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The Martin Potential Kernel for Improperly Essential Chains*

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0. Introduction

The Martin potential kernel $K(x, y)$ for a Markov chain X_n, $n=0, 1, \dots$ with discrete state space was defined in [3]. The importance of this function is due to the fact that lim $K(x, X_n)$ exists a.s., and the limiting random variable has close connections with the σ-field of invariant sets of the chain.

Similar theories are known for more general Markov processes but usually it is assumed that transition densities are defined.

In the present paper, we develop a theory of the Martin potential kernal for improperly essential Markov chains, with no assumption that transition densities exist. We prove the appropriate limit theorem, using a method of Hunt [5]. We identify this limit in a way which helps to explain the intimate connection between the Martin potential kernel and the invariant field.

1. Inessential Sets

We begin by introducing notations and definitions from the general theory of Markov chains. Let $(X, \mathscr{B})$ be a measurable space. Let Ω denote the space of countable sequences $(x_0, x_1, \dots)$ of elements of X, and let $\mathscr{F}$ denote the σ-field of subsets of Ω induced by the usual product topology. For each nonnegative integer n, define X_n mapping Ω into X by letting $X_n(x_0, x_1, \dots)=x_n$. Let the "shift" function θ mapping Ω into itself be defined by $\theta(x_0, x_1, \dots)=(x_1, x_2, \dots)$.

Let p be a real valued function on $X \times \mathscr{B}$, such that for each $x \in X$, $p(x, \cdot)$ is a probability measure on $\mathscr{B}$, and for each $A \in \mathscr{B}$, $p(\cdot, A)$ is a measurable function on X.

Let $A_0, A_1, \dots, A_n$ be elements of $\mathscr{B}$. A subset Λ of Ω of the form

$$\Lambda=\{\omega \in \Omega \mid X_0(\omega) \in A_0, \dots, X_n(\omega) \in A_n\}$$

is called a *cylinder*, which we usually express in an abbreviated form:

$$\Lambda=\{X_0 \in A_0, \dots, X_n \in A_n\}.$$

By definition, $\Lambda \in \mathscr{F}$. Let μ be a probability measure on $\mathscr{B}$. We define a probability measure P_μ on $\mathscr{F}$ by defining it on the cylinder Λ given above by the expression

$$P_\mu(\Lambda)=\int_{A_n} \cdots \int_{A_0} \mu(dx_0)\, p(x_0, dx_1) \dots p(x_{n-1}, dx_n).$$

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(The integral is read from inside out.) In the case where $\mu(\{x\})=1$ for some point $x\in X$, we will write P_x instead of P_μ.

With respect to the probability space $(\Omega, \mathscr{F}, P_\mu)$, the sequence $\{X_n;\, n\geqq 0\}$ is a temporally homogeneous Markov chain, with $P_\mu(X_0\in A)=\mu(A)$ for $A\in\mathscr{B}$, and for each nonnegative integer n, $P_\mu(X_{n+1}\in A\mid X_n)=p(X_n, A)$ almost surely with respect to the measure P_μ. We note without proof the following expression regarding the shift function θ: for $A\in\mathscr{F}$, $P_\mu(\theta^{-n}A\mid X_1,\dots,X_n)=P_{X_n}(A)$ P_μ-a.s. If $A\in\mathscr{B}$, we define

$$\{X_n\in A \text{ i.o.}\}=\bigcap_{m\geqq 0}\bigcup_{n\geqq m}\{X_n\in A\}$$

where "i.o." means "infinitely often". A set $A\in\mathscr{B}$ is *inessential* if

$$P_x(X_n\in A \text{ i.o.})=0$$

for all $x\in X$; otherwise, A is *essential.* An essential set which is the union of a countable collection of inessential sets is said to be *improperly essential.*

We define a set $A\in B$ to be *strongly transient* if for some $M<\infty$,

$$\sum_{n=0}^{\infty} P_x(X_n\in A)<M$$

for all $x\in A$. This phrase was used in [6] to describe sets $A\in\mathscr{B}$ such that

$$\sum P_x(X_n\in A)<\infty \qquad \text{for all } x\in X.$$

Our slightly strengthened version has the advantage that if A is a strongly transient set, then for any probability measure μ on $\mathscr{B}$, $\sum P_\mu(X_n\in A)<\infty$.

