ROTATIONAL DISTORTION ON THE SURFACES OF JUPITER AND SATURN

M. MARIE
Department of Astronomy, Cairo University, Egypt

and

M. B. EL-SHAARAWY
Department of Astronomy, El-Azhar University, Egypt

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Abstract. Having formulated the Clairaut second-order differential equations up to the fourth order in superficial distortion due to Hensen's coefficients in the previous article (El-Sharawy *et al.*, 1989 III, hereafter denotes by SM3), we are now in a position to solve them. In this paper we shall discuss the methods of solving the Clairaut theory, to give an explicit form about the distortion of the surfaces of Jupiter and Saturn, numerically up to the fourth-order.

1. Introduction

In the previous article SM3, we derived the differential equations governing the form and exterior potential of rotating stars. These equations are the Clairaut theory of the rotational distortion of self-gravitating configuration of arbitrary structure, arising from axial rotation with constant angular velocity, and extended to quantities of fourth-order in superficial distortion due to Hensen's coefficients.

A particular numerical solution of the previous Clairaut second-order theory (derived by Kopal, 1960) was made by James and Kopal (1963) for planets such as the Earth, as well as Jupiter and Saturn. They used Rung–Kutta's method for step-by-step numerical integration. In the present work, we shall investigate the numerical solutions of the foregoing differential equation formed in the previous article SM3 and governing f_2, f_4, f_6, and f_8, applying Hamming's predictor corrector method for numerical integration. We restrict our analysis to the configurations characterizing the planets Jupiter and Saturn with density distribution given by Zharkov *et al.* (1973).

2. Boundary Conditions

In this section we derive the formulae defining the boundary conditions, which are necessary for complete specification of the numerical solutions of the fourth-order Clairaut theory (Equation (1) in SM3), since we have a second-order differential equations two boundary values are needed. These are at the centre ($a = 0$) and at the external boundary ($a = a_1$). From these particular solutions we can evaluate the amplitudes $f(a)$, the radial part of the equipotential surfaces, which expresses the deformation of the shape of the equipotential. So,

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1. at the center ($a = 0$), all the $f_n(a)$ as are to be a minimum, which requires that their derivatives must identically vanish, i.e., all

$$f_n'(a) = 0, \quad \text{for} \quad n = 2,\ 4,\ 6,\ \text{and } 8, \tag{1}$$

2. at the external equipotential ($a = a_1$), all the expressions $E_n(a)$ (Equations (I1)–(I5) in Appendix I) are equal to zero, but the expressions $F_n(a)$ continue to be defined as before by Equations (I6)–(I10) in Appendix I also, replacing a by a_1, except for $j = 0$. The exterior potential V at $n = 0$ take the form, (see El-Shaarawy *et al.,* 1989I)

$$V_0 = G \int_0^{a_0} \mathrm{d}m = Gm_0, \tag{2}$$

where

$$m_0 = 4 \int_0^{a_0} \rho a^2 \mathrm{d}a. \tag{3}$$

Replacing a_0 by a_1, and using Equation (I6), Equation (3) become

$$m_1 = 4\Pi F_0(a_1), \tag{4}$$

where m_1 is the total mass of our distorted configuration.

Applying the previous condition (2) on the equations of the radial functions described the actual form of the equipotential surfaces as given by Equations (5)–(8) in Marie *et al.,* (1993), with the aid of Equations (9)–(13) in the same reference, we obtain, for $a = a_1$

$$2f_2 + a_1 f_2' + \frac{3\omega^2 a_1^2}{2Gm_2} = ({-_1}\beta_1 f_2^2 +_1 \beta_2 a_1 f_2 f_2' -_1 \beta_3 a_1^2 f_2'^2 +_1 \beta_4 f_4 +$$

$$+ {_1}\beta_5 a_1 f_4' -_1 \eta_1 f_2^3 -_1 \eta_2 a_1 f_2^2 f_2' -_1 \eta_3 a_1^2 f_2 f_2'^2 -_1 \eta_4 a_1^3 f_2'^3 -_1 \eta_5 f_2 f_4 +$$

$$+ {_1}\eta_6 a_1 f_2 f_4' +_1 \eta_7 a_1 f_2' f_4 +_1 \eta_8 a_1^2 f_2' f_4' -_1 \eta_9 f_6 -_1 \eta_{10} a_1 f_6' -_1 \xi_1 f_2^4 -$$

