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Approximation for the Measures of Ergodic Transformations of the Real Line

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Summary. Let $P_\varphi: L^1(R) \to L^1(R)$ be the Frobenius-Perron operator corresponding to a nonsingular point-transformation φ of the real line R into itself and let for each natural number n, P_n be the discrete analogue of P_φ. It is shown that under fairly weak restrictions on φ, the equation $f = P_\varphi f$ has an unique solution f_0 such that $f_0 > 0$ (a.e.), $\|f_0\| = 1$, and that this solution can be approximated in $L^1(R)$ in two different ways: (1) by the sequence $\left\{1/n \sum_{k=0}^{n-1} P_\varphi^k f\right\}$, where $f \geqq 0$, $\|f\| = 1$, and (2) by the sequence $\{s_{0n}\}$ of simple functions such that $s_{0n} = P_n(s_{0n})$.

1. Introduction

The purpose of this paper is to establish results analogous to those of Theorem 1 in [6] and Theorem 1.1 in [7], for a class of transformations of the real line R into itself. Our method of approach to this problem is based on some ideas due to [6, 12, 7], and also [9]. The main results of this paper are contained in Theorem 3.1 and Theorem 4.1.

The subject of our study are transformations of $\varphi: R \to R$, whose Frobenius-Perron operator P_φ shrinks the variation of functions. Because of this property of P_φ we obtain for any $f \in L^1$ with $V_R f < \infty$ that the sequence $\{V_R P_\varphi^k f\}$ is bounded. (Here and in what follows the symbol $V_J f$ denotes the variation of the function f over a closed interval $J \subset R$.)

Unfortunately, in the case of transformations φ of an unbounded interval J, the boundedness of $\{V_J P_\varphi^k f\}$ does not imply the compactness (in L^1) of the sequence $\{P_\varphi^k f\}$. (For example, if $\varphi(x) = (nx+1)/(x-n)$ for $x \in (n, n+1)$, $n = 1, 2, \ldots$, then $\limsup_{k\to\infty} V_{[1,\infty)} P_\varphi^k f < \infty$ for any $f \in L^1$, $V_{[1,\infty)} f < \infty$. At the same time, $\{P_\varphi^k f\}$ is without any non-trivial limit point.) However under certain additional conditions on the transformations under consideration, this is the case.

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Condition (3.1) guarantees that the desired implication holds. It turns out that condition (3.1) not only guarantees the compactness of $\{P_\varphi^k f\}$ but also the compactness of $\{P_n^k(s_n)\}_{k=1}^\infty$ where P_n is the discrete analogue of P_φ and $s_n \in L_n^1$ (see the proofs of Theorems 3.1 and 4.1 (i)). As a result, (3.1) guarantees that both the operators P_φ and P_n possess fixed points. (We note that in the case of transformations of a bounded interval, the existence of a fixed point of P_n follows from Brouwer's Fixed Point Theorem.)

In Sect. 2 we describe a class of transformations which is the subject of our study (see Definition 2.1). Then we show that the Frobenius-Perron operator associated with any transformation from this class has the property of shrinking the variation of functions (see Proposition 2.1). Sect. 3 is devoted to the proof of Theorem 3.1. This theorem states that transformations which satisfy in addition the condition (3.1) are ergodic. In Sect. 4 we introduce the operator P_n (for every natural $n \geqq 1$) which is the discrete analogue of P_φ. Then we show that P_n has a fixed point s_{0n}, and that the sequence $\{s_{0n}\}$ converges to f_0, the unique fixed point of P_φ (see Theorem 4.1).

2. Preliminaries

We start with the definition of the class of point-transformations with which we shall be concerned.

Definition 2.1. Let $\{\tilde{I}_r\}_{r\in Z}$ be a partition of the real line R such that:

$$\tilde{I}_r = [a_r, a_{r+1}), \quad C_1 = \inf_{r\in Z} |\tilde{I}_r| > 0, \tag{2.1}$$

where $|\tilde{I}_r| = a_{r+1} - a_r$ (Z denotes the integers).