We define a *time* to be a measurable function $\tau\colon \Omega\to\{0,1,\dots,\infty\}$ satisfying

$$\begin{aligned}\tau\theta&=\tau-1 && 0<\tau<\infty\\ &=\infty && \tau=0,\infty.\end{aligned}$$

If $A\in\mathscr{B}$, then let

$$\tau_A(\omega)=\sup\{n\mid X_n(\omega)\in A\}$$

when the sets on the right are not all empty, and $\tau_A(\omega)=\infty$ when they are. τ_A is a time, called the *exit time from A*. Not every time is an exit time.

If τ is a time, define $X_\tau\colon \Omega\to X$ by setting $X_\tau(\omega)=X_n(\omega)$ if $\tau(\omega)=n$. $X_\tau(\omega)$ is undefined if $\tau(\omega)=\infty$. X_τ is measurable, because for $A\in\mathscr{B}$,

$$\{X_\tau\in A\}=\bigcup_{n=0}^{\infty}\{X_n\in A\}\cap\{\tau=n\}.$$

Lemma 1. *Let τ be a time, n a nonnegative integer, and select $A\in\mathscr{B}$.*

a) $\theta^{-n}\{\tau<\infty\}=\{n\leqq\tau<\infty\}$.

b) $X_\tau\theta=X_\tau$ if $0<\tau<\infty$.

c) $\theta^{-1}\{X_\tau\in A\}=\{X_\tau\in A\}\cap\{0<\tau\}$.

d) $\int_A P_\mu(X_n\in dy)\, P_y(\tau=0)=P_\mu(X_\tau\in A,\, \tau=n)$.

Proof. For any nonnegative integer k

$$\theta^{-1}\{\tau=k\}=\{\tau\,\theta=k\}$$
$$=\{\tau=k+1\}$$

and from this, a) follows directly.

Suppose $0<k<\infty$, $\tau(\omega')=k$, and $X_k(\omega)=x_k$. Then $X_\tau(\omega)=x_k$, $\tau(\theta(\omega))=k-1$, and $X_{k-1}(\theta(\omega))=x_k$. Hence $X_\tau(\theta(\omega))=x_k$. This proves b), and c) follows directly from a) and b). Finally, we note

$$\begin{aligned}\int_A P_\mu(X_n\in dy)\,P_y(\tau=0)&=E_\mu(P_{X_n}(\tau=0);\,X_n\in A)\\&=E_\mu(P_\mu(\theta^{-n}\{\tau=0\}\,|\,X_n);\,X_n\in A)\\&=E_\mu(P_\mu(\tau=n\,|\,X_n);\,X_n\in A)\\&=P_\mu(\{X_n\in A\}\cap\{\tau=n\})\end{aligned}$$

where $E_\mu(\;;A)$ denotes expectation with respect to P_μ, over the set $A\in\mathscr{F}$. This completes the proof.

Lemma 2. *Let τ be a time. Choose $\varepsilon>0$ and a nonnegative integer k. The set*

$$A=\{x\,|\,P_x(\tau=k)>\varepsilon\}$$

is strongly transient.

Proof. Define $u_0(x)=P_x(k\leqq\tau<\infty)$ and for $j\geqq 0$, let

$$\begin{aligned}u_j(x)&=\int P_x(X_j\in dy)\,u_0(y)\\&=E_x(P_x(\theta^{-j}\{k\leqq\tau<\infty\}\,|\,X_j))\\&=P_x(k+j\leqq\tau<\infty).\end{aligned}$$

Hence, $u_0(x)-u_1(x)=P_x(\tau=k)$. For any n,

$$\begin{aligned}u_0(x)&\geqq u_0(x)-u_n(x)\\&=\sum_{j=0}^{n-1}\int P_x(X_j\in dy)(u_0(y)-u_1(y))\\&\geqq\sum_{j=0}^{n-1}\int P_x(X_j\in dy)(u_0(y)-u_1(y))\\&\geqq\varepsilon\sum_{j=0}^{n-1}P_x(X_j\in A).\end{aligned}$$

Hence

$$\sum_{j=0}^{\infty}P_x(X_j\in A)\leqq\frac{u_0(x)}{\varepsilon}\leqq\frac{1}{\varepsilon}$$

and the proof is complete.