$$- {_1}\xi_2 a_1 f_2^3 f_2' -_1 \xi_3 a_1^2 f_2^2 f_2'^2 -_1 \eta_4 a_1^3 f_2 f_2'^3 -_1 \xi_5 a_1^4 f_5'^4 -_1 \xi_6 f_2^2 f_4 +$$

$$+ {_1}\eta_7 a_1 f_2^2 f_4' - 1\xi_8 a_1 f_2 f_2' f_4 +_1 \eta_9 a_1^2 f_2 f_2' f_4' +_1 \xi_{10} a_1^2 f_2'^2 f_4 +$$

$$+ {_1}\xi_{11} a_1^3 f_2'^2 f_4' +_1 \xi_{12} f_2 f_6 -_1 \xi_{13} a_1 f_2 f_6' +_1 \xi_{14} a_1 f_2' f_6 -$$

$$- {_1}\xi_{15} a_1^3 f_2'^2 f_6' +_1 \xi_{16} f_4^2 -_1 \xi_{17} a_1 f_4 f_4' +_1 \xi_{18} a^2 f_4'^2) +$$

$$+ \frac{\omega^2 a_1^2}{4Gm_1} ({-_2}\alpha_1 f_2 +_2 \alpha_2 a_1 f_2' -_2 \beta_1 f_2^2 +_2 \beta_2 a_1 f_2 f_2' +_2 \beta_3 a_1^2 f_2'^2 +_2 \beta_4 f_4 +$$

$$+ {}_2\beta_5 a_1 f_4' - {}_2\eta_1 f_2^3 - {}_2\eta_2 a_1 f_2^2 f_2' + {}_2\eta_3 a_1^2 f_2 f_2'^2 + {}_2\eta_5 f_2 f_4 -$$
$$- {}_2\eta_6 a_1 f_2 f_4' - {}_2\eta_7 a_1 f_2' f_4 - {}_2\eta_8 a_1^2 f_2' f_4'), \qquad (5)$$

where the coefficients ${}_i\alpha_j$, ${}_i\beta_j$, ${}_i\eta_j$, and ${}_i\xi_j$ are given in Appendix II.

$$4f + a_1 f_4' = ({}_1\beta_1 f_2^2 + {}_1\beta_2 a_1 f_2 f_2' - {}_1\beta_3 a_1^2 f_2'^2 + {}_1\eta_1 f_2^3 - {}_1\eta_2 a_1 f_2^2 f_2' -$$
$$- {}_1\eta_3 a_1^2 f_2 f_2'^2 + {}_1\eta_4 a_1^3 f_1'^3 - {}_1\eta_5 f_2 f_4 - {}_1\eta_6 a_1 f_2 f_4' + {}_1\eta_7 a_1 f_2' f_4 - {}_1\eta_8 a_1^2 f_2' f_4'$$
$$+ {}_1\eta_9 f_6 + {}_1\eta_{10} a_1 f_6' + {}_1\xi_1 f_2^4 - {}_1\xi_2 a_1 f_2^3 f' + {}_1\xi_3 a_1^2 f_2^2 f_2'^2 + {}_1\xi_4 a_1^3 f_2 f_2'^3 +$$
$$+ {}_1\xi_5 a_1^4 f_2'^4 + {}_1\xi_6 f_2^2 f_4 - {}_1\xi_7 a_1 f_2^2 f_4' + {}_1\xi_8 a_1 f_2 f_2' f_4 - {}_1\xi_9 a_1^2 f_2 f_2' f_4' -$$
$$- {}_1\xi_{10} a_1^2 f_2'^2 f_4 - {}_1\xi_{11} a_1^3 f_2'^2 f_4' - {}_1\xi_{12} a_1 f_2 f_6' + {}_1\xi_{13} a_1 f_2 f_6' - {}_1\xi_{14} a_1 f_2' f_6$$
$$+ {}_1\xi_{15} a_1^2 f_2' f_6' - {}_1\xi_{16} f_4^2 + {}_1\xi_{17} a_1 f_4 f_4' - {}_1\xi_{18} a_1^2 f_4'^2) +$$
$$+ \frac{\omega^2 a_1^3}{4Gm_1}({}_2\beta_1 f_2^2 - {}_2\beta_2 a_1 f_2 f_2' - {}_2\beta_3 a_1^2 f_2'^2 + {}_2\beta_4 f_4 + {}_2\beta_5 a_1 f_4' - {}_2\eta_1 f_2^3 +$$
$$+ {}_2\eta_2 a_1 f_2^2 f_2' - {}_2\eta_3 a_1^2 f_2 f_2'^2 + {}_2\eta_4 a_1^3 f_2'^3 - {}_2\eta_5 f_2 f_4 + {}_2\eta_6 a_1 f_2 f_4' +$$
$$+ {}_2\eta_7 a_1 f_2' f_4 + {}_2\eta_8 a_1^2 f_2' f_4' - {}_2\eta_9 f_6) \qquad (6)$$

the coefficients ${}_i\beta_j$, ${}_i\eta_j$, and ${}_i\xi_j$ are given in Appendix II.

