DISTORTIONS OF THE MOON'S FIGURE DUE TO THE EARTH

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Abstract. The second zonal and the second sectorial Stokes parameters of the Moon's gravitational field and/or the polar and equatorial flattenings of the lunar triaxial level ellipsoid have been explained by the tidal and rotational distortions due to the Earth. The Epoch at which the Moon's figure formation was finished has been estimated as 1.6×10^9 y B. P. when the Earth-Moon distance was about 168 400 km and the orbital/rotational period of the Moon about 8 days.

1. Introduction

There are about forty synchronously orbiting satellites in the Solar System from the total number sixty one, and twenty of them at least are triaxial (Davies *et al.* 1991). The triaxiality of bodies orbiting in 1/1 rotational/orbital resonance is rather a rule than an exception in the Solar System. The topic will be treated in detail as regards the Moon. The question to be answered is as follows: In which extent is the Earth responsible for the triaxial figure of the Moon? By the way, the parameters of the triaxial level ellipsoid of the Moon will be derived.

2. Defining the Moon's Basic Equipotential Surface (Selenoid)

The basic equation for the solution is the selenopotential W at arbitrary point P (ρ, ϕ, Λ) at the lunar surface; ρ is selenocentric radius-vector, ϕ selenocentric latitude, Λ selenocentric longitude reckoned to the East. The plane of the prime meridian $\Lambda = 0$ passes through the Earth's mass center $\oplus$, the librations will not be taken into account.

$$\begin{aligned} W = \frac{GM_{☾}}{\rho} \Bigg\{ & 1 + \sum_{n=2}^{\overline{n}} \left(\frac{a_0}{\rho}\right)^n \sum_{k=0}^{n} (J_n^{(k)} \cos k\Lambda + S_n^{(k)} \sin k\Lambda) P_n^{(k)}(\sin\phi) + \\ & + \frac{1}{3}\left(\frac{a_0}{\rho}\right)^{-3} q[1 - P_2^{(0)}(\sin\phi)] + \\ & + \frac{GM_{\oplus}}{GM_{☾}} \sum_{n=2}^{N} \left(\frac{\rho}{\Delta_{\oplus ☾}}\right)^{n+1} \Bigg[P_n^{(0)}(\sin\delta_{\oplus}) P_n^{(0)}(\sin\phi) + \sum_{k=1}^{n} \\ & + 2\sum_{k=1}^{n} \frac{(n-k)!}{(n+k)!} P_n^{(k)}(\sin\delta_{\oplus}) P_n^{(k)}(\sin\phi) \cos k(\Lambda - \Lambda_{\oplus}) \Bigg] \Bigg\}, \end{aligned} \tag{1}$$

$$q = \frac{\omega^2 a_0^3}{GM_{☾}} = \frac{G(M_{\oplus} + M_{☾})}{GM_{☾}} \left(\frac{a_0}{\Delta_{\oplus ☾}}\right)^3 = 7.5731 \times 10^{-6}, \tag{2}$$

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$$GM_\oplus = (398\,600.4405 \pm 0.001) \times 10^9\ \mathrm{m}^3\ \mathrm{s}^{-2}, \tag{3}$$

$$GM_{☾} = (4902.799 \pm 0.003) \times 10^9\ \mathrm{m}^3\ \mathrm{s}^{-2}, \tag{4}$$

$J_n^{(k)}$, $S_n^{(k)}$ are the Stokes parameters of the Moon's gravitational field in the conventional definition scaled for $a_0 = 1737$ km, not normalized, of degree n and order k; $P_n^{(k)}$ is the Legendre associated function, also conventional, not formalized; $\delta_\oplus$ is the selenocentric declination of the Earth's mass center, $\Delta_{\oplus ☾}$ the distance of the Moon's and Earth's mass centers;

$$\omega = n = 2.6616995 \times 10^{-6}\ \mathrm{rad\ s}^{-1}$$

is the angular velocity of the Moon's rotation equal to its mean motion n. In Equation (1) the Earth's gravitational field model giving rise to the tides on the Moon is spherically symmetrical. Note that, the indirect tidal influence of the Earth, as regards the zero-frequency tides on the Moon, is included in the Stokes parameters $J_n^{(k)}$, $S_n^{(k)}$ as determined from artificial satellite orbit dynamics.

