Probab. Th. Rel. Fields 74, 271–287 (1987)

Probability Theory and Related Fields

Self-Avoiding Random Walk: A Brownian Motion Model with Local Time Drift

J.R. Norris, L.C.G. Rogers, and David Williams

Statistical Laboratory, 16 Mill Lane, Cambridge CB2 1SB, Great Britain

Summary. A natural model for a 'self-avoiding' Brownian motion in $\mathbb{R}^d$, when specialised and simplified to $d=1$, becomes the stochastic differential equation $X_t=B_t-\int_0^t g(X_s, L(s, X_s))\,ds$, where $\{L(t,x)\colon t\geqq 0,\ x\in\mathbb{R}\}$ is the local time process of X. Though X is not Markovian, an analogue of the Ray-Knight theorem holds for $\{L(\infty,x)\colon x\in\mathbb{R}\}$, which allows one to prove in many cases of interest that $\lim_{t\to\infty} X_t/t$ exists almost surely, and to identify the limit.

1. Introduction

One of the most challenging problems of probability is to construct and analyse models for self-avoiding random walks. The physics literature contains numerous such models (we list some references at the end of this section), which display considerable ingenuity and which, at the same time, underline the difficulties in performing a rigorous analysis. Our interest here is in a model which is specified by the stochastic differential equation

$$X_t=B_t-\int_0^t g(X_s, L(s, X_s))\,ds, \tag{1}$$

where $\{L(t,x)\colon t\geqq 0,\ x\in\mathbb{R}\}$ is the local time process of X, and B is Brownian motion.

The model one would most like to look at would be the three-dimensional SDE

$$X_t=B_t+\int_0^t\left(\int_0^s f(X_s-X_u)\,du\right)ds,$$

where, say, f was the electrostatic potential, $f(x)=x/|x|^2$. It is not clear that this SDE has a unique solution, let alone what its properties might be. Even in the case when f is globally Lipschitz (when it is easy to show that there is a unique solution), the study of properties of the solution is extremely difficult, so as a first step one looks at the one-dimensional analogue. This is still so difficult in general that one looks for particular f for which more can be said, and one such choice is to take f to be the Dirac δ-function, in which case the SDE becomes

$$X_t = B_t + \int_0^t L(s, X_s)\, ds.$$

This one-dimensional SDE has the qualitative behaviour that X is pushed up strongly from levels where it has already spent a lot of time; by mixing the laws of X and $-X$ one could get a process pushed away from levels where it had already spent a lot of time.

The techniques by which this SDE are handled cope equally well with the generalisation (1). We remark that, if $g(x, l) = l$, then the process X is a Brownian motion with a downward drift equal to the (local) time spent at the present position; it is known (see Barlow [3]) that the process $(L(t, X_t))_{t \geqq 0}$ is *not* a semi-martingale if X is Brownian motion, so the drift is a potentially awkward object. Nonetheless, the construction of a law $\tilde{P}$ on the canonical path space $\Omega = C(\mathbb{R}^+, \mathbb{R})$ under which the canonical process X solves (1), via Cameron-Martin-Girsanov change of measure, presents no problems. At least, it presents no problems if g is bounded, but the *unbounded* drift case, which one encounters most commonly in practice, is encountered least commonly in theory, and needs a little care.

In §2, using ideas of McGill [10] and Jeulin [8], we investigate the behaviour (as a process in the space variable a) of

$$N_a \equiv \int_0^T u_s I_{\{X_s \leqq a\}}\, dX_s$$

under Wiener measure P. Here, $T \equiv \inf\{t\colon X_t = -K\}$, for some fixed $K \in \mathbb{N}$, and u is a suitable previsible process. The main point is essentially that N *is a martingale in the excursion filtration* $(\mathscr{E}_a)$ provided u satisfies a measurability condition forcing N to be adapted, and an integrability condition (see Theorem 1). The classical Ray-Knight theorem is a corollary.

In §3, we assume we are in a position to use the Cameron-Martin-Girsanov change of measure to convert the canonical process X into a solution of (1); the key point then is that the Cameron-Martin-Girsanov density can also be identified as a change of measure *in the* $(\mathscr{E}_a)$ *filtration*, converting the P-martingale N into a $\tilde{P}$-martingale plus drift, which can be identified explicitly.

In §4, we specialise to g depending on $L(s, X_s)$ alone. Here we take the trouble to prove the existence and properties of $\tilde{P}$, and we deduce that if g is non-negative and increasing then the process $\{\zeta_{-x} \equiv L(\infty, -x)\colon x \geqq 0\}$ *is a stationary diffusion*, and $t^{-1} X_t \to -\frac{1}{\mu}$, where μ is the mean of the stationary distribution. As a special case, if

$$X_t = B_t - \int_0^t L(s, X_s)\, ds,$$

then

$$\frac{X_t}{t} \xrightarrow{\text{a.s.}} -\sqrt{\frac{\pi}{4}}.$$

The physics literature discusses numerous models for interacting random walks, usually in discrete time. One common feature of such models is that if $\Omega_N = \{\text{functions } x\colon \{0, 1, \ldots, N\} \to \mathbb{Z}^d\}$, one constructs a measure P_N on Ω_N and calculates the P_N-expectation of some functional of the path (for example, $E_N(|x_N|^2)$) and investigates the behaviour as $N \to \infty$. However, frequently the measures P_N are not consistent in the sense that $P_{N+1} \circ \pi_{N+1}^{-1} \neq P_N$, where $\pi_{N+1}\colon \Omega_{N+1} \to \Omega_N$ is the natural restriction, so there is no connection between the problem for N and the problem for $N+1$. Likewise, 'Markovian' models, where the jump probabilities depend on the times spent in the different states, can result in the sample path entering a 'cage' and being unable to continue; both of these features seem to us to be undesirable drawbacks. See the references [1, 4, 5, 9, 14, 17] for more information.

In a note submitted to Physics Letters [12], we have given some of the heuristics which motivated this work, relating it to work of Edwards [5] and Westwater [17].