Denote $R_0 = \bigcup_{r\in Z} I_r$, where $I_r = (a_r, a_{r+1})$, and let φ be a transformation from R_0 onto R. We say φ belongs to the class $\tilde{\Phi}_1$ if it satisfies the following conditions:

(2.2) For each $r\in Z$, the restriction φ_r of φ to the interval I_r is a bijective map of I_r onto R, and for each $x \in I_r$, there exists an $a>0$ such that:

(i) φ_r is continuous on $[x, x+a)$, or on $(x-a, x]$;

(ii) the derivative $D\varphi_r$ of φ_r exists, and is finite, on both the intervals $(x, x+a)$ and $(x-a, x)$;

(iii) there exist (finite or infinite) the limits

$$\lim_{y\downarrow x} D\varphi_r(y) = D\varphi_r(x+0) \quad \text{and} \quad \lim_{y\uparrow x} D\varphi_r(y) = D\varphi_r(x-0);$$

(iv) $\inf\{|D\varphi_r(x+0)|, |D\varphi_r(x-0)| : x\in I_r, r\in Z\} = C_2 > 0$.

For any $r\in Z$, let $\sigma_r : R \to [0, C_2^{-1}]$ be a mapping associated with φ_r in the following way: for each $x \in R$ we put $\sigma_r(x) = 1/|D_-\varphi_r(\varphi_r^{-1}(x))|$ if $D_-\varphi_r(\varphi_r^{-1}(x))$ exists, or $\sigma_r(x) = 1/|D_+\varphi_r(\varphi_r^{-1}(x))$ otherwise (from (2.2i)–(2.2iii) it follows that for any $x\in I_r$, there exists at least one of the right- and the left-hand derivatives $D_+\varphi_r(x)$ and $D_-\varphi_r(x)$, respectively).

(2.3) For each $r \in Z$, the mapping σ_r has the following properties:

(i) if $A_r = \{x: D\varphi_r(x+0) \neq D\varphi_r(x-0)\} = \emptyset$, then σ_r is locally Lipschitzean, and

$$C_3 = \sup_{r \in Z_1} \sup_{x \in R} (|D\sigma_r(x)|/\sigma_r(x)) < \infty,$$

where $Z_1 = \{r \in Z: A_r = \emptyset\}$;

(ii) if $A_r \neq \emptyset$, then σ_r is of bounded variation, and

$$C_4 = \sup_{r \in Z_2} V_R \sigma_r < \infty,$$

where $Z_2 = \{r \in Z: A_r \neq \emptyset\}$;

(iii) $C_5 = 1/C_2 + C_4 < 1$.

Now we recall the definition and basic properties of the Frobenius-Perron operator (F.-P. operator).

Definition 2.2 (see [10]). Let (R, Σ, m) be a measure space where Σ is the σ-algebra of all Borel sets of R, and m is the Lebesgue measure. Let $(L^1, \|\cdot\|)$ be the space of all integrable functions defined on R. The F.-P. operator P_τ (associated with a given nonsingular transformation $\tau: R \to R$) is defined by the following formula:

(2.4) $$\int_B P_\tau f\, dm = \int_{\tau^{-1}(B)} f\, dm, \qquad \text{for any } f \in L^1 \text{ and } B \in \Sigma.$$

It is known that P_τ is a linear positive isometry of L^1 into itself. Hence in particular $P_\tau(G) \subset G$, where G is the set of all densities (all $f \in L^1$ such that $f \geqq 0$ and $\|f\| = 1$).

Another important property of P_τ is that if $P_\tau f = f$ for some $f \in G$, then the measure $d\nu = f\,dm$ is τ-invariant (i.e., $\nu(\tau^{-1}(B)) = \nu(B)$ for each $B \in \Sigma$) and conversely.

In the following we shall need an explicit formula for the F.-P. operator P_φ, associated with $\varphi \in \tilde{\Phi}_1$. It can be checked that P_φ may be written explicitly in the following form:

(2.5) $$P_\varphi f(x) = \sum_{r \in Z} g_r(x) \qquad \text{(a.e.); where for any } r \in Z,$$

(2.6) $$g_r(x) = f(\varphi_r^{-1}(x))\, \sigma_r(x) \qquad \text{for any } x \in R.$$

Henceforth we shall write P instead of P_φ for $\varphi \in \tilde{\Phi}_1$.

