6.2}$$

Proof. It is easily seen that Φ is lower semicontinuous on M. Let φ_ε be a continuous, nonvanishing, function on $(-\infty, \infty]$ such that $|\varphi_\varepsilon(x) - x| < \varepsilon$ for all x and $\varphi_\varepsilon(x) = x$ for $|x| > \varepsilon$. Then $\varphi_\varepsilon \circ \Phi$ is lower semicontinuous and $(\varphi_\varepsilon \circ \Phi)^{-1}$ is upper semicontinuous on M. By Theorem 5.2 we get

$$\liminf_n \varphi_\varepsilon \left(\frac{1}{n} \sum_{j=0}^{n-1} V(S_j/c(n)) \right) \geqq \inf_{\beta \in C_G} (\varphi_\varepsilon \circ \Phi)(\beta), \qquad \text{a.s.} \tag{6.3}$$

Therefore

$$\liminf_n \frac{1}{n} \sum_{j=0}^{n-1} V(S_j/c(n)) \geqq \inf_{\beta \in C_G} \Phi(\beta) - 2\varepsilon. \tag{6.4}$$

Since $\Phi(\beta)=\infty$ on $M-\mathcal{M}$, the inf on the right in (6.4) can be taken over $C_G \cap \mathcal{M}$.

We now prove the inequality in the opposite direction: Let β be a probability measure on R such that $\beta\{x: |x| \leqq a\}=1$ and $I_G(\beta)<1$. By Theorem 4.5 this β is a limit point of $\bar{L}_n(\omega, \cdot)$ along sequences where $\bar{L}_n(\omega, \cdot)$ is supported by $[-a', a']$ a.s. Therefore

$$\liminf_n \frac{1}{n} \sum_{j=0}^{n-1} V(S_j/c(n)) \leqq \int V(x)\, d\beta(x) \tag{6.5}$$

a.s. for each such β. As observed in the proof of Theorem 5.1, let D_1 be a countable set of such β's dense in C_G. Then the right side in (6.5) can be replaced by

$$\inf_{\beta \in D_1} \int V(x)\, d\beta(x). \tag{6.6}$$

Now suppose that $\beta \in \mathcal{M} \cap C_G$, and $\int V(x)\, d\beta(x) < \infty$. Then if $\beta_n \in D_1$, $\beta_n \to \beta$, we have

$$\int_{[a,b]} V(x)\, d\beta_n(x) \to \int_{[a,b]} V(x)\, d\beta(x),$$

where a, b are points of continuity of β. It is thus clear that

$$\liminf_n \int V(x)\, d\beta_n(x) \geqq \int V(x)\, d\beta(x).$$

Therefore the quantity in (6.6) is the same as the right side in (6.2). This finishes the proof.

If we take $V(x)=|x|^a$ for $a>0$, we get Theorem 1.3 of the introduction. For $a=\alpha=2$, the constant on the right side in (6.2) is identified in [8] to be 1/4.

Example 6.3. For $c>0$ let

$$\Phi_c(\beta)=\beta\{x: |x| \leqq c\},$$

and

$$\Phi'_c(\beta)=\beta\{x: |x| < c\},$$

the $\Phi_c(\Phi'_c)$ is upper (lower) semicontinuous. If $I_G(\beta)<\infty$, then β is absolutely continuous with respect to Lebesgue measure [5], hence

$$\sup_{\beta \in C_G} \Phi_c(\beta) = \sup_{\beta \in C_G} \Phi'_c(\beta) = k_{c,G}. \tag{6.7}$$

It is clear that for fixed G, the limit distribution, $k_{c,G}$ increases in c. Let c_G be the smallest c such that $k_{c,G}=1$ for $c \geqq c_G$. The existence of such a c_G is pointed out in [8]. It is also shown there that if $\alpha=2$ (i.e., G is $N(0,1)$) then $c_G=8^{-1/2}\pi$

(this corresponds to standard Brownian motion). This constant c_G is identified with the constant of (1.11) in [8].

By Theorem 5.2 applied to Φ_c and Φ'_c we get

$$\limsup_n \frac{1}{n} \sum_{j=0}^{n-1} \chi_{[0,c]}(|S_j|/c(n)) = k_{c,G}, \qquad \text{a.s.,} \tag{6.8}$$

and for $c \geqq c_G$ the right side equals 1.

Example 6.4. This example is Theorem 1.2, and we now give its proof. Suppose $c < c_G$, then by (6.8) we have

$$\limsup_n \frac{1}{n} \sum_{j=0}^{n-1} \chi_{[0,c]}(|S_j|/c(n) < 1, \qquad \text{a.s.} \tag{6.9}$$

Therefore almost surely there exists $\varepsilon > 0$ such that for all sufficiently large n

$$\sum_{j=0}^{n-1} \chi_{[0,c]}(|S_j|/c(n)) \leqq (1-\varepsilon) n \tag{6.10}$$

and this implies

$$\liminf_n (A_n(\omega)/c(n)) \geqq c, \qquad \text{a.s.} \tag{6.11}$$

It remains to prove the opposite inequality. Let $c_G < c < c'$. Suppose there exists a $\beta \in \mathscr{M}$ such that $\beta\{x: |x| \leqq c\} = 1$ and $I_G(\beta) < 1$, then by Theorem 4.5 we have

$$\liminf_n (A_n(\omega)/c(n)) \leqq c', \qquad \text{a.s.}$$

It is thus sufficient to show that for any $c > c_G$ such a β exists. By the definition of c_G, there exists a sequence $\{\beta_n\}$ such that $\beta_n \in C_G$, so β_n is absolutely continuous with respect to Lebesgue measure, and $\beta_n\{x: |x| \leqq c\} \to 1$. It follows that along a subsequence $\beta_n \Rightarrow \beta_0$ and $\beta_0\{x: |x| \leqq c\} = 1$, and since I_G is lower semicontinuous, $I_G(\beta_0) \leqq \liminf_n I_G(\beta_n) \leqq 1$. Now let $\theta > 1$ and define

$$\hat{\beta}(A) = \beta_0(\theta^{-1/\alpha} A), \qquad A \in \mathscr{B}. \tag{6.12}$$

Then by the scaling property

$$\theta I_G(\hat{\beta}) = I_G(\beta_0). \tag{6.13}$$

If $c_1 > c$, then $\theta > 1$ can be picked so $\hat{\beta}\{x: |x| \leqq c_1\} = 1$, and by (6.13) we have $I_G(\hat{\beta}) < 1$. The constant c_G is identified with the constant of (1.11) in [8]. This finishes the proof.

Acknowledgements. (i) This work was essentially completed while I was visiting the Department of Mathematics, University of Illinois, Champaign-Urbana in 1979–80. I very deeply appreciate the hospitality of the department.

(ii) I am indebted to S.R.S. Varadhan for pointing out an error in the original version of this paper.

(iii) I would also like to thank the referee and K.L. Chung for several useful comments.

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Received November 1, 1980