$$6f_6 + a_1 f_6' = ({}_1\eta_1 f_2^3 + {}_1\eta_2 a_1 f_2^2 f_2' + {}_1\eta_3 a_1^2 f_2 f_2'^2 - {}_1\eta_4 a_1^3 f_2'^3 + {}_1\eta_5 f_2 f_4 +$$
$$+ {}_1\eta_6 a_1 f_2 f_4' + {}_1\eta_7 a_1 f_2' f_4 + {}_1\eta_8 a_1^2 f_2' f_4' - {}_1\xi_1 f_2^4 - +1\xi_2 a_1 f_2^3 f_2' -$$
$$- {}_1\xi_3 a_1^2 f_2^2 f_2'^2 - {}_1\xi_4 a_1^3 f_2 f_2'^3 - {}_1\xi_5 a_1^4 f_2'^4 + {}_1\xi_6 f_2^2 f_4 + {}_1\xi_7 a_1 f_2^2 f_4' -$$
$$- {}_1\xi_8 a_1 f_2 f_2' f_4 - {}_1\xi_9 a_1^2 f_2 f_2' f_4' + {}_1\xi_{10} a_1^2 f_2'^2 f_4 + {}_1\xi_{11} a_1^3 f_2'^2 f_4' -$$
$$- {}_1\xi_{12} f_2 f_6 - {}_1\xi_{13} a_1 f_2 f_6' - {}_1\xi_{14} a_1 f_2' f_6 + {}_1\xi_{15} a^2 f_2' f_6' - {}_1\xi_{16} f_4^2 -$$
$$- {}_1\xi_{17} a_1 f_4 f_4' + {}_1\xi_{18} a_1^2 f_4'^2) + \frac{\omega^2 a_1^3}{4m_1 G}(-{}_2\eta_1 f_2^3 - {}_2\eta_2 a_1 f_2^2 f_2' +$$
$$+ {}_2\eta_3 a_1^2 f_2 f_2'^2 - {}_2\eta_5 f_2 f_4 - {}_2\eta_6 a_1 f_2 f_4' - {}_2\eta_7 a_1 f_2' f_4 - {}_2\eta_8 a_1^2 f_2' f_4' -$$
$$- {}_2\eta_9 f_6 - {}_2\eta_{10} a_1 f_6'), \qquad (7)$$

where the coefficients ${}_i\eta_j$ and ${}_i\xi_j$ are given in Appendix II, and,

$$8f_8 + a_1 f_8' = -\xi_1 f_2^4 - \xi_2 a_1 f_2^3 f_2' - \xi_3 a_1^2 f_2^2 f_2'^2 + \xi_4 a_1^3 f_2 f_2'^3 +$$
$$+\xi_5 a_1^4 f_2'^4 + \xi_6 f_2^2 f_4 + \xi_7 a_1 f_2^2 f_4' - \xi_8 a_1 f_2 f_2' f_4 - \xi_9 a_1^2 f_2 f_2' f_4' -$$
$$-\xi_{10} a_1 f_2 f_4 - \xi_{11} a_1^3 f_2'^2 f_4' + \xi_{12} f_2 f_6 - \xi_{13} a_1 f_2 f_6' -$$
$$-\xi_{14} a_1 f_2' f_6 - \xi_{15} a_1^2 f_2' f_6' + \xi_{16} f_4^2 - \xi_{17} a_1 f_4 f_4' + \xi_{18} a_1^2 f_4'^2, \qquad (8)$$

the coefficients ξ_i are given in Appendix II.

These four first-order differential equations of the amplitudes $f_j(a)$ specify the particular solutions of Clairaut equations (SM3, Equation (1)) and will be needed on dealing with the numerical solutions afterwards.