Now we want to show that roughly speaking P has the property of shrinking the variation of a function. The following result is the precise statement of this property.

Proposition 2.1. *If $f \in L^1$ and $V_R f < \infty$, then*

(i) $V_R P^k f \leqq C_5^k V_R f + (C_5^{k-1} + \ldots + C_5 + 1)\, C_6 \|f\|$

for $k = 1, 2, \ldots$, where $C_6 = \max\{C_3, C_4/C_1\}$ consequently;

(ii) $\limsup_{k \to \infty} V_R P^k f \leqq C_7 \|f\|$, *where $C_7 = C_6/(1 - C_5)$.*

Proof. Let us fix an arbitrary number $N>0$, and let $-N=x_1<x_2<\ldots<x_n<x_{n+1}=N$ be any finite number of the points of the interval $\tilde{J}_N=[-N,N]$. A partition of $\tilde{J}_N$ into the intervals $J_1=[x_1,x_2),\ldots,J_{n-1}=[x_{n-1},x_n)$, $J_n=[x_n,x_{n+1}]$ will be denoted by π.

For any $r\in Z$ let us form the following three sums:

$$v(g_r,\pi)=\sum_{j=1}^{n}|g_r(x_{j+1})-g_r(x_j)|, \tag{2.7}$$

$$B_{1r}(\pi)=\sum_{j=1}^{n}\sigma_r(x_{j+1})|f(\varphi_r^{-1}(x_{j+1}))-f(\varphi_r^{-1}(x_j))|, \tag{2.8}$$

$$B_{2r}(\pi)=\sum_{j=1}^{n}|f(\varphi_r^{-1}(x_j))|\ |\sigma_r(x_{j+1})-\sigma_r(x_j)|. \tag{2.9}$$

From the formulas (2.5) and (2.6) it follows that these sums satisfy the following inequality

$$v(g_r,\pi)\leqq B_{1r}(\pi)+B_{2r}(\pi). \tag{2.10}$$

We now break up the rest of this proof into four steps.

Step 1. For any $r\in Z$, $B_{1r}\leqq C_2^{-1}V_{I_r}f$.

Proof. From the definition of σ_r and the condition (2.2iv) it follows that for any $r\in Z$, $\sigma_r\leqq 1/C_2$. This inequality, together with equality (2.8), implies the required inequality.

Step 2. For any $r\in Z_1$, $V_R g_r\leqq C_2^{-1}V_{I_r}f+C_3\int_{I_r}|f|\,dm$; and for any $r\in Z_2$, $V_R g_r\leqq C_2^{-1}V_{I_r}f+C_4|f(y_r)|$, where

$$y_r\in I_r, \quad \text{and} \quad |f(x)|\leqq|f(y_r)| \quad \text{for each } x\in I_r. \tag{2.11}$$

Proof. Let us begin with the case $r\in Z_1=\{r\in Z\colon A_r=\emptyset\}$. By condition (2.3i) we have for $k=1,2,\ldots,n$,

$$|\sigma_r(x_{k+1})-\sigma_r(x_k)|\leqq C_3(\sup_{x\in J_k}\sigma_r(x))(x_{k+1}-x_k).$$

These inequalities together with the equality (2.9), yield the following inequality:

$$B_{2r}(\pi)\leqq C_3 B_{3r}(\pi), \tag{2.12}$$

where

$$B_{3r}(\pi)=\sum_{k=1}^{n}(\sup_{x\in J_k}|f(\varphi_r^{-1}(x))|\sup_{x\in J_k}\sigma_r(x))(x_{k+1}-x_k). \tag{2.13}$$

Let us now choose for $n=1,2,\ldots$, any two partitions

$$\pi_n^1=\{J_{nk}^1\colon k=1,\ldots,i_n\} \quad \text{and} \quad \pi_n^2=\{J_{nl}^2\colon l=1,\ldots,j_n\},$$

of the interval $\tilde{J}_N$ such that,

$$\begin{aligned} &v(g_r, \pi_n^1) \to V_{\tilde{J}_N} g_r && \text{as } n \to \infty; \\ (2.14) \qquad &\max_{1 \leqq 1 \leqq j_n} |J_{nl}^2| \to 0 && \text{as } n \to \infty, \text{ and consequently} \\ &B_{3r}(\pi_n^2) \to \int_{\tilde{J}_N} |g_r|\, dm && \text{as } n \to \infty; \end{aligned}$$

where $v(g_r, \pi_n^1)$ and $B_{3r}(\pi_n^2)$ have been defined by formulas (2.7) and (2.13), respectively.