2. Brownian Local Times and the Excursion Filtration

We collect here a number of definitions and well known results. Let $\Omega \equiv C(\mathbb{R}^+, \mathbb{R})$, with canonical process X and natural filtration $\{\mathscr{F}_{t+}^0\}$ made right-continuous. Let P denote Wiener measure on $(\Omega, \mathscr{F}_\infty^0)$. Now define for $a \in \mathbb{R}$

$$A(t,a) \equiv \int_0^t I_{\{X_s \leq a\}}\, ds, \qquad \tau(t,a) \equiv \inf\{u\colon A(u,a) > t\},$$

$$X(t,a) \equiv \int_0^t I_{\{X_s \leq a\}}\, dX_s, \qquad \xi(t,a) \equiv X(\tau(t,a), a),$$

$$\mathscr{F}(t,a) \equiv \sigma(\{\xi(s,a)\colon s \leq t\}), \qquad \tilde{X}(t,a) \equiv X(\tau(t,a)),$$

$$\mathscr{E}_a \equiv \mathscr{F}(\infty, a).$$

The times $\tau(t,a)$ are finite P-a.s., and are $\{\mathscr{F}_{t+}^0\}$-stopping times. Walsh [16] shows that $(\mathscr{E}_a)_{a \in \mathbb{R}}$ is a right-continuous increasing family of σ-fields. Tanaka's formula (which may be taken as the definition of semimartingale local times) says that

$$\begin{aligned} X_t \wedge a &= X_0 \wedge a + \int_0^t I_{\{X_s \leq a\}}\, dX_s - \tfrac{1}{2} L(t,a) \\ &= (X_0 \wedge a) + X(t,a) - \tfrac{1}{2} L(t,a); \end{aligned} \tag{2}$$

see Azéma-Yor [2] or Meyer [11], p. 365.

To prove the main result of this section, Theorem 1, we need to time change stochastic integrals: the following result, quoted from Rogers-Williams

[13], covers all the cases we need. The filtration $\{\mathscr{F}_t\}$ is assumed right continuous.

Theorem A. *Let A be a continuous increasing process with right-continuous inverse τ. Let $\tilde{\mathscr{F}}_t \equiv \mathscr{F}(\tau_t)$, and suppose $A_0 = 0$.*

(i) *If T is an $\{\mathscr{F}_t\}$-stopping time, then A_T is an $\{\tilde{\mathscr{F}}_t\}$-stopping time.*

(ii) *If S is an $\{\tilde{\mathscr{F}}_t\}$-stopping time, then τ_S is an $\{\mathscr{F}_t\}$-stopping time.*

(iii) *If H is an $\{\tilde{\mathscr{F}}_t\}$-previsible process, then $H \circ A$ is an $\{\mathscr{F}_t\}$-previsible process, assuming $H_0 = 0$.*

(iv) *Suppose M is an $\{\mathscr{F}_t\}$-local martingale with the property that*

$\exists \{\tilde{\mathscr{F}}_t\}$-stopping times $S_n \uparrow \infty$ such that

$M^{\tau(S_n)}$ is a uniformly integrable martingale.

Then $\tilde{M}_t \equiv M(\tau_t)$ is an $\{\tilde{\mathscr{F}}_t\}$-local martingale, and for any locally bounded $\{\tilde{\mathscr{F}}_t\}$-previsible process H, $H_0 = 0$,

$$\int_{(0,t]} H_s \, d\tilde{M}_s = \int_{(0,\tau t]} H(A_s) \, dM_s. \tag{3}$$

Using Theorem A(iv) on the local martingales $X(t, a)$ and $X(t, a)^2 - A(t, a)$, with the stopping times

$$S_n \equiv \inf\{t\colon\ \tilde{X}(t, a) < -n, \text{ or } L(\tau(t, a), a) > n\},$$

we deduce that *$\{\xi(t, a)$: $t \geqq 0)$ is an $\{\mathscr{F}(\tau(t, a))\}$-Brownian motion.* In particular, $\mathscr{E}_a$ is generated by a Brownian motion, *so every $L^2(\mathscr{E}_a)$ random variable has a stochastic integral representation.* This is the central idea behind the key lemma of McGill [10].

Now let u be a $\{\mathscr{F}_t\}$-previsible process, and for each $a \in \mathbb{R}$, let

$$\tilde{u}(t, a) \equiv u(\tau(t, a)).$$

Define

$$L(\mathscr{E}) \equiv \{\text{previsible } u \text{ such that } \tilde{u}(t, a) \text{ is } \{\mathscr{F}(t, a)\}\text{-previsible } \forall a\}.$$

In our development, the analogue of McGill's lemma is the following.

Theorem 1. *Suppose that $u \in L(\mathscr{E})$ satisfies the condition*

$$E \int_0^\infty u_s^2 I_{\{X_s \leqq a\}} \, ds < \infty \quad \textit{for all } a \in \mathbb{R}. \tag{4}$$

Then the process

$$N_a \equiv \int_0^\infty u_s I_{\{X_s \leqq a\}} \, dX_s$$

is an $\{\mathscr{E}_a\}$-martingale with

$$\langle N \rangle_a = \int_0^\infty u_s^2 I_{\{X_s \leqq a\}} \, ds.$$

Proof. Let $C_a \equiv \int_0^\infty u_s^2 I_{\{X_s \leqq a\}} ds$. Firstly, we prove that N and C are $\{\mathscr{E}_a\}$ adapted. By Theorem A(iv), for each $a \in \mathbb{R}$,

$$N_a \equiv \int_0^\infty u_s I_{\{X_s \leqq a\}} dX_s = \int_0^\infty \tilde{u}(s, a)\, d\xi(s, a)$$

which is in $\mathscr{E}_a$. Likewise,

$$C_a \equiv \int_0^\infty u_s^2 I_{\{X_s \leqq a\}} ds = \int_0^\infty \tilde{u}(s, a)^2\, ds$$

is in $\mathscr{E}_a$, since $u \in L(\mathscr{E})$.