Corollary. *Let τ be a time. The set*

$$A=\{x\,|\,P_x(\tau<\infty)>0\}$$

is the union of a countable collection of strongly transient sets.

Proof. For each pair of integers $n>0$, $k\geqq 0$, let

$$A_{nk}=\left\{x\,|\,P_x(\tau=k)>\frac{1}{n}\right\}.$$

A_{nk} is strongly transient, and $A=\bigcup A_{nk}$.

Theorem 1. *An inessential set is the union of a countable collection of strongly transient sets.*

Proof. Let A be inessential, and let τ be the exit time from A. Clearly $P_x(\tau<\infty)=1$ for $x\in A$. Now define $B=\{x\,|\,P_x(\tau<\infty)>0\}$. $A\subseteq B$ and B is the union of a countable collection of strongly transient sets. Therefore, so is A.

Corollary. *An improperly essential set is the union of a countable collection of strongly transient sets.*

2. The Invariant Field

The invariant field $\mathscr{G}$ is the class of sets $\Lambda\in\mathscr{F}$, such that $\theta^{-1}\Lambda=\Lambda$. It is easy to verify that $\mathscr{G}$ is indeed a σ-field. A characterization of $\mathscr{G}$ has been given in [1]. In this section we will derive a new one.

If $Z_1, Z_2, \ldots$ is a sequence of random variables over $(\Omega,\mathscr{F})$, we let $\mathscr{B}(Z_1, Z_2, \ldots)$ denote the smallest σ-field with respect to which the sequence is measurable. We let

$$\mathscr{B}_\infty(Z_1, Z_2, \ldots)=\bigcap_{n\geqq 0}\mathscr{B}(Z_n, Z_{n+1}, \ldots)$$

which we call the *tail field* of the sequence $Z_1, Z_2, \ldots$. We will show how under certain conditions, $\mathscr{G}$ is the tail field of a sequence of random variables.

A *time sequence* is a sequence of times $\tau_1, \tau_2, \ldots$ satisfying

a) $\{\tau_k<\infty\}\subseteq\{\tau_{k+1}<\infty\}$ all k

b) $\tau_k\leqq\tau_{k+1}$ on $\{\tau_k<\infty\}$, all k

c) $\bigcup_{k\geqq 1}\{\tau_k<\infty\}=\Omega$

d) $\lim_{k\to\infty}\tau_k=\infty$

where each of these statements is taken almost surely with respect to P_x for each $x\in X$.

If X is improperly essential, there is a sequence $A_1\subseteq A_2\subseteq\cdots$ of inessential sets in $\mathscr{B}$ such that $\bigcup A_j=X$. If τ_k is the exit time from A_k, then $\{\tau_1, \tau_2, \ldots\}$ is a time sequence.

Theorem 2. *X is improperly essential if and only if a time sequence exists.*

Proof. Necessity is shown by the preceding example. Suppose a time sequence $\tau_1, \tau_2, \ldots$ exists. Let

$$A_k=\{x\,|\,P_x(\tau_k<\infty)>0\}.$$

By the corollary to Lemma 2, A_k is the union of a countable collection of strongly transient sets. Since $X=\bigcup A_k$, the theorem follows.

