EULERIAN LIBRATION POINTS OF RESTRICTED PROBLEM OF THREE OBLATE SPHEROIDS

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Abstract. In this paper we consider the circular restricted problem of three oblate spheroids. The collinear equilibrium solutions are obtained. Finally a numerical study of the influence of the non-sphericity in the location of the libration points is made.

1. Introduction

It is well known that there are five equilibrium solutions in the restricted three bodies problem. Three are collinear with the primaries and the other two are in equilateral triangular configuration with the primaries. The restricted problem of three rigid bodies (S_1, S_2, S_3) is defined as the study of the motion of S_3 of infinitesimal mass (which does not perturb the motion of the others) under the action of S_1, S_2 – called the primaries – as in the classic case. This study was begun by Nikolaev (1970), which obtained equilibrium solutions when the more massive primary S_1 is an oblate spheroid, and the other two are spheres. This work is extended by Sharma and Subbarao (1975) in the case where S_1, S_2 are oblate spheroids, and where the infinitesimal S_3 is a sphere. Other authors that must be mentioned are Robinson (1979), Markov (1980), Vidyakin (1979). Duboshin (1982, 1984) has given conditions for the existence of collinear and equilateral equilibrium solution in the case of rigid bodies having an exis and plane of symmetry. Elipe and Ferrer (1985) studied the circular planar restricted problem of three axisymmetric ellipsoids S_1, S_2, S_3 such that their equatorial planes coincide with the plane of the motion of the three centers of mass.

In this paper we consider the restricted problem of three axisymmetric rigid bodies. The shift of the equilibrium position (due to the non-sphericity) with respect to the classical restricted problem is checked numerically.

2. Equations of Motion

Let S_1, S_2, S_3 be three axisymmetric rigid bodies, with masses m_1, m_2, m_3 and O_1, O_2, O_3 their centers of masses. We suppose that m_3 is infinitesimal, i.e., it does not influence the motion of S_1 and S_2 (primaries), whose centers of masses describe circular orbits around O, its common centre of mass, with mean motion n. Besides we suppose that O_3 is moving in the plane defined by the motion of the centers of masses O_1, O_2, of the primaries.

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Now, we consider the following orthogonal system of reference:

(1) A fixed system $OXYZ$ such that the OXY plane coincides with the fixed plane which contains the points O_1, O_2, O_3.

(2) O_iXYZ, parallel to $OXYZ$, with origin at O_i.

(3) $O_i\xi_i\eta_i\zeta_i$, defined by the principal axes of inertia of S_i.

The relation between the system $O_i\xi_i\eta_i\zeta_i$ and O_iXYZ is given by the Euler angles $(\theta_i, \phi_i, \psi_i)$.

The potential which acts on the solid body S_3 (Duboshin, 1975) is given by

$$V = -\sum_{i=1}^{2} fm_2m_i\left\{\frac{1}{p_i} + \frac{1}{p_i^3}[a_i^2 - c_i^2) + (a_3^2 - c_3^2)]\right\}, \tag{2.1}$$

where $p_i = |r_i| = |O_iO_3|$; f is the gravitational constnat; a_j, c_j $(j = 1, 2, 3)$ are the semi-axes of S_j. Then the equations of motion of S_3 are

$$\ddot{\mathbf{r}} = -fm_1g_1\frac{\mathbf{r}_1}{p_1^3} - fm_2g_2\frac{\mathbf{r}_2}{p_2^3}, \tag{2.2}$$

$$\dot{\mathbf{L}} = \mathbf{W} \wedge \mathbf{L} = -(\delta^{-1})^T\frac{\partial V}{\partial q}; \tag{2.3}$$

where

$$g_i = 1 + \frac{3}{10p_i^2}[(a_i^2 - c_i^2) + (a_3^2 - c_3^2)], \quad \mathbf{r} = \overline{O_3O},$$

$$\mathbf{q} = (\psi_3, \theta_3, \phi_3), \qquad \delta = \begin{vmatrix} \sin\theta_3 & \sin\phi_3 & \cos\phi_3 & 0 \\ \sin\theta_3 & \cos\phi_3 & -\sin\phi_3 & 0 \\ \cos\theta_3 & 0 & 0 & 1 \end{vmatrix},$$

and $\mathbf{W}$ is the angular velocity vector and $\mathbf{L}$ and angular momentum vector of S_3, referred to O_3, ξ_3, η_3, ζ_3.