Now fix $a < b \in \mathbb{R}$. For each $F \in L^\infty(\mathscr{E}_a)$, there is an $\{\mathscr{F}(t, a)\}$-previsible process v such that $E \int_0^\infty v_s^2\, ds < \infty$, and

$$\begin{aligned} F &= E(F) + \int_0^\infty v_s\, d\xi(s, a) \\ &= E(F) + \int_0^\infty v(A(t, a))\, I_{\{X_t \leqq a\}} dX_t, \end{aligned}$$

by Theorem A(iii)-(iv). Thus

$$\begin{aligned} E((N_b - N_a) F) &= E\left[\int_0^\infty u_s I_{\{a < X_s \leqq b\}} dX_s \cdot \int_0^\infty v(A(t, a))\, I_{\{X_t \leqq a\}} dX_t\right] \\ &= 0. \end{aligned}$$

Hence $(N_a, \mathscr{E}_a)$ is a martingale. Next, if

$$M_t \equiv \int_0^t u_s I_{\{a < X_s \leqq b\}} dX_s,$$

we have by Itô's formula

$$\begin{aligned} E[(N_b - N_a)^2 F] &= E\left(2 \int_0^\infty M_s u_s I_{\{a < X_s \leqq b\}} dX_s \cdot F\right) + E\left(\int_0^\infty u_s^2 I_{\{a < X_s \leqq b\}} ds \cdot F\right) \\ &= E\left(\int_0^\infty u_s^2 I_{\{a < X_s \leqq b\}} ds \cdot F\right), \end{aligned}$$

the first term vanishing by a similar argument to that used to prove that N is a martingale.

Hence $N_a^2 - C_a$ is an $\{\mathscr{E}_a\}$-martingale, and since C is plainly increasing and continuous $\left(\text{since } \int_0^\infty u_s^2 I_{\{X_s = a\}} ds = 0 \text{ for all } a\right)$, C is $\{\mathscr{E}_a\}$-previsible, and $C = \langle N \rangle$. □

Remark. The hypothesis that $u \in L(\mathscr{E})$ was only used to prove that N and C are adapted; without this assumption, the same argument proves that the $\{\mathscr{E}_a\}$-

optional projection of N is an $\{\mathscr{E}_a\}$-martingale, with quadratic variation process equal to the $\{\mathscr{E}_a\}$-dual previsible projection of C.

At first sight, Theorem 1 may have the appearance of a technical lemma; but to convince you that it is not, we remark as a first consequence that we are now only a few steps away from the celebrated Ray-Knight theorem on Brownian local time.

Fix $K \in \mathbb{N}$, and let $T \equiv \inf\{t: X_t \leqq -K\}$. Defining now

$$Z_a \equiv L(T, a),$$
$$Y_a \equiv \int_0^T I_{\{X_s \leqq a\}} dX_s \equiv X(T, a),$$

we use Theorem 1 to deduce a result of Jeulin [8].

Lemma 1. *The process* $\{Y_a\colon a \geqq -K\}$ *is a continuous local* $(\{\mathscr{E}_a\}, P)$*-martingale, with quadratic variation*

$$\langle Y \rangle_a = \int_{-K}^a Z_x \, dx.$$

Proof. The process $u_t \equiv I_{\{t \leqq T\}}$ is in $L(\mathscr{E})$, and satisfies (4). Indeed

$$\tilde{u}(t, a) = I_{\{\tau(t,a) \leqq T\}} = I_{\{t \leqq A(T,a)\}}$$

and $A(T, a) = \inf\{t\colon \tilde{X}(t, a) = -K\}$ is an $\{\mathscr{F}(t, a)\}$-stopping time, since $\tilde{X}(t, a)$ is adapted to the filtration $\{\mathscr{F}(t, a)\}$ of $\xi(t, a)$. Indeed, $\tilde{X}(t, a) = a \wedge 0 + \xi(t, a) - \frac{1}{2} L(\tau(t, a), a)$ and (see, for example, Ikeda-Watanabe [6], Lemma III-4.2) $\frac{1}{2} L(\tau(t, a), a) = \sup_{s \leqq t} (\xi(s, a) - a^+)^+$. The condition (4) is easy to check, so we apply Theorem 1; Y is an $\{\mathscr{E}_a\}$-martingale, with

$$\langle Y \rangle_a = \int_0^\infty I_{\{s \leqq T\}} I_{\{X_s \leqq a\}} ds = \int_{-\infty}^a Z_x \, dx$$

by the fact that local time is the occupation density. □

The Ray-Knight theorem now follows immediately.

Corollary (Ray-Knight). *Under* P, *the process* Z *solves the stochastic differential equation*

$$Z_a = 2 \int_{-K}^a (Z_x^+)^{1/2} \, dW_x + 2((a \wedge 0) + K) \qquad (a \geqq -K)$$

where W *is a Brownian motion.*

Proof. Tanaka's formula at time T says

$$\tfrac{1}{2} Z_a = (0 \wedge a) + K + Y_a. \quad \square$$

3. Local Time Process of Drifting Brownian Motion

We continue to use the notation of §2, but now we suppose given some locally bounded measurable $g\colon \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$, and a probability $\tilde{P}$ on $(\Omega, \mathscr{F}_\infty^0)$ with the property that

$$\tilde{P} \ll P \text{ on each } \mathscr{F}^0(T_n+), \quad \tilde{P}(T_n \to \infty) = 1,$$

and

$$\left.\frac{d\tilde{P}}{dP}\right|_{\mathscr{F}^0(T_n+)} = \rho(T_n) \equiv \exp(M(T_n) - \tfrac{1}{2}\langle M \rangle_{T_n}) \quad P\text{-a.s.}, \tag{5}$$

where

$$T_n \equiv n \wedge \inf\{t\colon |X_t| > n, \text{ or } L(t, X_t) > n\},$$

$$M_t \equiv \int_0^t u_s \, dX_s, \quad \rho_t \equiv \exp(M_t - \tfrac{1}{2}\langle M \rangle_t),$$

and where

$$u_s \equiv -g(X_s, L(s, X_s)). \tag{6}$$

As is well known,

$$B_t \equiv X_t + \int_0^t g(X_s, L(s, X_s))\, ds \quad \text{is a } \tilde{P}\text{-Brownian motion.} \tag{7}$$

(Since $\langle M \rangle_{T_n}$ is bounded, $\int \rho(T_n)\, dP = 1$, and the only difficulty which (5) could present is that there might be *no* measure $\tilde{P}$ which projects down in this way. For the rest of this section, though, we assume that such a $\tilde{P}$ does exist.) Write $Q = \frac{1}{2}(P + \tilde{P})$, and let $\{\mathscr{F}_t\}$ be the usual Q-augmentation of $\{\mathscr{F}^0_{t+}\}$.

We shall further assume that

$$\tilde{P}(H_{-n} < \infty) = 1 \quad \text{for all } n \in \mathbb{N}, \tag{8}$$

where $H_x \equiv \inf\{t\colon X_t = x\}$.