Next we take into account the partition π_n^3 consisting of the intervals $J_{n,kl}^3 = J_{nk}^1 \cap J_{nl}^2$ (omitting those $J_{n,kl}^3$ which are empty). Since π_n^3 is finer than π_n^1 and π_n^2, one has

$$v(g_r, \pi_n^1) \leqq C_2^{-1} V_{I_r} f + C_3 B_{3r}(\pi_n^3),$$

by (2.10), Step 1, and (2.12).

Finally, we obtain the required inequality by passing to the limits (in the last inequality) and taking into account (2.14), and the following fact:

$$B_{3r}(\pi_n^3) \to \int_{\tilde{J}_N} |g_r|\, dm \leqq \int_{I_r} |f|\, dm, \quad \text{as } n \to \infty.$$

Let us now consider the case $r \in Z_2 = \{r \in Z : A_r \neq \emptyset\}$. From condition (2.3ii) and (2.9) it follows that,

$$B_{2r} \leqq C_4 |f(y_r)|.$$

This inequality and the inequality (2.10) imply the required inequality. Step 2 has been proved.

Step 3. For any $Z_0 \subset Z$,

$$(2.15) \qquad \sum_{r \in Z_0} |f(y_r)| \leqq V_R f + C_1^{-1} \sum_{r \in Z_0} \int_{I_r} |f|\, dm,$$

where $y_r (r \in Z_0)$ satisfies (2.11).

Proof. Let us choose for each $r \in Z_0$, a point $z_r \in I_r$ such that,

$$|f(z_r)| \leqq |I_r|^{-1} \int_{I_r} |f|\, dm.$$

By this inequality and condition (2.1) we have

$$\begin{aligned} \sum_{r \in Z_0} |f(y_r)| &\leqq \sum_{r \in Z_0} \left(|f(y_r) - f(z_r)| + C_1^{-1} \int_{I_r} |f|\, dm\right) \\ &\leqq V_R f + C_1^{-1} \sum_{r \in Z_0} \int_{I_r} |f|\, dm, \end{aligned}$$

which is what we set out to prove.

Step 4. $V_R Pf \leqq C_5 V_R f + C_6 \|f\|$.

Proof. The required inequality follows immediately from the equality (2.5), and Steps 2 and 3. Step 4 has been proved.

We are now ready to finish the proof of Proposition 2.1. To this end let us observe that for $k=1,2,\ldots$, one has

$$V_R P^k f \leqq C_5 V_R P^{k-1} f + C_6 \|f\|,$$

by the preceding step and the equality $\|P^k f\| = \|f\|$ $(k=1,2,\ldots)$. This finishes the proof of Proposition 2.1.

A corollary of Proposition 2.1 is the following:

Corollary 2.1. *For any $f \in L^1$ with $V_R f < \infty$, there exists a subsequence $\{P^{k_j} f\}$ of $\{P^k f\}$ and $g \in L^1$ which is the pointwise limit (a.e.) of $\{P^{k_j} f\}$.*

Proof. By Proposition 2.1 (i) and Helly's Selection Theorem, $\{P^k f\}$ contains a subsequence $\{P^{k_j} f\}$ such that, $\lim_{j \to \infty} P^{k_j} f = g$ (g is defined as the pointwise limit (a.e.) of $\{P^{k_j} f\}$).

Since, $\|g\| \leqq \liminf_{j \to \infty} \|P^{k_j} f\| = \|f\|$, by Fatou's Lemma, then $g \in L^1$.