Finally, we choose another orthogonal system of reference $Oxyz$. Ox is the direction O_1O_2 of the primaries, Oz perpendicular to the OXY plane and Oy the third axis of a right oriented system. In this system the Equations (2.2) have the form

$$\begin{aligned} \ddot{x} - 2n\dot{y} &= n^2x - fm_1g_1\frac{x - x_1}{p_1^3} - fm_2g_2\frac{x - x_2}{p_2^3}, \\ \ddot{y} + 2n\dot{x} &= n^2y - fm_2g_1\frac{y}{p_1^3} - fm_2g_2\frac{y}{p_2^3}, \end{aligned} \tag{2.4}$$

$(x_1, 0)$, $(x_2, 0)$ being the coordinates of O_1, O_2, respectively, in the system $Oxyz$. Introducing the function

$$\Omega = \tfrac{1}{2}n^2(x^2 + y^2) + fm_1G_1 + fm_2G_2\,, \tag{2.5}$$

where

$$G_i = \frac{1}{p_i} + \frac{1}{10p_i^3}\,[(a_i^2 - c_i^2) + (a_3^2 - c_3^2)]\,.$$

The equations (2.4) may be written as

$$\begin{aligned} \ddot{x} - 2n\dot{y} &= \Omega_x \\ \ddot{y} + 2n\dot{x} &= \Omega_y\,. \end{aligned} \tag{2.6}$$

Equations (2.6) together with (2.3) describe the motion of S_3.

3. Particular Solutions

Let us seek the solutions of (2.6) analogous to the Euler equilibrium points of the classic restricted problem.

Making $\Omega_x = \Omega_y = 0$, we have

$$n^2x - fm_1g_1\frac{x - x_1}{p_1^3} - fm_2g_2\frac{x - x_2}{p_2^3} = 0\,, \tag{2.7}$$

$$y\left(n^2 - fm_1\frac{g_1}{p_1^3} - fm_2\frac{g_2}{p_2^3}\right) = 0\,. \tag{2.8}$$

This system admits of two types of solutions.

$$y = 0, \qquad y \neq 0\,.$$

We will study the first case, i.e., O_1, O_2, O_3 are collinear points. According to the usual notation, we choose the constants such that

$$fm_2 = \mu, \qquad f(m_1 + m_2) = 1\,.$$

Then the coordinates of O_1, O_2 are

$$O_1 = (x_1, O) = (-\mu, O), \qquad O_2 = (x_2, O) = (1 - \mu, O)\,.$$

The positions of the collinear points, are determined by the solution of Equation (2.7). We will consider three cases, corresponding to the equilibrium point situated: at the right (L_1), in the middle (L_2), and to the left (L_3) of the primaries, Elipe and Ferrer (1985).

(L_1) In this case, the Equation (2.7) can be solved approximately and obtain

$$P_2 = S\left[1 - \frac{\beta}{4\alpha}S + \frac{1}{4}\left(\frac{7}{8}\frac{\beta^2}{\alpha^2} + \frac{1}{3I_2} - \frac{\gamma}{\alpha}\right)S^2 + \cdots\right],$$

$$P_1 = 1 - P_2, \quad S = \sqrt[4]{\frac{3I_2}{\alpha} \cdot \frac{\mu}{1-\mu}}, \quad \alpha = 1 - n^2 + 3I_1 ,$$

$$\beta = 2 + n^2 + 12I_1 , \qquad \gamma = 3(1 + 10I_1) .$$

(L_2) In this case, the Equation (2.7) can be solved and we get

$$P_2 = S\left[1 + \frac{\beta}{4\alpha}S + \frac{1}{4}\left(\frac{7}{8}\frac{\beta^2}{\alpha^2} + \frac{1}{3I_2} - \frac{\gamma}{\alpha}\right)S^2 + \cdots\right],$$

$$P_1 = 1 + P_2 ,$$

where S, α, β and γ are defined as in (L_1).

(L_3) From Equation (2.7), the libration point L_3 can be determined as following

$$P_1 = 1 + \delta, \quad P_2 = 2 + \delta, \quad \delta = S_0 + S_1\mu + S_2\mu^2 + S_3\mu^3 + \cdots ,$$

$$S_0 = \frac{\lambda_0^3}{\lambda_1^5}(4\lambda_2^2 - 2\lambda_1\lambda_2 - 3) - \frac{\lambda_0}{\lambda_1^3}(\lambda_0\lambda_2 + \lambda_1^2)$$

$$S_1 = \frac{1}{\lambda_1^3}(\lambda_1^2 + 2\lambda_0\lambda_2) + \frac{3\lambda_0^2}{\lambda_1^5}(2\lambda_2^2 - \lambda_1\lambda_3)$$

$$S_2 = \frac{-1}{\lambda_1^3}\left[\lambda_2 + \frac{3\lambda_0^2}{\lambda_1^2}(2\lambda_2^2 - \lambda_1\lambda_3)\right]$$

$$S_3 = \frac{1}{\lambda_1^5}(2\lambda_2^2 - \lambda_1\lambda_3), \quad \lambda_0 = -\frac{A}{G}, \quad \lambda_1 = -\frac{1}{G}\left(B - \frac{AH}{G}\right)$$

$$\lambda_2 = -\frac{1}{G}\left[C - \frac{HB}{G} + A\left(\frac{H^2}{G^2} - \frac{K}{G}\right)\right],$$