As in §2, the aim is to find the law of $\{Z_a\colon a \geqq -K\} \equiv \{L(T, a);\ a \geqq -K\}$, where $T \equiv H_{-K}$. We firstly show that $\tilde{P} \ll P$ on $\mathscr{F}_T$ with density ρ_T. The exponential martingale expression (5) for ρ_T allows us to see how local martingales transform when P changes to $\tilde{P}$, but *by expressing ρ_T as the exponential of an $\{\mathscr{E}_a\}$-local martingale*, we can also see how $\{\mathscr{E}_a\}$-local martingales (especially Y!) transform when P changes to $\tilde{P}$.

Lemma 2. *Let $T \equiv H_{-K}$. Then $\tilde{P} \ll P$ on $\mathscr{F}_T$, and*

$$\left.\frac{d\tilde{P}}{dP}\right|_{\mathscr{F}_T} = \rho_T. \tag{9}$$

Proof. Continuity of L and the hypothesis (5) imply that

$$P(T_n \uparrow \infty) = \tilde{P}(T_n \uparrow \infty) = 1. \tag{10}$$

Take $A \in \mathscr{F}_T$, $P(A) = 0$. Then

$$A \subseteq \{T = \infty\} \cup \{\sup_n T_n < \infty\} \cup \left(\bigcup_n (A \cap \{T \leqq T_n\})\right),$$

a union of $\tilde{P}$-null sets, by (8), (10), and (5), respectively. Thus $\tilde{P} \ll P$ on $\mathscr{F}_T$, and (9) follows from the martingale convergence theorem and (5). □

The key to expressing ρ_T as the exponential of an $\{\mathscr{E}_a\}$-local martingale is the following result.

Lemma 3. *The process* $\{N_a\colon a\in\mathbb{R}\}$ *defined by*

$$N_a \equiv \int_0^T I_{\{X_s \leqq a\}} u_s \, dX_s$$

is an $(\{\mathscr{E}_a\}, P)$*-local martingale, and*

$$\langle N, Y\rangle_a = \int_0^T I_{\{X_s \leqq a\}} u_s \, ds, \tag{11}$$

$$= -\int_{-K}^{a} f(x, Z_x)\, dx, \tag{12}$$

where

$$f(x, z) \equiv \int_0^z g(x, y)\, dy. \tag{13}$$

Proof. The process $u_t \equiv -g(X_t, L(t, X_t))$ is in $L(\mathscr{E})$ because

$$\tilde{u}(t, a) = -g(\tilde{X}(t, a), L(\tau(t, a), \tilde{X}(t, a)))$$

and $L(\tau(t, a), \tilde{X}(t, a))$ is the local time of $\tilde{X}$ at level $\tilde{X}(t, a)$ at time t, which is $\{\mathscr{F}(t, a)\}$-previsible.

Although $u I_{(0,T]}$ may not satisfy (4), if we let $u_t^k \equiv (-k) \vee (u_t) \wedge k$, and define correspondingly

$$N_a^k \equiv \int_0^T I_{\{X_s \leqq a\}} u_s^k \, dX_s,$$

then $u^k I_{(0,T]}$ satisfies (4), and, by Theorem 1, each N^k is an $(\{\mathscr{E}_a\}, P)$-martingale.

Since g is locally bounded, for each $m\in\mathbb{N}$ there is some $k\in\mathbb{N}$ such that $x\in[-K, m]$, $z\in[0, m] \Rightarrow |g(x, z)| \leqq k$. Define the $\{\mathscr{E}_a\}$-stopping place

$$\alpha_m = \inf\{x\colon Z_x > m\} \wedge m.$$

Then

$$\begin{aligned} \alpha_m > a &\Rightarrow I_{\{X_t \leqq a\}} |g(X_t, L(t, X_t))| \leqq k \quad &&\text{for } 0 \leqq t \leqq T \\ &\Rightarrow I_{\{X_t \leqq a\}} (u_t - u_t^k) = 0 &&\text{for } 0 \leqq t \leqq T. \end{aligned} \tag{14}$$

Now the stochastic integrands $I_{\{X_t \leqq a\}} u_t$ and $I_{\{X_t \leqq a\}} u_t^k$ can be approximated by simple functions which (by (14)) may be assumed to agree on $[0, T]$ when $\alpha_m > a$. Thus the stochastic integrals N_a and N_a^k are approximated by Riemann sums which, on the event $\{\alpha_m > a\}$, agree. Hence $\alpha_m > a \Rightarrow N_a = N_a^k$, and $N_{a \wedge \alpha_m} = N^k_{a \wedge \alpha_m}$, and N is an $\{\mathscr{E}_a\}$-local martingale, reduced by the α_m.

Finally, it follows from Theorem 1 that

$$\langle N^k, Y\rangle_a = \int_0^T u_s^k I_{\{X_s \leqq a\}} \, ds,$$

from which (11) follows, and for (12), by (an extension of) the occupation density formula for local time

$$\int_0^T I_{\{X_s \leqq a\}} g(X_s, L(s, X_s))\, ds = \int_{-\infty}^{a} dx \left(\int_0^T g(x, L(s, x)), L(ds, x)\right)$$
$$= \int_{-K}^{a} f(x, Z_x)\, dx. \quad \square$$

We can now very easily easily obtain the analogue of the Ray-Knight theorem for the law $\tilde{P}$.