3. The Ergodicity of Some Transformations From $\tilde{\Phi}_1$

As was already mentioned in Introduction, the boundedness of $\{V_R P^k f\}$ in general does not imply the compactness of $\{P^k f\}$. Hence the function g from Corollary 2.1 need not be the L^1-limit of $\{P^{k_j} f\}$.

In order to ensure the compactness of $\{P^k f\}$, we impose on $\varphi \in \tilde{\Phi}_1$ the following additional condition:

Definition 3.1. We say that a transformation $\varphi \in \tilde{\Phi}_1$ belongs to the class Φ_1 if it satisfies the following condition:

(3.1) $$\lim_{N \to \infty} C_8(N) = 0,$$

where $C_8(N) = \sup_{r \in Z} |I_r| \int_{J_N} \sigma_r \, dm, \qquad J_N = \{x \in R : |x| > N\}$.

We wish to show that under this condition φ is an ergodic transformation. In order to prove this however we need an auxiliary result.

Proposition 3.1. *If $\varphi \in \Phi_1$ then for any $f \in L^1$ with $V_R f < \infty$, and for any $N > 0$,*

$$\int_{J_N} |P^k f| \, dm \leqq C_8(N) \, C_9 \|f\| \quad \textit{for sufficiently large } k,$$

where C_9 is a constant (depending on f).

Proof. From formulas (2.5), (2.6) and condition (3.1) it follows that

$$\int_{J_N} |Pf|\,dm \leqq C_8(N) \sum_{r\in Z} |I_r|^{-1} |f(y_r)|,$$

where C_8 and $f(y_r)$ have been defined by (3.1) and (2.11), respectively. Next combining this inequality with (2.15), we get

$$\int_{J_N} |Pf|\,dm \leqq C_8(N)(C_1^{-1} V_R f + C_1^{-2} \|f\|). \tag{3.2}$$

This inequality together with the inequality (ii) of Proposition 2.1 implies that for a large enough k,

$$\int_{J_N} |P^k f|\,dm \leqq C_8(N)\, C_9 \|f\|,$$

where $C_9 = (C_1^{-1} C_{07} + C_1^{-2})$, and C_{07} is any number greater than C_7. This proves the proposition.

Now we show that Φ_1 consists of the ergodic transformations. This result generalizes some previous results of J.H.B. Kemperman [4, 5], F. Schweiger [11], M. Jabłoński, A. Lasota [3] and P. Bugiel [1].

Theorem 3.1. *If $\varphi \in \Phi_1$, then there is precisely one solution $f_0 \in G$ of the equation $Pf = f$ such that:*

(1) *$f_0 > 0$ (a.e.), $V_R f_0 \leqq C_7$ for some constant $C_7 > 0$, and (in consequence) φ is an ergodic transformation with respect to the (φ-invariant) measure $d\mu = f_0\,dm$;*

(2) *for each $f \in G$, $f_0 = \lim_{n\to\infty} 1/n \sum_{k=0}^{n-1} P^k f$.*

Proof. Existence: From Corollary 2.1 and Proposition 3.1 it follows routine argumentation that for any $f \in L^1$ with $V_R f < \infty$ the sequence $\{P^k f\}$ is relatively compact. This fact implies, by Yosida-Kakutani Ergodic Theorem, the existence of some fixed point of P.

Uniqueness: We now show that there exists in G, exactly one fixed point of P. We do this in stages. First note that for any $f \in L^1$ the sequence $\{S_n f\}$ where $S_n = 1/n \sum_{k=0}^{n-1} P^k$ converges in L^1. Indeed, since for any $f \in L^1$, $V_R f < \infty$ the sequence $\{P^k f\}$ is relatively compact, and since the set of the integrable functions of bounded variation is dense in L^1, the convergence of $\{S_n f\}$ follows from the Yosida-Kakutani Ergodic Theorem.

Secondly we show that for each $f \in G$, the limit function of $\{S_n f\}$ is of bounded variation. Note that for each $f \in G$ with $V_R f < \infty$ we have

$$V_R S f \leqq C_7, \qquad \text{where } S = \lim_{n\to\infty} 1/n \sum_{k=0}^{n-1} P^k, \tag{3.3}$$

by Proposition 2.1 (i) and Helly's Theorem.