Theorem 3. *Let $\tau_1, \tau_2, \ldots$ be a time sequence. Then*

$$\mathscr{G}=\mathscr{B}_\infty(X_{\tau_1}, X_{\tau_2}, \ldots)\ P_x\text{-}a.s.,\qquad x\in X.$$

Proof. Let $\Lambda \in \mathscr{G}$, and let $A = \{x \mid P_x(\Lambda) > \frac{1}{2}\}$. Let

$$\{X_n \in A \text{ ult.}\} = \bigcup_{n \geqq 0} \bigcap_{m \geqq n} \{X_m \in A\}$$

where "ult." stands for "ultimately." It is shown in [1] that the sets Λ, $\{X_n \in A \text{ i.o.}\}$ and $\{X_n \in A \text{ ult.}\}$ are P_x equivalent for every $x \in X$. Clearly,

$$\{X_n \in A \text{ ult.}\} \subseteq \{X_{\tau_k} \in A \text{ ult.}\} \subseteq \{X_n \in A \text{ i.o.}\}$$

hence $\Lambda = \{X_{\tau_k} \in A \text{ ult.}\}$. This shows that $\mathscr{G} \subseteq \mathscr{B}_\infty(X_{\tau_1}, X_{\tau_2}, \ldots)$.

To show the converse, suppose that Λ is a cylinder of $\mathscr{B}(X_{\tau_k} \ldots)$. This means there are sets $C_0, C_1, \ldots, C_n$ in $\mathscr{B}$, and for some $j \geqq k$,

$$\Lambda = \{X_{\tau_j} \in C_0\} \cap \cdots \cap \{X_{\tau_{j+n}} \in C_n\}.$$

From Lemma 1,

$$\theta^{-1}\Lambda = [\{X_{\tau_j} \in C_0\} \cap \{\tau_j > 0\}] \cap \cdots \cap [\{X_{\tau_{j+n}} \in C_n\} \cap \{\tau_{j+n} > 0\}].$$

Since $\tau_i \leqq \tau_{i+1}$ on $\{\tau_i < \infty\}$, we have

$$\theta^{-1}\Lambda = \Lambda \cap \{0 < \tau_j < \infty\}. \tag{1}$$

From this it follows that for an arbitrary set $\Lambda \in \mathscr{B}(X_{\tau_k}, \ldots)$, (1) holds for every $j \leqq k$. Hence, if $\Lambda \in \mathscr{B}_\infty(X_{\tau_1}, X_{\tau_2}, \ldots)$, taking the union of both sides of (1) over the index j, we obtain

$$\theta^{-1}\Lambda = \Lambda \cap \Big(\bigcup_{j \geqq 0} \{0 < \tau_j < \infty\}\Big).$$

Our assumptions on the sequence $\tau_1, \tau_2, \ldots$ imply that $\bigcup_{j \geqq 0} \{0 < \tau_j < \infty\} = \Omega$, hence $\Lambda \in \mathscr{G}$. This completes the proof.

3. The Martin Potential Kernel

We assume that X is improperly essential. Let μ be a fixed probability measure on X. For each $x \in X$, $A \in \mathscr{B}$, define

$$g(x, A) = \sum_{n=0}^{\infty} P_x(X_n \in A),$$

$$g(A) = \sum_{n=0}^{\infty} P_\mu(X_n \in A).$$

It follows from the corollary to Theorem 1 that g and $g(x, \cdot)$ are σ-finite measures on X. For each $x \in X$, we can write

(2) $g(x, dy) = K(x, y)\, g(dy) + s(x, dy)$ where $K(x, \cdot)$ is $\mathscr{B}$-measurable, and $s(x, \cdot)$ and g are mutually singular on $\mathscr{B}$. We call K the *Martin potential kernel.*

K can be defined in such a way that it is jointly measurable, if we assume that $\mathscr{B}$ is countably generated. A standard argument can be found in [2, p. 616]. However, we do not need joint measurability in this paper.

Suppose temporarily that there is a measure m on $\mathscr{B}$, such that for each integer n there exists a function p^n on $X \times X$ and a function p^n_μ in X, such that

$$P_x(X_n \in dy) = p^n(x, y)\, m(dy),$$
$$P_\mu(X_n \in dy) = p^n_\mu(y)\, m(dy).$$

It is not hard to verify that the function K defined above satisfies

$$K(x, y) = \sum_{n=0}^{\infty} p^n(x, y) \Big/ \sum_{n=0}^{\infty} p^n_\mu(y)$$

for g-a.e. x and y in X, which is the usual definition of the Martin potential kernel.