Theorem 2. *Under $\tilde{P}$ the local time process $(Z_a)_{a \geqq -K} \equiv (L(T, a))_{a \geqq -K}$ satisfies the stochastic differential equation*

$$Z_a = \int_{-K}^{a} 2(Z_x^+)^{\frac{1}{2}}\, dW_x + 2\int_{-K}^{a} (I_{\{x \leqq 0\}} - f(x, Z_x))\, dx. \tag{15}$$

Proof. If we take a measure $\bar{P}$ on $(\Omega, \mathscr{F}_\infty^0)$ which is absolutely continuous with respect to P, defined by $(d\bar{P}/dP) = \rho_T$, then under $\bar{P}$,

$$B_t \equiv X_t + \int_0^{t \wedge T} g(X_s, L(s, X_s))\, ds \quad \text{is Brownian motion,}$$

and $\bar{P}$ agrees with $\tilde{P}$ on $\mathscr{F}_T$. But (Z_a) is $\mathscr{F}_T$-measurable, so it suffices to obtain the $\bar{P}$-law of Z_a. However,

$$\frac{d\bar{P}}{dP} = \rho_T = \exp(M_T - \tfrac{1}{2}\langle M\rangle_T)$$
$$= \exp(N_\infty - \tfrac{1}{2}\langle N\rangle_\infty)$$

so, by the Cameron-Martin theorem, under $\bar{P}$, $\tilde{Y}_a \equiv Y_a - \langle N, Y\rangle_a$ is a local martingale, with quadratic variation $\langle Y\rangle_a = \int_{-K}^{a} Z_x\, dx$. From (2) and (12),

$$Z_a = 2Y_a + 2\int_{-K}^{a} I_{\{x \leqq 0\}}\, dx$$
$$= 2\tilde{Y}_a + 2\int_{-K}^{a} (I_{\{x \leqq 0\}} - f(x, Z_x))\, dx,$$

from which (15) follows immediately. $\square$

As a first application, we consider the diffusion given by the solution of the SDE

$$dY_t = \sigma(Y_t)\, dB_t + b(Y_t)\, dt,$$

where σ is C^1 and everywhere positive, and σ, b are such that $Y_t \to -\infty$ a.s. as $t \to \infty$. Defining $h \in C^2$ by $h' = \frac{1}{\sigma}$, $h(0) = 0$, and letting $X_t = h(Y_t)$, then

$$dX_t = dB_t - g(X_t)\, dt,$$

where

$$g(x) = \left(\tfrac{1}{2}\sigma' - \frac{b}{\sigma}\right) \circ h^{-1}(x).$$

If $\{L(t, a): t \geqq 0, a \in \mathbb{R}\}$ is the (semimartingale) local time of Y (related to the occupation density $\{\tilde{L}(t, a): t \geqq 0, a \in \mathbb{R}\}$ by $L(t, a) = \sigma(a)^2 \tilde{L}(t, a)$), then

$$L(t, a) = \Lambda(t, h(a))\, \sigma(a)^2\, h'(a) = \Lambda(t, h(a))\, \sigma(a),$$

where Λ is the (semimartingale) local time of X. We can apply Theorem 2 to deduce the law of $(\Lambda(T, x))_{x \geqq h(-K)}$, and by the familiar techniques of scale and speed we can translate this back into a statement about the law of $(L(T, x))_{x \geqq -K}$. We find that, if $Z_a \equiv L(T, a)$,

$$Z_a = \int_{-K}^{a} 2(Z_x^+)^{\frac{1}{2}}\, dW_x + 2 \int_{-K}^{a} [I_{\{x \leqq 0\}} + b\, \sigma^{-2}(x)\, Z_x]\, dx.$$

4. Homogeneous Drift: Asymptotic Behaviour

We now specialise by assuming that g does not depend on X, and that g is $\geqq 0$. Thus we suppose given some continuous $g: \mathbb{R}^+ \to \mathbb{R}^+$, and define

$$f(x) \equiv \int_0^x g(y)\, dy.$$

Theorem 3. *Suppose* $\lim_{x \to \infty} f(x) = 1 + 2\varepsilon > 1$. *Then there exists a probability* $\tilde{P}$ *on* $(\Omega, \mathscr{F}_\infty^0)$ *such that*

$$X_t = B_t - \int_0^t g(L(s, X_s))\, ds, \tag{16}$$

where B *is a* $\tilde{P}$*-Brownian motion, and*

$$\tilde{P}(X_t \to -\infty \ \text{as}\ t \to \infty) = 1. \tag{17}$$

If $\zeta_x \equiv L(\infty, x)$, *then* $\{\zeta_{-x}: x \geqq 0\}$ *is a stationary diffusion on* $\mathbb{R}^+$ *with generator*

$$\mathscr{G} = 2z \frac{d^2}{dz^2} + 2(1 - f(z)) \frac{d}{dz} \tag{18}$$

and invariant density

$$\pi(z) = c \cdot \exp\left[-\int_0^z f(y) \frac{dy}{y} \right]. \tag{19}$$

Proof. The existence of $\tilde{P}$ is no problem if g is bounded; this is a simple application of the Cameron-Martin-Girsanov change of measure (see, for example, Ikeda-Watanabe [6], IV.4).

To handle the case of unbounded g, let $g_n(x) \equiv I_{[0,n]}(x)\, g(x)$, where we only consider n so large that $f(n) > 1 + \varepsilon$. Then, since g_n is bounded, there exists a measure $\tilde{P}_n$ on $(\Omega, \mathscr{F}_\infty^0)$ defined by

$$\left. \frac{d\tilde{P}_n}{dP} \right|_{\mathscr{F}_t^0} = \exp\{M_t^{(n)} - \tfrac{1}{2} \langle M^{(n)} \rangle_t\}, \tag{20}$$

where $M_t^{(n)} = -\int_0^t g_n(L(s, X_s))\,dX_s$, and under which

$$B_t^{(n)} \equiv X_t + \int_0^t g_n(L(s, X_s))\,ds \tag{21}$$

is a Brownian motion. Now if $T_n = n \wedge S_n$, where $S_n \equiv \inf\{t: L(t, X_t) > n\}$, then on $[0, T_n]$,

$$B_t^{(n)} = X_t + \int_0^t g(L(s, X_s))\,ds, \qquad M_t^{(n)} = M_t,$$

where $M_t \equiv -\int_0^t g(L(s, X_s))\,ds$, and $\tilde{P}_{n+1}$ agrees with $\tilde{P}_n$ on $\mathscr{F}^0(T_n)$.

The following result (see Stroock-Varadhan [15], Theorem 1.3.5) applies to the present situation.