Imposing W = constant, the lunar equipotential surface is defined. However, the problem to be solved requires surface

$$W = W_0 \tag{5}$$

be defined as best fitting (representing) the boundary surface defining the lunar body. We impose the condition as folows: the integral mean value of the radius of surface Equation (1) be equal to that of lunar topography $\overline{\rho}_t$.

In the linear approximation, i.e., neglecting terms q^2, $J_2^{(0)}q$, $(J_2^{(0)})^2$, it then holds that

$$W_0 = \frac{GM_{☾}}{\overline{\rho}_t}\left(1 + \frac{1}{3}q\right). \tag{6}$$

On the basis of lunar topography data by Bills and Ferrari (1977)

$$\overline{\rho}_t = 1\,737\,530\ \mathrm{m}, \tag{7}$$

one gets

$$W_0 = 2\,821\,727\ \mathrm{m}^2\ \mathrm{s}^{-2}, \tag{8}$$

and the selenopotential scale factor

$$R_0 = \frac{GM_{☾}}{W_0} = 1\,737\,517\ \mathrm{m}. \tag{9}$$

3. Parameters of the Best Fitting Triaxial Level Ellipsoid of the Moon

The surface of the selenoid defined by Equation (1) and Equation (5) is not stationary because $\delta_\oplus$ and $\Delta_{\oplus☾}$ are functions of time. However, the inclination of the plane of the Moon's equator to the ecliptic is small, about $I = 1° \; 32'01''$ $\pm \; 7''$ (Kopal 1966). That is why, $\delta_\oplus = \pm I$ and we put $\delta_\oplus = 0$ in the model. Distance $\Delta_{\oplus☾}$ varies about (356 400 – 406 700) km and we adopt, in deriving the Moon's elipsoid,

$$\Delta_\oplus \quad = 384\,400\,\text{km}. \tag{10}$$

When $\delta_\oplus = 0$, the tidal terms vanish if n or $(n - k)$ is odd: ($n = 2$, $k = 1$), (3, 0), (3, 2), (4, 1), (4, 3) etc. We neglect (3, 1) and all terms $n > 3$.

The selenocentric triaxial level ellipsoid E is defined by four parameters to be determined: a (the longest semiaxis), α (polar flattening of meridian containing a), α_1 (equatorial flattening), Λ_a (longitude of meridian containing a). The radius vector of E is function of the above parameters:

$$\rho_E = \rho_E(a, \alpha, \alpha_1, \Lambda_a). \tag{11}$$

After inserting $\rho = \rho_E$ in Equation (1) we get selenopotential W_E on E and the basic condition can be imposed as

$$W_E = W_0. \tag{12}$$

Then all the harmonics in W_E should equal zero.

These conditions yield the unknowns in Equation (11) as functions of the parameters of the actual Moon's field. However, only four of the parameters can be adopted. The best solution is, the four be selected as follows:

$$R_0, J_2^{(0)}, q, J_{2,2} = [(J_2^{(2)})^2 + (S_2^{(2)})^2]^{1/2}. \tag{13}$$

However, in that case, if $n > 2$

$$(J_n^{(k)})_E \neq J_n^{(k)}, \qquad (S_n^{(k)})_E \neq S_n^{(k)}. \tag{14}$$

The solution is laborious, only the final formulas will be written here, and neglecting terms 10^{-9} in order of magnitude and smaller, for brevity:

$$\begin{aligned} a \; = \; R_0 \bigg[1 - \frac{1}{2}\nu^2 J_2^{(0)} + 3\nu^2 J_{2,2} + \frac{3}{2}\nu^{-3} q + \frac{7}{40}\nu^4 (J_2^{(0)})^2 - \frac{3}{2}\nu^4 J_2^{(0)} J_{2,2} + \\ + \frac{1033}{70}\nu^4 (J_{2,2})^2 - \frac{13}{14}\nu^{-1} J_2^{(0)} q + \frac{457}{35}\nu^{-1} J_{2,2} q + \frac{461}{140}\nu^{-6} q^2 \bigg], \end{aligned} \tag{15}$$