3. Numerical Solutions

A construction of the numerical solutions of the Clairaut equations specified by the boundary conditions (5)–(8) can be accomplished by successive approximation using the same technique followed by Kopal (1974): first solving for to accuracy of first-order, next solving for f_2 and f_4 to quantities of second-order, and continuing until a solution for f_2, f_4, f_6 and f_8 accurate to fourth-order has been established. Over all steps of our calculations the Hamming's predictor-corrector method is used to report the numerical integration of the foregoing Clairaut differential equations governing f_2, f_4, f_6, and f_8 for the density distribution $\rho(a)$ characterizing our configuration. This method consists of solving a system of n-first order ordinary differential equations.

The Clairaut equation (see SM3) takes the form

$$a^2 f_j'' + 6D(f_j + af_j') - j(j+1)f_j = T_j(a), \tag{9}$$

which is a second order differential equation, where D is the ratio between the arbitrary density $\rho(a)$ and the mean density which is given by

$$D = \frac{\rho(a)}{\bar{\rho}(a)} \text{ and } \bar{\rho} = \frac{3}{a^3}\int_0^{a_0} \rho a^2 \, \mathrm{d}a = \frac{3}{a^3} F_0. \tag{10}$$

The ratio D can be expanded, in the proximity of the origin, in a series as

$$D(a) = 1 - \lambda a + \cdots \tag{11}$$

even power only occurring on the r.h.s on account of spherical symmetry of our configuration in its undistorted state, where by definition

$$D(a) = 1 \text{ and } D(a_1) = 0. \tag{12}$$

By successive approximation method, we can extend with difficulties to solve for each amplitude $f_j(a)$ of jth harmonic distortion to the required degree of accuracy. The structure of the differential equations of the form (9) governing these, make it evident that near the origin ($a = 0$), the complementary function of each f_j (factored by an arbitrary positive constant K_j) will vary as $K_j a^{j-1}$, while its particular integral will be factored by $K_j^{j/2}$, while the constants $K_j (j = 2, 4, 6\&8)$ constitute the eigen-parameters of our problem, and their values must be determined with the aid of the boundary conditions (5)–(8) valid at $a = a_1$.

Thus, to put Equations (9) as a two first order ordinary differential equations, consider that

$$y_1 = f_j \quad \text{and} \quad y_2 = f_j' \tag{13}$$

on differentiation with respect to a, thus

$$y_1' = f_j' \quad \text{and} \quad y_2 = f_j'' \tag{14}$$

Equations (11) can be written as,

$$y_1' = y_2 \tag{15}$$

and,

$$a^2 y_2' = [j(j+1) - 6D]y_1 - 6aDy_2 + T_j, \tag{16}$$

which are two differential equations of first-order, subject to the initial conditions

$$y_1(0) = \text{constant } (f_j(0)) \quad \text{and} \quad y_2(0) = 0 \tag{17}$$

The numerical integration of the boundary value problem (13) for Jupiter and Saturn has been carried out on the basis of their models of internal structure worked out by Zharkov (1975) as given diagrammatically in Figures 1 and 2. These planets will underlying under the assumption that the mass of these planets being in hydrostatic equilibrium and are known to rotate so fast that the higher-order than first-order effects of centrifugal force become of much greater relative importance.

Using the computing program, written by one of us (Marie), based on solving n-first order differential equations, which applying the successive approximation steps as shown before and through every step, it solved the system of differential Equations (15) and (16) under the boundary conditions (17) – up to the order of evaluation – using Hamming predictor – corrector method for the numerical integrations.

As long as terms of first-order only are retained throughout Equation (9), the problem for computation is simple: The first-order equation being linear and homogeneous in the sole dependent variable f_2, its numerical integration can be started from an arbitrary value $f_2(0)$, and its appropriate scale constant adjusted by means of the outer boundary condition (5). To do so we consider as a first approximation $K_2 = 1$, to begin the numerical integration. Using the values of f_2 and f_2' from the first step and satisfy the boundary condition at $a = a_1$, the program use iteration method to K_2 until the values of f_2 and f_2' will give appropriate accuracy to fulfill the boundary conditions (5), where the value of constants occurring in Equation (5) can be specified without difficulty for our planets. These constants were given in Table I.