Theorem B. *Let $\Omega = C(\mathbb{R}^+, \mathbb{R})$ be the canonical path space with its canonical filtration $\{\mathscr{F}_t^0\}$, and let (T_n) be an increasing sequence of stopping times. Suppose there is a sequence $(\tilde{P}_n)$ of probabilities on $(\Omega, \mathscr{F})$ such that*

(i) $\tilde{P}_{n+1}$ *agrees with* $\tilde{P}_n$ *on* $\mathscr{F}^0(T_n)$;

(ii) *for each* $t \in (0, \infty)$,

$$\tilde{P}_n(T_n < t) \to 0 \quad \text{as } n \to \infty.$$

Then there exists a probability $\tilde{P}$ on $(\Omega, \mathscr{F})$ such that for each n,

$$\tilde{P} = \tilde{P}_n \quad \text{on} \quad \mathscr{F}^0(T_n).$$

Thus we have only to verify the second condition to prove the existence of $\tilde{P}$ satisfying (5). This will follow immediately from the two statements:

$$\text{for each } K, \quad \tilde{P}_n(S_n < H_{-K}) \to 0 \quad \text{as } n \to \infty; \tag{22}$$

for each $t \in (0, \infty)$ and $\varepsilon > 0$, there is some K such that (23)

$$\tilde{P}_n(H_{-K} < t) < \varepsilon \quad \text{for all } n.$$

We firstly prove (22). For each n, the measure $\tilde{P}_n$ satisfies (5) by construction, and (8) since $g_n \geqq 0$, so we can apply Theorem 2. Letting Z denote the solution of the SDE

$$Z_a = \int_{-K}^a 2(Z_x^+)^{\frac{1}{2}}\,dW_x + 2\int_{-K}^a (I_{\{x \leqq 0\}} - f(Z_x))\,dx, \quad (a \geqq -K) \tag{24}$$

and letting $Z^{(n)}$ denote the solution of

$$Z_a^{(n)} = \int_{-K}^a 2(Z_x^{(n)+})^{\frac{1}{2}}\,dW_x + 2\int_{-K}^a (I_{\{x \leqq 0\}} - f(n \wedge Z_x))\,dx,$$

then $Z = Z^{(n)}$ on the event $\{\sup\{Z_a: a \geqq -K\} < n\}$, and under $\tilde{P}_n$, $\{L(H_{-K}, a): a \geqq -K\}$ has, by Theorem 2, the law of $Z^{(n)}$. Thus

$$\tilde{P}_n(\sup L(H_{-K}, a) \geqq n) = P(\sup_a Z_a \geqq n) \to 0 \quad \text{as } n \to \infty,$$

proving (22).

To prove (23), we use a stochastic comparison theorem (see, for example, Ikeda-Watanabe [6], VI.1) to see that (for any n)

$$P(Z_a^{(n)} \geqq Z_a \text{ for all } a) = 1.$$

Now

$$H_{-K} = \int_{-K}^{\infty} L(H_{-K}, x)\, dx,$$

so for any n,

$$P_n(H_{-K} < t) \leqq P\left(\int_{-K}^{\infty} Z_a\, da < t\right) \to 0$$

as $K \to \infty$, proving (23), and the existence of $\tilde{P}$. The property (16) is immediate, and $\tilde{P}(H_{-K} < \infty) = 1$ for all $K \in \mathbb{N}$, since g is non-negative.

To prove that $\tilde{P}(X_t \to -\infty) = 1$, we shall prove that for all $a < b \in \mathbb{R}$,

$$\tilde{P}(\sup_n \sup_{a \leqq x \leqq b} L(H_{-n}, x) < \infty) = 1. \tag{25}$$

The result follows from this, because if we define $\tau_k = \inf\{t > H_{-k}\colon X_t = 0\}$, and if every τ_k were finite, then for infinitely many k, $|X(\tau_k + t)| \leqq 1$ for all $t \in [0, 1]$, since X is a Brownian motion with a drift which (when X is in $[-1, 1]$) is bounded, by (16) and (25). This contradicts (25), because (25) implies

$$\sup \int_0^{H_{-n}} I_{[-1,1]}(X_s)\, ds < \infty \quad \tilde{P}\text{-a.s.}.$$

To prove (25), let $\zeta_a^{(n)} \equiv L(H_{-n}, a)$, and let U be a random variable independent of X with law π. Now $\zeta^{(n)}$ solves the SDE

$$\zeta_a^{(n)} = \int_{-n}^{a} 2(\zeta_x^{(n)})^{\frac{1}{2}}\, dW_x + 2\int_{-n}^{a} (I_{\{x \leqq 0\}} - f(\zeta_x^{(n)}))\, dx,$$

and if $\bar{\zeta}^{(n)}$ is the (pathwise unique) solution to

$$\bar{\zeta}_a^{(n)} = U + \int_{-n}^{a} 2(\bar{\zeta}_x^{(n)})^{\frac{1}{2}}\, dW_x + 2\int_{-n}^{a} (I_{\{x \leqq 0\}} - f(\zeta_x^{(n)}))\, dx,$$

then by the stochastic comparison theorem,

$$\tilde{P}(\bar{\zeta}_a^{(n)} \geqq \zeta_a^{(n)} \text{ for all } a \geqq -n) = 1.$$

But $\{\bar{\zeta}_a^{(n)}\colon -n \leqq a \leqq 0\}$ is a stationary diffusion with generator $\mathscr{G}$, so for each $\lambda > 0$,

$$\begin{aligned}\tilde{P}(\sup_n \sup_{a \leqq x \leqq b} L(H_{-n}, x) > \lambda) &= \uparrow \lim_{n \to \infty} \tilde{P}(\sup_{a \leqq x \leqq b} L(H_{-n}, x) > \lambda)\\ &\leqq \tilde{P}(\sup_{a \leqq x \leqq b} \bar{\zeta}_x^{(n)} > \lambda), \text{ which is the same for all } n > -a,\end{aligned}$$

and which tends to 0 as $\lambda \to \infty$. This proves (25), from which follows (17).

Consequently, $\zeta_x = \uparrow \lim_{n\to\infty} \zeta_x^{(n)}$ is a continuous process, and for fixed $K \in \mathbb{N}$, $\{\zeta_x : x \geqq -K\}$ solves the same SDE (24) as $\zeta^{(n)}$, with different initial conditions (each $\{\zeta_x^{(n)} : x \geqq -K\}$ solves the same martingale problem, and dominated convergence implies the same is true in the limit). Now the law of $\zeta^{(n)}_{-K}$ is the law at time $n-K$ of a diffusion with generator $\mathscr{G}$, started at 0. Since $\mathscr{G}$ has an invariant distribution π, this invariant measure is the limit (in norm) of the laws $P_t(0, \cdot)$ (where P_t is the transition semigroup of the diffusion with generator $\mathscr{G}$), so $\{\zeta_x : -K \leqq x \leqq 0\}$ is a stationary diffusion with generator $\mathscr{G}$. Since P_t is reversible with respect to π, (see, for example, Itô-McKean [7]), the final statement of the Theorem is proved. □

The final result is an ergodic theorem which gives the speed at which X tends to $-\infty$.