Now we can extend inequality (3.3) to the whole of G in the following way. For any $f \in G$ let $\{f_n\}$ be a sequence such that $f_n \in G$, $V_R f_n < \infty$ and $\lim_{n\to\infty} \|f_n - f\| = 0$. By inequality (3.3), Helly's Selection Theorem and Fatou's Lemma, the sequence $\{Sf_n\}$ contains a subsequence $\{Sf_{n_j}\}$ such that,

$$\lim_{j\to\infty} Sf_{n_j} = g \text{ (a.e.)}, \qquad \text{where } g \in L^1 \text{ and } V_R g \leqq C_7. \tag{3.4}$$

By Proposition 3.1 we get

$$\int_{J_N} Sf_{n_j}\, dm \leqq C_8(N)\, C_9, \qquad \text{for } j = 1, 2, \ldots.$$

From this, (3.4), and the equality $\lim_{n\to\infty} \|Sf_n - Sf\| = 0$ it follows that $g = Sf$. Thus, each density satisfies the inequality (3.3).

Thirdly, we show that if $g \in G$ and $Pg = g$, the set $H = \{x : g(x) > 0\}$ is equal (a.e.) to the whole real line R.

As we have seen g is of bounded variation, thus in particular there exists an interval $J \subset H$. So that from the definition of Φ_1 it follows that for a large enough k, $\varphi^k(J) = R$. These two fact, together with the equality $\varphi(H) = H$ (see [8], Lemma 2.3), imply that $H = R$ which was to be proved.

We are now ready to finish the proof of the uniqueness of the solutions for the equation $Pf = f$. Take arbitrary $g_i \in G$ such that $Pg_i = g_i$, for $i = 1, 2$. If $g_1 \leqq g_2$ on a set $B \in \Sigma$ with $m(B) > 0$ then, by Lemmas 2.2 and 2.3 in [8], the density $g = (g_2 - g_1)\, 1_B / \|(g_2 - g_1)\, 1_B\|$ is a fixed point of P (1_B is the characteristic function of B). Thus $B = R$ (a.e.) i.e., $g_1 \leqq g_2$ (a.e.). This last inequality, together with the equality $\|g_1\| = \|g_2\|$ implies $g_1 = g_2$ (a.e.).

The uniqueness part has been proved, and the proof of the theorem has been finished.

4. Approximation for the Invariant Measures by the Fixed Points of $P_n (n = 1, 2, \ldots)$

In this section we wish to show that f_0, the unique fixed point of P, can be approximated (in L^1) by a sequence $\{s_{0n}\}$ of fixed points of $P_n (n = 1, 2, \ldots)$, the discrete analogous of P.

Definition 4.1. Let $\pi_n = \{I_{nr}\}_{r \in Z}$, $n = 1, 2, \ldots$, be a partition of R on nonoverlapping equal intervals such that:

$$a_n = m(I_{nr}) \to 0 \quad \text{as } n \to \infty, \qquad a_1 \leqq C_1; \tag{4.1}$$

$$\pi_n \quad \text{is finer than } \pi_{n-1}, \qquad \text{for } n = 1, 2, \ldots \tag{4.2}$$

(here, for each n, I_{n0} contains zero).

Let Σ_n be the smallest σ-algebra containing all atoms of π_n, and let L_n^1 be the space of all integrable functions over (R, Σ_n, m) which are multiplied by a_n^{-1}, thus every $s_n \in L_n^1$ can be expressed in the following form:

$$s_n = a_n^{-1} \sum_{r \in Z} a_{nr} 1_{I_{nr}}, \qquad \text{where } \sum_{r \in Z} |a_{nr}| < \infty.$$

Now, for any $s_n \in L_n^1$, let us put

$$P_n(s_n) = a_n^{-1} \sum_{r \in Z} b_{nr} 1_{I_{nr}}, \tag{4.3}$$

where

$$b_{nr} = \sum_{j \in Z} a_{nj} P_{n,jr},$$

and

$$P_{n,jr} = a_n^{-1} \int 1_{I_{nr}} P 1_{I_{nj}} dm. \tag{4.4}$$

These above last three formulas define an operator of L_n^1 into itself. It will be called the discrete analogue of P.