$$\alpha = -\frac{3}{2}\nu^2 J_2^{(0)} + 3\nu^2 J_{2,2} + 2\nu^{-3} q - \frac{3}{8}\nu^4 (J_2^{(0)})^2 + \frac{69}{28}\nu^{-1} J_2^{(0)} q + \\ +\frac{9}{2}\nu^4 J_2^{(0)} J_{2,2} + \frac{15}{14}\nu^4 (J_{2,2})^2 + \frac{3}{14}\nu^{-1} J_{2,2} q - \frac{1}{28}\nu^{-6} q^2, \tag{16}$$

$$\alpha_1 = 6\nu^2 J_{2,2} + \frac{3}{2}\nu^{-3} q + \frac{15}{7}\nu^{-1} J_{2,2} q - \frac{66}{7}\nu^4 (J_{2,2})^2 + \frac{9}{8}\nu^{-6} q^2, \tag{17}$$

$$\tan 2\Lambda_a = \frac{S_2^{(2)}}{J_2^{(2)}}, \tag{18}$$

$$\nu = \frac{a_0}{R_0}. \tag{19}$$

The Stokes parameters $n > 2$ of the selenopotential normal model defined by the level triaxial ellipsoid Equations (15)–(18) are functions of the four arguments in Equation (11), e.g.:

$$(J_4^{(0)})_E = \frac{9}{5}\nu^4 (J_2^{(0)})^2 + \frac{15}{28}\nu^{-6} q^2 - \frac{15}{7}\nu^{-1} J_2^{(0)} q, \tag{20}$$

$$(J_4^{(2)})_E = -\frac{5}{14}\nu^{-1} J_{2,2} q + \frac{3}{2}\nu^4 J_2^{(0)} J_{2,2} - \frac{11}{120}\nu^{-6} q^2, \tag{21}$$

etc.

Numerical solution based on parameters Equations (2), (9) and (IERS Standards 1989)

$$J_2^{(0)} = -202.151 \times 10^{-6}, \tag{22}$$

$$J_2^{(2)} = 22.302 \times 10^{-6}, \qquad S_2^{(2)} = 0, \tag{23}$$

is as follows:

$$a = 1\,737\,830\ \text{m}, \tag{24}$$

$$1/\alpha = 2\,596, \tag{25}$$

$$1/\alpha_1 = 6\,889, \tag{26}$$

$$\Lambda_a = 0. \tag{27}$$

The corresponding parameters of the Moon's level rotational ellipsoid ($J_{2,2} = 0$) come out as

$$\overline{a} = 1\,737\,710\ \text{m}, \tag{28}$$

$$1/\overline{\alpha} = 3\,142. \tag{29}$$

4. Tidal Distortions Due to the Earth

The indirect tidal effect due to the Earth is included in the second sectorial Stokes parameter Equation (23) of the Moon

$$\delta J_2^{(2)} = \frac{1}{12} k_{☾} \frac{GM_\oplus}{GM_{☾}} \left(\frac{a_0}{\Delta_{\oplus ☾}} \right)^3 P_2^{(2)}(\sin \delta_\oplus) = (7.50 \times 10^{-6}) k_{☾} \ ; \tag{30}$$

$k_{☾}$ is the secular tidal Love number of the Moon. It means, about 1/3 of the actual value Equation (23) can be explained in this way. Note that $k_{☾} = 3/2$ for homogeneous model; $k_s < 3/2$ for models with increasing densities in the deep interiors. It is quite different from the second Love number of the Moon $k_2 = 0.020$ (Kaula 1968), $k_2 = 0.0222$ (IERS Standards 1989). The actual value Equation (2) is too small. Hence stresses within the body of the Moon are not hydrostatic.