Theorem 4. *Suppose that g is increasing. Then $\tilde{P}$-a.s.,*

$$\frac{X_t}{t} \to -\frac{1}{\mu} \tag{26}$$

where μ is the mean of the invariant density π (19).

Remarks. The condition that g is increasing implies that $\lim_{t\to\infty} f(t) = \infty$, and hence the invariant distribution *has* a mean. The proof of Theorem 4 rests on the following lemma, of interest in its own right.

Lemma 4. *For each $\lambda > 0$, and each $K > 0$*

$$\tilde{P}(\sup_{t \geqq 0} \{X(H_{-K}+t) - X(H_{-K})\} > \lambda) \leqq \tilde{P}(\sup_t X_t > \lambda). \tag{27}$$

Proof. Fixing $K > 0$, abbreviating H_{-K} to H, and defining the process $\tilde{X}$ by

$$\tilde{X}_t \equiv X(H+t) + K \qquad (t \geqq 0), \tag{28}$$

we see that

$$\tilde{X}_t = \tilde{B}_t - \int_0^t \tilde{g}(\tilde{X}_s, \tilde{L}(s, \tilde{X}_s))\, ds,$$

where

$$\tilde{B}_t \equiv B(H+t) - B(H), \qquad \tilde{L}(s,x) = L(H+s, -K+x) - L(H, -K+x),$$

and

$$\begin{aligned} \tilde{g}(x,l) &= g(L(H, -K+x) + l) \\ &\geqq g(l), \end{aligned}$$

since g is increasing, and $\tilde{g}(x,l) = g(l)$ if $x \leqq 0$. Thus $\tilde{X}$ satisfies an SDE of the type considered in §3. Letting $\tilde{\zeta}_x \equiv \tilde{L}(\infty, x)$, by Theorem 2 and the comparison theorem

$$\begin{aligned} \tilde{P}(\sup_{t \geqq 0} \{X(H+t) - X(H)\} > \lambda) &= \tilde{P}(\tilde{\zeta}_\lambda > 0) \\ &\leqq \tilde{P}(\zeta_\lambda > 0), \end{aligned}$$

since the drift in the SDE for $\tilde{\zeta}$ is dominated by the drift in the SDE for ζ, and equal to it in $(-\infty, 0)$ (and hence the law of $\tilde{\zeta}_0$ is the law of ζ_0, the invariant distribution). □

Proof of Theorem 4. For each $K>0$, let $\sigma_{-K} \equiv \sup\{t: X_t = -K\}$. For each $\varepsilon>0$,

$$\tilde{P}(H_{-(1+\varepsilon)K} < \sigma_{-K}) \leqq \tilde{P}(\sup_t X_t > \varepsilon K) \tag{29}$$

by the inequality (27).

Now

$$\begin{aligned}\sigma_{-K} &= \int L(\sigma_{-K}, a)\, da \\ &= \int_{-K}^{\infty} \zeta_a\, da + \int_{-\infty}^{-K} L(\sigma_{-K}, a)\, da \\ &= \int_{-K}^{0} \zeta_a\, da + \int_{-\infty}^{-K} L(\sigma_{-K}, a)\, da + \int_0^{\infty} \zeta_a\, da,\end{aligned}$$

so that

$$\begin{aligned}\liminf_{K\to\infty} \frac{1}{K}\sigma_{-K} &\geqq \lim_{K\to\infty} \frac{1}{K} \int_{-K}^{0} \zeta_a\, da \\ &= \mu \equiv \int_0^{\infty} x\,\pi(x)\, dx,\end{aligned} \tag{30}$$

by the well known ergodic theorem for positive recurrent one-dimensional diffusions (see, for example, Itô-McKean [7], § 6.8). Similarly

$$H_{-K} = \int_{-K}^{\infty} L(H_{-K}, a)\, da \leqq \int_{-K}^{\infty} \zeta_a\, da$$

so

$$\limsup_{K\to\infty} \frac{1}{K} H_{-K} \leqq \mu. \tag{31}$$

We shall show that

$$\tilde{E}[\sup_t X_t] < \infty, \tag{32}$$

from which, by the first Borel-Cantelli lemma and (29) it follows that for each $\varepsilon>0$, P-a.s. $H_{-(1+\varepsilon)K} \geqq \sigma_{-K}$ for all large enough K. Together with (30) and (31), this implies that P-a.s.

$$\lim_{K\to\infty} \frac{1}{K}\sigma_{-K} = \lim_{K\to\infty} \frac{1}{K} H_{-K} = \mu. \tag{33}$$

To prove (32), notice that $\sup_t X_t = \inf\{x>0: Z_x = 0\} \equiv \tau$ is the first hitting time of zero of a diffusion with generator $2x\frac{d^2}{dx^2} - 2f(x)\frac{d}{dx}$. Familiar scale and speed calculations reveal that for $a>0$

$$E^a(\tau) = \tfrac{1}{2} \int_0^{\infty} s(a \wedge x) \frac{dx}{x\, s'(x)},$$

where s is the scale function, $s'(x)=\exp\left(\int_0^x f(t)\,\frac{dt}{t}\right)=\frac{c}{\pi(x)}$, and E^a denotes expectation with starting point a for the diffusion $\{Z_x\colon x\geqq 0\}$. But the law of Z_0 is the invariant distribution with density π, so

$$\begin{aligned}
E(\sup_t X_t) &= \int_0^\infty da\,\pi(a)\,E^a(\tau)\\
&= c\int_0^\infty \frac{da}{s'(a)}\int_0^\infty \frac{dx}{x\,s'(x)}\int_0^x I_{\{y\leqq a\}}\,s'(y)\,dy\\
&= c\int_0^\infty s'(y)\,dy\left(\int_y^\infty \frac{da}{s'(a)}\right)\left(\int_y^\infty \frac{dx}{x\,s'(x)}\right)\\
&= c\int_0^\infty dy\left(\int_y^\infty \exp\left(-\int_y^a \frac{f(t)\,dt}{t}\right)da\right)\left(\int_y^\infty \frac{dx}{x\,s'(x)}\right)
\end{aligned}$$

and since $\lim_{t\to\infty} f(t)>2$ by hypothesis, the integral is finite.