Owing to this we can obtain the value of amplitudes f_2 and its derivative at the different levels inside the planet as a first approximation. When, however the terms of second, third and fourth order are considered, we face a non-linear boundary value problem of the jury type the solution of which proceeded as follows: consider Equation (6) at the boundary $a = a_1$ of the amplitude f_4, using the values of f_2 and f_2' which obtained as a first approximation to get initial value to K_4, as well as f_4. Take the numerical values of f_2 and f_4 to repeat the numerical integration of Equations (9) up to second-order and so continue

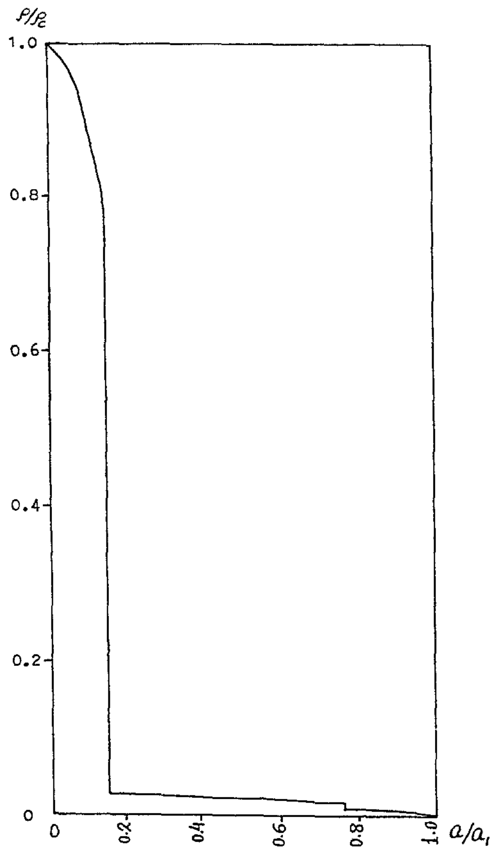

Fig. 1. Internal density distribution of Jupiter.

TABLE I

Constants characterize our planets Jupiter and Saturn

	Jupiter	Saturn
Mass m_1	1.8985×10^{30} gm	0.56846×10^{30} gm
Mean radius a_1	6.9911×10^{9} cm	5.8232×10^{9} cm
Mean density $\bar{\rho}$	1.326 gm/cm^3	0.6873 gm/cm^3
Angular velocity of rotation ω	0.00017585 sec^{-1}	0.000163785 sec^{-1}

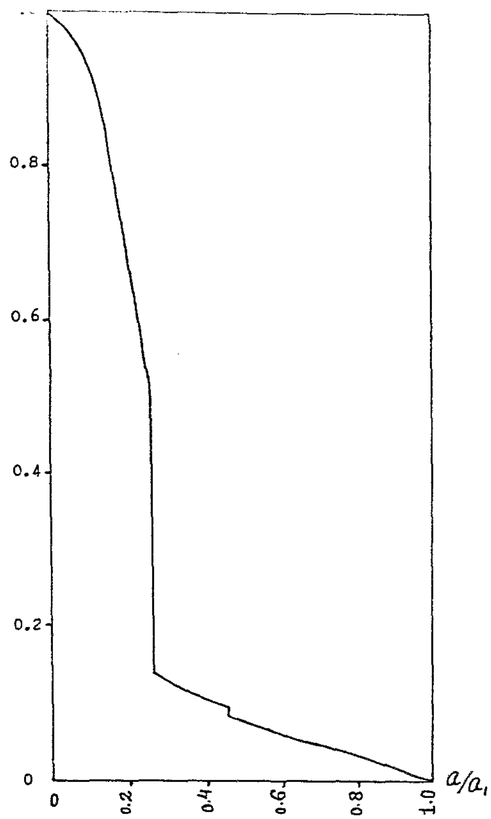

Fig. 2. Internal density distribution of Saturn.

as mentioned before until we solve up to four approximations, where the initial values of K_j's, j = 2, 4, 6, and 8, are obtained from the boundary condition equations.

In this investigation, we take into consideration the different shape of ratio D, Equation (11), which given by James and Kopal (1968) as: this ratio of the form of Equation (11) could be used only to a/a_1 = 0.15 and 0.267, respectively. Outside the core, empirical polynomial were used again to represent the theoretical variation of ρ/ρ_c: For Jupiter, polynomials of degree 3, 2, and 2 were found adequate in the inner, intermediate, and outer shell, respectively; while for Saturn, distinct polynomials had to be used in the range, $0.267 < a/a_1 < 0.465$, $0.465 < a/a_1 < 0.549$ and $0.549 < a/a_1 < 1.0$.