$$\lambda_3 = -\frac{2}{G}\left[\left(D - \frac{CH}{G}\right) + B\left(\frac{H^2}{G^2} - \frac{K}{G}\right) + A\left(\frac{2KH}{G^2} - \frac{L}{G} - \frac{H^3}{G^3}\right)\right]$$

$$A = 16(n^2 - 3I_1 - 1), \quad B = 16(7n^2 - 6I_1 - 4) ,$$

$$C = 8(43n^2 - 9I_1 - 13), \quad D = 8(76n^2 - 3I_1 - 11) ,$$

$$G = 3(16I_1 - I_2) + 16n^2 + 12, \quad H = 4(24n^2 + 24I_1 - 3I_2 + 11) ,$$

$$L = 12(2L_1 - L_2) + 360n^2 + 44, \quad K = 18(4I_1 - I_2) + 248n^2 + 63 .$$

4. Numerical Results

We have made a numerical study of the non-sphericity of the axisymmetric primaries S_1, S_2 in the location of the Euler's libration points of an axisymmetric satellite S_3. The shift of the equilibrium position with respect to the classical restricted problem is checked numerically for several values of I_i $(i = 1, 2)$ and $\mu = 0.0009536896$ as in the following tables.

I_1	L_1	ΔL_1	L_2	ΔL_2	L_3	ΔL_3
$I_1 = I_2 = 0$	0.932369999		1.068826078		−1.000397371	
10^{-2}	0.900745382	−0.031624617	1.097347238	0.028521160	−1.000376894	−0.000020477
10^{-3}	0.899871890	−0.032498109	1.098220730	0.029394652	−1.000367494	−0.000029877
10^{-4}	0.899781940	−0.032588095	1.098310680	0.029484602	−1.000366504	−0.000030867
10^{-5}	0.899772789	−0.032597210	1.098319831	0.029493753	−1.000366404	−0.000030967

I_1	L_1	ΔL_1	L_2	ΔL_2	L_3	ΔL_3
$I_1 = I_2 = 0$	0.932369999		1.068826078		−1.000397371	
10^{-2}	0.937128548	0.004758549	1.060964073	−0.007862005	−1.000402990	−0.000005619
10^{-3}	0.936583199	0.004213200	1.061509421	−0.007316657	−1.000395360	−0.000002011
10^{-4}	0.936527055	0.004157056	1.061565565	−0.007260513	−1.000394559	−0.000002812
10^{-5}	0.936521424	0.004151425	1.061571196	−0.007254882	−1.000394479	−0.000002892

I_1	L_1	ΔL_1	L_2	ΔL_2	L_3	ΔL_3
$I_1 = I_2 = 0$	0.932369999		1.068826078			−1.000397371
10^{-2}	0.959996460	0.027626461	1.038096160	−0.030729918	−1.000405428	−0.000008057
10^{-3}	0.959652826	0.027282827	1.038439794	−0.030386284	−1.000397954	−0.000000583
10^{-4}	0.959617450	0.027247451	1.038475170	−0.030350908	−1.000397170	−0.000000201
10^{-5}	0.959613902	0.027243903	1.038478718	−0.030347360	−1.000397091	−0.000000280

I_1	L_1	ΔL_1	L_2	ΔL_2	L_3	ΔL_3
$I_1 = I_2 = 0$	0.932369999		1.068826078		−1.000397371	
10^{-2}	0.974339407	0.041969408	1.023753213	−0.045072865	−1.000405670	−0.000008299
10^{-3}	0.974122008	0.041752009	1.023970612	−0.044855466	−1.000398212	−0.000000841
10^{-4}	0.974099628	0.041729629	1.023992992	−0.044855466	−1.000397429	−0.000000058
10^{-5}	0.974097383	0.041727384	1.023995237	−0.044830841	−1.000397391	−0.000000020

I_1	L_1	ΔL_1	L_2	ΔL_2	L_3	ΔL_3
$I_1 = I_2 = 0$	0.932369999		1.068826078		−1.000397371	
10^{-2}	0.983457335	0.051087336	1.014635285	−0.054190793	−1.000405694	−0.000008323
10^{-3}	0.983213176	0.050843177	1.014879444	−0.053946634	−1.000398238	−0.000000867
10^{-4}	0.983198959	0.050828960	1.014893661	−0.053932417	−1.000397455	−0.000000084
10^{-5}	0.983197533	0.050827534	1.014895087	−0.053930991	−1.000397377	−0.000000006

Some of the conclusions are as follows

(1) The shifts of L_1 and L_2 are greater than the shift of L_3.

(2) When I_2 is in the order of 10^{-2}, the libration point L_1 directs towards the center of masses M_1 and M_2 while L_2 moves faraway from M_2.

(3) When I_2 is smaller than 10^{-2}, the libration points L_1 and L_2 directs towards M_2.

(4) When I_2 is greater than 10^{-4}, the libration point L_3 directs towards M_1. On the other hand when I_2 is smaller than 10^{-4}, L_3 moves for away from M_1.

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