Finally, we prove (26). For each $\varepsilon>0$, a.s. for all large enough K

$$\sigma_{-K/(1+\varepsilon)}<H_{-K}<\sigma_{-K}<H_{-K(1+\varepsilon)}$$

so for $t\in(H_{-K},\sigma_{-K})$ (for large enough K)

$$-K(1+\varepsilon)<X_t<\frac{-K}{(1+\varepsilon)}$$

and hence for $t\in(H_{-K},H_{-K-1})$

$$-K(1+\varepsilon)<X_t<\frac{-K}{(1+\varepsilon)}.$$

Thus for $t\in(H_{-K},H_{-K-1})$

$$\frac{X_t}{t}<\frac{-K}{(1+\varepsilon)\,H_{-K}},\qquad \frac{X_t}{t}>\frac{-K(1+\varepsilon)}{H_{-K-1}},$$

from which (26) follows, by (33). □

Example. Take $g(x)=a\,x^\alpha$, where a and α are positive constants. Then, writing β for $1+\alpha$

$$f(x)=\frac{a\,x^\beta}{\beta}$$

and the invariant density π solves the adjoint differential equation

$$\mathcal{G}^*\pi\equiv\frac{d^2}{dz^2}[2\,z\,\pi(z)]-\frac{d}{dz}[2(1-f(z))\,\pi(z)]=0.$$

This can be solved easily: we find

$$\pi(z)=c\cdot\exp(-a\,z^\beta/\beta^2).$$

Hence we can calculate moments of the invariant distribution in this instance:

$$\int_0^\infty z^k \pi(z)\,dz = \left(\frac{\beta^2}{a}\right)^{k/\beta} \Gamma\left(\frac{k+1}{\beta}\right) \Big/ \Gamma\left(\frac{1}{\beta}\right).$$

In particular, if $a=\alpha=1$, the first moment is $\sqrt{(4/\pi)}$. Hence from Theorem 4

$$\frac{X_t}{t} \to -\sqrt{\frac{\pi}{4}} \quad \text{a.s.}.$$

5. Computer Simulation

Rather to our surprise, a very rough-and-ready computer simulation by Peter Townsend and one of us (D.W.) gave results in very close agreement with our theorems.

To simulate the equation

$$dX = dB - L(t, X_t)\,dt \tag{34}$$

a first-order Euler method was used, with time-increment of size h. The X-values were restricted to multiples of $Kh^{\frac{1}{2}}$, where K is a constant not much bigger than 1. The value of $L(t,x)$, where t is a multiple of h and x a multiple of $Kh^{\frac{1}{2}}$, was taken to be

$$Nh/(Kh^{\frac{1}{2}})$$

where N is the number of multiples of h less than or equal to t at which X is at x. The following equation was taken as the discretized form of (34):

$$X_{t+h} - X_t = \Delta,$$

where Δ takes one of the values $-Kh^{\frac{1}{2}}$, 0, $Kh^{\frac{1}{2}}$, and it is arranged that

$$E(\Delta) = -L(t, X_t)\,h, \qquad \operatorname{Var}(\Delta) = h.$$

Acknowledgements. Our thanks to Peter Townsend for his part in the simulation; and to Ian Davies and Aubrey Truman for many interesting discussions.

References

1. Amit, D.J., Parisi, G., Peliti, L.: Asymptotic behaviour of the "true" self-avoiding random walk. Phys. Rev. B, **27**, 1635–1645 (1983)
2. Azéma, J., Yor, M. (eds.) Temps locaux. Astérisque **52–53**. Société Mathématique de France 1978
3. Barlow, M.T.: $L(B_t, t)$ is not a semimartingale. Séminaire de Probabilités XVI, pp. 206–211. Berlin Heidelberg New York: Springer 1982
4. Duxbury, P.M., de Queiroz, S.L.A., Stinchcombe, R.B.: A comparative study of interacting random-walk models. J. Phys. A, **17**, 2113–2118 (1984)
5. Edwards, S.F.: The statistical mechanics of polymers with excluded volume. Proc. Phys. Sci., **85**, 613–624 (1965)

6. Ikeda, N., Watanabe, S.: Stochastic differential equations and diffusion processes. Amsterdam Tokyo: North Holland-Kodansha 1981
7. Itô, K., McKean, H.P.: Diffusion processes and their sample paths. Berlin Heidelberg New York: Springer 1965
8. Jeulin, T.: Ray-Knight's theorem on Brownian local times and Tanaka's formula. In: Çinlar, E., Chung, K.L., Getoor, R.K. (eds.) Seminar on Stochastic Processes, 1983, pp. 131–142. Boston: Birkhauser 1984
9. Kremer, K., Lyklema, J.W.: Indefinitely growing self-avoiding walk. Phys. Rev. Lett. **54**, 267–269 (1985)
10. McGill, P.: Markov properties of diffusion local times; a martingale approach. Adv. Appl. Prob. **14**, 789–810 (1982)
11. Meyer, P.A.: Un cours sur les intégrales stochastiques. Séminaire de Probabilités X, pp. 245–400. Berlin Heidelberg New York: Springer 1976
12. Norris, J.R., Rogers, L.C.G., Williams, D.: An excluded volume problem for Brownian motion. Phys. Letters A **112**, 16–18 (1985)
13. Rogers, L.C.G., Williams, D.: Diffusions, Markov processes, and martingales, vol. 2. Chichester: Wiley 1987
14. Stella, A.L., de Queiroz, S.L.A., Duxbury, P.M., Stinchcombe, R.B.: Series study of the one-dimensional "true" self-avoiding walk. J. Phys. A **17**, 1903–1912 (1984)
15. Stroock, D.W., Varadhan, S.R.S.: Multidimensional diffusion processes. Berlin Heidelberg New York: Springer 1979
16. Walsh, J.B.: Excursions and local time. Astérisque **52–53**, pp. 159–192. Société Math. de France 1978
17. Westwater, J.: On Edwards' model for long polymer chains. Comm. Math. Phys. **72**, 131–174 (1980)

Received July 17, 1985; in revised form September 5, 1986