Take into consideration the above different ratio of ρ/ρ_c; first to Jupiter and secondly for Saturn, the outcome of our solutions of the fourth order Clairaut Theory is then contained in Tables II and III for Jupiter and Saturn, respectively,

TABLE II
Coefficients of the internal distorton of Jupiter

a/a_1	$-10^2 f_2$	$-10^2 f_4$	$-10^2 f_6$
0	0.781	0.0047	0.000009
0.095	0.781	0.0048	0.000020
0.146	0.782	0.0049	0.000039
0.150	0.783	0.0050	0.000070
0.150	0.784	0.0051	0.000073
0.193	0.806	0.0061	0.00062
0.241	0.822	0.0069	0.00112
0.291	0.844	0.0083	0.00190
0.337	0.872	0.0099	0.00291
0.385	0.907	0.0120	0.00440
0.443	0.961	0.0154	0.00685
0.490	1.021	0.0194	0.0099
0.54	1.099	0.0247	0.0146
0.588	1.195	0.0318	0.0212
0.641	1.321	0.0417	0.0316
0.690	1.475	0.0546	0.0466
0.732	1.643	0.0697	0.0656
0.765	1.866	0.0896	0.0914
0.765	1.867	0.0897	0.0921
0.800	1.039	0.1079	0.1193
0.850	2.344	0.1422	0.1752
0.874	2.608	0.1712	0.2239
0.899	2.829	0.1985	0.2741
0.919	3.033	0.2246	0.3244
0.938	3.249	0.2533	0.3818
0.959	3.508	0.2888	0.4558
0.968	3.712	0.3152	0.5097
0.976	3.859	0.3350	0.5517
0.988	4.060	0.3642	0.6168
0.9917	4.262	0.3893	0.6683
0.9953	4.350	0.4013	0.6944
1.0000	4.461	0.4172	0.7304

which shows the deformation at the equipotential agrees with the density distribution level given in Figures 1, 2. These solutions are diagrammatic in Figures 3 and 4.

Once the values of $f_j(a)$ are thus numerically for two planetary configurations, Jupiter and Saturn, then insertion in Equation (9) gives the respective approxi-

TABLE III

Coefficients of the internal distorton of Saturn

a/a_1	$-10^2 f_2$	$-10^2 f_4$	$-10^2 f_6$
0	1.086	0.0081	0.00018
0.005	1.078	0.0097	0.00019
0.091	1.079	0.0098	0.00020
0.147	1.080	0.0100	0.00030
0.198	1.082	0.0105	0.00035
0.247	1.088	0.0110	0.00061
0.267	1.100	0.0120	0.00071
0.267	1.107	0.0128	0.00080
0.298	1.215	0.0170	0.00257
0.338	1.264	0.0200	0.00376
0.388	1.339	0.0250	0.00582
0.445	1.447	0.0320	0.00924
0.465	1.510	0.0360	0.01130
0.465	1.520	0.0370	0.01150
0.500	1.616	0.0440	0.01518
0.549	1.763	0.0550	0.02182
0.599	1.956	0.0710	0.03182
0.650	2.190	0.0910	0.04651
0.699	2.455	0.1170	0.06657
0.749	2.772	0.1500	0.09569
0.799	3.140	0.1930	0.13689
0.848	3.590	0.2500	0.19698
0.873	3.930	0.2930	0.24464
0.899	4.235	0.3350	0.29635
0.917	4.514	0.3750	0.34470
0.934	4.783	0.4150	0.39537
0.960	5.194	0.4800	0.48344
0.969	5.506	0.523	0.53997
0.979	5.721	0.5570	0.58805
0.9864	5.903	0.5870	0.62919
0.9919	6.045	0.6090	0.66194
0.9958	6.131	0.6240	0.68315
1.0000	6.208	0.6370	0.70485

mation to the form of the equipotential surfaces of any layer of constant density and mean radius a inside the planets.

In order to test the accuracy of our results, the following two checks have been carried out:

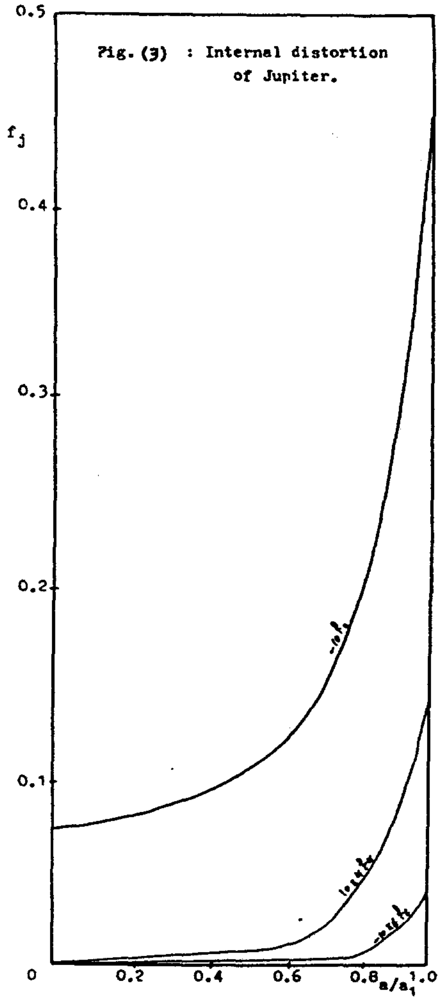

Fig. 3.

1. Considering the density distribution of Jupiter and Saturn reported by James and Kopal (1963), and repeating the computation of the distortions up to second-order only, we notice that our results are identical up to the third decimal with that of James and Kopal.

2. The value of the constant λ in Equation (11) was arbitrarily reduced by 1% and the integration repeated; the corresponding ranges in the terminal values of f_j's were likewise changed about 1%.

Note that the departure of our planets from (Jupiter and Saturn) due to these fourth-order rotational distortion terms was interesting up to the amplitude f_6 only. Thus any greater accuracy would, however, require an increased precision

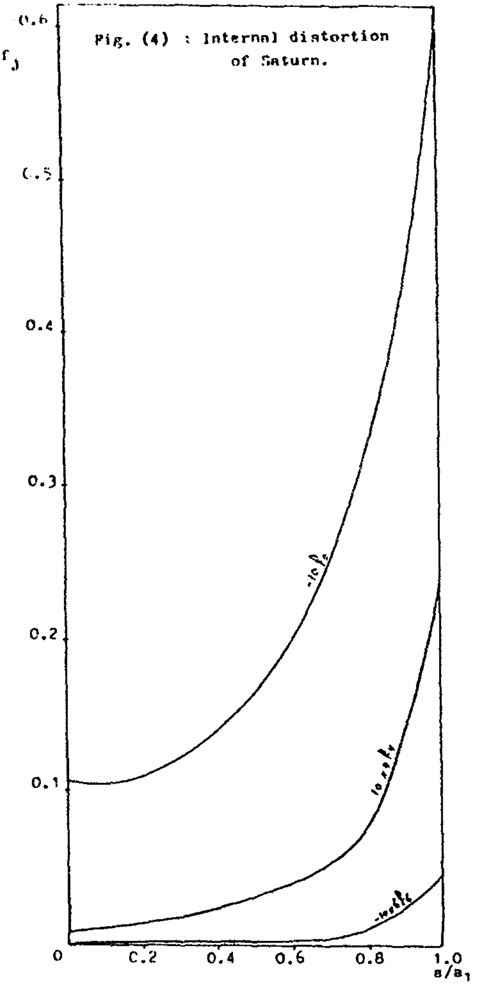

Fig. 4.

in our knowledge of the density distribution inside these planets. So the third-order rotational distortion terms was sufficient to study the deformation of these planets owing to the density distribution in our hand.

4. Appendix I

Radial function of internal potential

$$E_0(a) = \int_{a_0}^{a_1} \rho \frac{\partial}{\partial a} \left[a^2 \left(1 - \frac{7}{2} f_2^2 + \frac{1}{4} f_2^4 - \frac{7}{2} f_4^2 - 2 f_2^2 f_4 \right) \right] \, da, \tag{I1}$$

$$E_2(a) = \int_{a_0}^{a_1} \rho \frac{\partial}{\partial a} \left(f_2 + \frac{3}{4} f_2^3 - \frac{1}{2} f_2 f_4 \right) \mathrm{d}a, \tag{I2}$$

$$E_4(a) = \int_{a_0}^{a_1} \rho \frac{\partial}{\partial a} \left[a^{-2} \left(f_4 - \frac{3}{4} f_2^2 + \frac{3}{4} f_2^3 - \frac{3}{2} f_2^2 f_4 \right) \right] \mathrm{d}a, \tag{I3}$$

$$E_6(a) = \int_{a_0}^{a_1} \rho \frac{\partial}{\partial a} \left[a^{-4} \left(f_6 - \frac{5}{4} f_2^3 - \frac{5}{2} f_2 f_4 \right) \right] \mathrm{d}a, \tag{I4}$$

and

$$E_8(a) = \int_{a_0}^{a_1} \rho \frac{\partial}{\partial a} \left[a^{-6} \left(f_8 - \frac{49}{24} f_2^4 - \frac{7}{2} f_2 f_6 - \frac{7}{4} f_4 \right) \right] \mathrm{d}a. \tag{I5}$$