However, the triaxial figure was formed in the past, when the Moon was orbiting closer to the Earth. Let us compute orbital parameters $\overline{\Delta}_{\oplus ☾}$, $\overline{\mathbf{n}} = \overline{\omega}$ for this case:

$$\frac{GM_\oplus}{GM_{☾}} \left(\frac{a_0}{\overline{\Delta}_{\oplus ☾}} \right)^3 = \frac{\overline{n}^2 a_0^3}{GM_{☾}} = \frac{4 J_2^{(2)}}{k_{☾}} = (8.92 \times 10^{-5}) / k_{☾} \tag{31}$$

$$\overline{\Delta}_{\oplus ☾} = (168\,408 \text{ km}) \sqrt[3]{k_{☾}} \tag{32}$$

$$\overline{n} \sqrt{k_{☾}} = (9.135 \times 10^{-6}) \text{ rad s}^{-1}, \qquad (\overline{T} / \sqrt{k_{☾}} = 7.96 \text{ d}); \tag{33}$$

$\overline{T}$ is orbital/rotational period of the Moon.

Time Δt can be estimated as (Kopal 1978)

$$\frac{\mathrm{d}\Delta_{\oplus ☾}}{\mathrm{d}t} = 6 \frac{[G(M_\oplus + M_{☾})]^{1/2}}{\Delta_{\oplus ☾}^{1/2}} \left[\frac{GM_{☾}}{GM_\oplus} \left(\frac{a_\oplus}{\Delta_{\oplus ☾}} \right)^5 (k_2 \epsilon)_\oplus + \right. \\ \left. + \frac{GM_\oplus}{GM_{☾}} \left(\frac{a_0}{\Delta_{\oplus ☾}} \right)^5 (k_2 \epsilon)_{☾} \right]; \tag{34}$$

$a_\oplus = 6\ 378\ 136.5$ m is the mean equatorial radius of the Earth, k_2 is the Love number, ϵ the phase lag angle. Because the orbital/rotational resonance 1/1 is supposed to exist also in the past (time Δt B.P.), $(k_2 \epsilon)_{☾} = 0$ and after integrating Equation (34) one gets (at $k_{☾} = 1$)

$$\Delta t = \frac{1}{39} \frac{\Delta_{\oplus ☾}^{13/2} - \overline{\Delta}_{\oplus ☾}^{13/2}}{a_\oplus^5 (k_2 \epsilon)_\oplus} \frac{GM_\oplus}{GM_{☾}} \frac{1}{[G(M_\oplus + M_{☾})]^{1/2}} = 1.595 \times 10^9 \text{ y}, \tag{35}$$

if $(k\epsilon)_\oplus = 0.0123$ = present value = constant.

That is why, the origin of the triaxial figure of the Moon can be explained by the tidal distortions due to the Earth. The rough estimation can be refined after specifying secular tidal number $k_☾$ at Epoch Δt B.P.

Note that at Δt the direct radial tidal effect exerted by the Earth

$$\frac{GM_\oplus}{GM_☾}\frac{\rho^4}{\Delta^3_{\oplus ☾}} = 154.95\ \mathrm{m}, \tag{36}$$

is about 12 times larger than that at present

$$\frac{GM_\oplus}{GM_☾}\frac{\rho^4}{\Delta^3_{\oplus ☾}} = 13.03\ \mathrm{m}. \tag{37}$$

Also the direct tidal effect in α_1 Equation (17) was much larger at Epoch Δt than at present. Equatorial flattening at Δt was about

$$1/\alpha_1 = 3740;$$

50% of value α_1 is due to $J_2^{(2)}$ and 50% due to direct tidal effect exerted by the Earth.

As regards the polar flattening of the Moon, it is partly due to the Moon's proper rotation, partly due to the Earth's zonal tide. At Epoch Δt, about 2/3 of value α was due to the Moon's rotational deformations, 1/3 due to the Earth's zonal tidal deformation.

5. Conclusions

(1) The actual figure of the Moon can be explained by the tidal deformations due to the Earth and the rotational deformations due to the lunar proper rotation in the past.

(2) The Epoch at which the Moon's figure formation was finished, may be estimated as about 1.6×10^9 y B.P.

(3) If the secular Love and tidal number of the Moon is about unity, both the second zonal Equation (2) and the second sectorial Equation (23) Stokes parameters can be explained by the tidal and rotational deformations at Epoch 1.6×10^9 y B.P.

(4) The distance of the Earth's and Moon's centers at Epoch above may be estimated about 168 400 km, the rotational/orbital period of the Moon about 8 days. The distance at Epoch is inside the Earth's gravitational sphere of influence which is about 260 000 km.

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