Since P_n is a positive linear isometry the set G_n of all densities of L_n^1 is invariant under P_n.

Note also that the k-th iterate of P_n can be represented by the following formula:

$$P_n^k(s_n) = a_n^{-1} \sum_{r \in Z} c_{nr}^{(k)} 1_{I_{nr}}, \tag{4.5}$$

where

$$c_{nr}^{(k)} = \sum_{j \in Z} a_{nj} (\tilde{P}_n^k)_{jr}, \qquad \text{and } \tilde{P}_n^k \text{ is the } k\text{-th}$$

iterate of the matrix $\tilde{P}_n = [P_{n,jr}]_{j,r \in Z}$.

We now state the basic properties of P_n (assuming in what follows that P is associated with $\varphi \in \Phi_1$).

Lemma 4.1. *For any $N > 0$, the following inequality holds true:*

$$\sup_{r \in Z} \Big(\sum_{|j| \geqq N} P_{n,rj} \Big) \leqq 2(C_1 a_n)^{-1} C_8(N) \qquad (n = 1, 2, \ldots).$$

Proof. Since $a_n \leqq C_1$ (see (4.1)), then for each $r \in Z$ there exists exactly one $k_r \in Z$ such that, $I_{nr} \subset I_{k_r} \cup I_{k_r+1}$. From this and formulas (2.5) and (2.6) it follows that,

$$P 1_{I_{nr}}(x) \leqq 1_{I_{nr}}(\varphi_{k_r}^{-1}(x))\, \sigma_{k_r}(x) + 1_{I_{nr}}(\varphi_{k_r+1}^{-1}(x))\, \sigma_{k_r+1}(x).$$

By this inequality, formula (4.4) and conditions (2.1), (3.1) we get

$$\sum_{|j| \geqq N} P_{n,rj} \leqq a_n^{-1} \int_{J_N} (\sigma_{k_r} + \sigma_{k_r+1})\, dm \leqq 2(C_1 a_n)^{-1} C_8(N),$$

for each $r \in Z$, and $n = 1, 2, \ldots$. The lemma has been proved.

In what follows the conditional expactation operator $E(\cdot | \Sigma_n)$ of L^1 into L_n^1 plays a very useful role. It is well known that it is a positive linear isometry.

The following properties of $E(\cdot | \Sigma_n)$ will be also used:

Lemma 4.2.

(i) *For each* $f \in L^1$, $\|E(f|\Sigma_n) - f\| \to 0$ *as* $n \to \infty$;
(ii) $V_R E(f|\Sigma_n) \leqq V_R f$;
(iii) *for each* $s_n \in L^1_n$, $E(Ps_n|\Sigma_n) = P_n(s_n)$.

Proof. (i) Property (i) follows from the martingale convergence theorem of Doob ([2], p. 319) and the following inequality:

$$\int_{J_N} |E(f|\Sigma_n)|\, dm \leqq \int_{J_N} |f|\, dm, \qquad \text{for any } N>0,\ n \geqq 1. \tag{4.6}$$

(ii) The proof of this property for $f \in L^1(R, m)$, does not differ from the proof in [7] of an analogous property for $f \in L^1([0,1], m)$ (see Lemma 2.6), and therefore it will be omitted.

(iii) From the definition of $E(\cdot|\Sigma_n)$ and the formulas (4.3), (4.4) it follows that for any $s_n = a_n^{-1} \sum_{r \in Z} a_{nr} 1_{I_{nr}}$, the following equalities are valid:

$$\begin{aligned} E(Ps_n|\Sigma_n) &= \sum_{i \in Z} \Big(a_n^{-1} \int_{I_{ni}} Ps_n\, dm\Big) 1_{I_{ni}} \\ &= a_n^{-1} \sum_{i \in Z} \Big(\sum_{r \in Z} a_{nr} P_{n,ri}\Big) 1_{I_{ni}} = P_n(s_n). \end{aligned}$$

This proves the lemma.