Radial function of external potential,

$$F_0(a) = \int_0^{a_0} \rho a^2 \, \mathrm{d}a, \tag{I6}$$

$$F_2(a) = \int_0^{a_0} \rho \frac{\partial}{\partial a} \left[a^5 \left(f_2 - \frac{1}{2} f_2^3 + 2 f_2 f_4 \right) \right] \mathrm{d}a, \tag{I7}$$

$$F_4(a) = \int_0^{a_0} \rho \frac{\partial}{\partial a} \left[a^7 \left(f_4 + \frac{3}{4} f_2^2 + \frac{9}{2} f_2^2 f_4 - \frac{5}{4} f_2^4 \right) \right] \mathrm{d}a, \tag{I8}$$

$$F_6(a) = \int_0^{a_0} \rho \frac{\partial}{\partial a} \left[a^9 \left(f_6 + \frac{7}{2} f_2^3 + 4 f_2 f_4 \right) \right] \mathrm{d}a, \tag{I9}$$

and,

$$F_8(a) = \int_0^{a_0} \rho \frac{\partial}{\partial a} \left[a^{11} \left(f_8 + \frac{15}{4} f_2^4 + 5 f_2 f_6 + \frac{5}{2} f_4^2 \right) \right] \mathrm{d}a. \tag{I10}$$

5. Appendix II

COEFFICIENTS OF THE BOUNDARY-CONDITION EQUATIONS

For $n = 2$, Equation (5)

$$\alpha_{21} = 0.960, \quad \alpha_{22} = 0.192$$

TABLE II-1

$_i\beta_j$

j/i	1	2
1	1.398	0.820
2	10.602	0.473
3	1.043	0.200
4	0.356	12.220
5	0.178	2.448

$_i\eta_j$

j/i	1	2
1	224.233	7.063
2	59.566	6.579
3	52.412	9.095
4	4.510	
5	71.443	122.447
6	62.827	23.369
7	569.811	42.242
8	43.134	2.252
9	2.250	
10	1.201	

ξ_i

i	ξ
1	23304.440
2	709.513
3	606.828
4	141.948
5	12.861
6	2642.228
7	1255.993
8	2298.639
9	266.595
10	920.509
11	251.313
12	150.349
13	30.355
14	7.073
15	3.313
16	112.872
17	37.173
18	7.696

For $n = 4$, Equation (6)

$\alpha_{21} = 0.960, \quad \alpha_{22} = 0.192$

TABLE (II-2)

$_i\beta_j$

j/i	1	2
1	13.883	2.989
2	18.886	1.434
3	5.971	1.146
4		96.526
5		13.783

$_i\eta_j$

j/i	1	2
1	1940.640	50.780
2	1370.270	321.015
3	99.990	85.083
4	35.038	0.697
5	3438.151	291.302
6	509.673	431.353
7	3757.689	360.193
8	325.157	6.291
9	36.540	0.843
10	9.135	

ξ_i

i	ξ
1	127522
2	54820
3	11784
4	2603
5	148.093
6	68124
7	24642
8	44906
9	15709
10	6429.622
11	2221.347
12	3009.495
13	457.206
14	485.365
15	51.153
16	1612.857
17	887.502
18	54.004

For $n = 6$, Equation (7)

TABLE (II-3)

	${}_i\beta_j$	
j/i	1	2
1	760.192	80.100
2	240.244	94.288
3	61.914	14.252
4	3.839	
5	512.779	127.235
6	52.141	154.946
7	294.380	65.863
8	35.85	6.761
9		3.258
10		0.363

	ξ_i
i	ξ
1	69158.640
2	13920.842
3	1521.300
4	201.909
5	3.354
6	14147.950
7	6176.276
8	10972.332
9	1091.333
10	929.157
11	64.228
12	223.132
13	82.640
14	194.812
15	5.738
16	21.113
17	143.331
18	6.268

For $n = 8$, Equation (8)

TABLE (II-3)
ξ_i

i	ξ
1	1190.734
2	910.458
3	426.987
4	5.991
5	0.319
6	157.532
7	19.697
8	357.118
9	49.530
10	9.385
11	3.245
12	66.473
13	7.400
14	1.482
15	0.406
16	4.314
17	2.131
18	0.045

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