The discrete analogue of Propositions 2.1 and 3.1 is the following:

Proposition 4.1. *If* $s_n \in L^1_n$, *then*

(i) $V_R P_n^k(s_n) \leqq C_5^k V_R s_n + (C_5^{k-1} + \ldots + C_5 + 1) C_6 \|s_n\|$ *for* $k = 1, 2, \ldots$, *consequently*
(ii) $\limsup_{k \to \infty} V_R P_n^k(s_n) \leqq C_7 \|s_n\|$,
(iii) $\int_{J_N} |P_n^k(s_n)|\, dm \leqq 2(C_1 a_n)^{-1} C_8(N) \|s_n\|$ for $k = 1, 2, \ldots$.

Proof. (i) The required inequality follows from Lemma 4.2(ii)–(iii), Proposition 2.1(i), and the equality $\|P_n^k(s_n)\| = \|s_n\|$.

(iii) From (4.5) it follows that for $k = 1, 2, \ldots, N > 0$,

$$\int_{J_N} |P_n^k(s_n)|\, dm \leqq \sum_{i \in Z} |a_{ni}| \sum_{r \in Z} (\tilde{P}_n^{k-1})_{ir} \sum_{|j| \geqq N} P_{n,rj}.$$

Now the required inequality follows from this inequality and Lemma 4.1. The proposition is proved.

We are now able to prove the following approximation theorem:

Theorem 4.1.

(i) *There exists, for each natural number* $n \geqq 1$, *a density* $s_{0n} \in G_n$ *such that,* $s_{0n} = P_n(s_{0n})$;

(ii) *the sequence* $\{s_{0n}\}$ *of the fixed points of* P_n *converges to* f_0, *the unique fixed point of* P.

Proof. (i) Put $s_{nk}=P_n^k(s_n)$, for $s_n \in G_n$. By Proposition 4.1(i), Helly's Selection Theorem and Fatou's Lemma, there exist a subsequence $\{s_{nk_j}\}_{j=1}^{\infty}$ of $\{s_{nk}\}_{k=1}^{\infty}$ and some $\tilde{s}_{0n} \in G$ which is the pointwise limit (a.e.) of $\{s_{nk_j}\}$.

From Proposition 4.1(iii) it can be easily seen that $\tilde{s}_{0n}$ is the L^1-limit of $\{s_{nk_j}\}$. This and the fact, that G_n is convex, and closed in L^1 it follows (by Yosida-Kakutani Ergodic Theorem) that there is a fixed point s_{0n} of P_n, and that $s_{0n} \in G_n$.

(ii) By Proposition 4.1(ii), we have

$$V_R s_{0n} \leqq C_7 \quad \text{for } n=1,2,\ldots. \tag{4.7}$$

Next by Lemma 4.2(iii) and inequalities (4.6), (3.2), (4.7) we have

$$\int_{J_N} s_{0n}\, dm \leqq C_8(N)\, C_{10} \qquad \text{for } n=1,2,\ldots,$$

where $C_{10}=(C_1^{-1} C_7 + C_1^{-2})$. This inequality together with (4.7) implies the existence of some limit point s_0 for $\{s_{0n}\}$.

Finally from the inequality $\|s_0 - P s_0\| \leqq 2\|s_0 - s_{0n}\| + \|E(P s_0|\Sigma_n) - P s_0\|$, and Lemma 4.2(i) it follows that, $P s_0 = s_0$. Hence $s_0 = f_0$. This completes the proof of (ii) and Theorem 4.1.

Final Remark. Let $\tilde{\tilde{\Phi}}_1$ be a class consisting of the transformations $\varphi \in \tilde{\Phi}_1$ such that for each $r \in Z$, the mapping φ_r is differentiable at every point of I_r (i.e., $A_r = \emptyset$). It can be shown (by using different methods than that used in this paper) that for such transformations the main results of this paper remain valid under the following (more general) condition:

(4.8) there exists $Z_0 \subset Z$ such that for every $y \in R$.

$$\inf_{j \in Z}\Big(\sum_{r \in Z_0} w_r\, \sigma_r(y)\, w_j \int_{I_r} \sigma_j\, dm\Big) > 0, \qquad \text{where } w_r = |I_r|^{-1}.$$

This and some other things (for instance, exactness) will be elaborated in a subsequent paper.

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