We return to our general setting. For $x \in X$, the measures $s(x, \cdot)$ defined in (2) and g are singular. Let $A_x \in \mathscr{B}$ be such that $s(x, A_x) = 0$ and $g(A_x^c) = 0$. Clearly, $P_\mu(X_n \in A_x) = 1$ for all n, hence for any time τ, $P_\mu(X_\tau \in A_x) = 1$.

Lemma 3. *Let $A_0, A_1, \ldots, A_n$ be measurable subsets of A_x. Let τ be a time, and let Λ be the set*

$$\Lambda = \{X_{\tau-n} \in A_0, \ldots, X_\tau \in A_n,\ n \leqq \tau < \infty\}.$$

Then

$$E_\mu[K(x, X_{\tau-n}); \Lambda] = P_x(\Lambda).$$

Proof. Let $u(x) = P_x(\tau = 0)$. For any nonnegative integer m, a proof like that of Lemma 1d) shows that

$$p^m(x, dx_0)\, p(x_0, dx_1) \ldots p(x_{n-1}, dx_n)\, u(x_n)$$
$$= P_x(X_{\tau-n} \in dx_0, \ldots, X_\tau \in dx_n,\ \tau = n+m)$$

where p^m is the usual mth iterate of p. Summing on m, we obtain

$$g(x, dx_0)\, p(x_0, dx_1) \ldots p(x_{n-1}, dx_n)\, u(x_n) = P_x(X_{\tau-n} \in dx_0, \ldots, X_\tau \in dx_n,\ \tau \geqq n).$$

The required relation now follows by integating both sides over $A_0 X \ldots X A_n$.

If $\mathscr{F}'$ is a sub-σ field of $\mathscr{F}$, let P'_x and P'_μ denote the restrictions, respectively, of P_x and P_μ to $\mathscr{F}'$. We know there exists an $\mathscr{F}'$-measurable function f, and a measure Q which is singular with respect to P'_μ over $\mathscr{F}'$, such that

$$P'_x(d\omega) = f(\omega)\, P'_\mu(d\omega) + Q(d\omega).$$

We adopt the following notation for f

$$f = \left.\frac{dP_x}{dP_\mu}\right|_{\mathscr{F}'}.$$

Corollary. *If τ is a time, then*

$$K(x, X_\tau) = \left.\frac{dP_x}{dP_\mu}\right|_{\mathscr{B}(X_\tau)} \quad P_\mu\text{-}a.s.$$

Now suppose $\tau_1, \tau_2, \ldots$ is a time sequence. In Theorem 3, we will show that $\lim_{k\to\infty} K(x, X_{\tau_k})$ exists, using a method of proof devised by Hunt [5]. In Theorem 4, we identify the limit, using a new technique based on the preceding corollary.

Let τ be a time. For $k=0, 1, 2, \ldots$, define

$$\begin{aligned} Z_k &= K(x, X_{\tau-k}) \quad & 0 \leqq k \leqq \tau \\ &= 0 & \tau < k. \end{aligned}$$

Let $\mathscr{C}_k$ denote the P_μ-completion of the σ-field generated by sets of the form

$$\{X_{\tau-j} \in A_0, \ldots, X_\tau \in A_j\} \tag{3}$$

for $j \leqq k$, $A_i \subseteq A_x$. Then $\mathscr{C}_0 \subseteq \mathscr{C}_1 \subseteq \cdots$ and Z_k is $\mathscr{C}_k$-measurable.

Lemma 4. *$\{Z_k, \mathscr{C}_k\}$ is a super-martingale, that is,*

$$E_\mu(Z_k | \mathscr{C}_{k-1}) \leqq Z_{k-1} \; P_\mu\text{-}a.s.$$

Proof. Let Λ be a cylinder of $\mathscr{C}_{k-1}$ of the form (3). Then

$$\begin{aligned} E_\mu(E_\mu(Z_k | \mathscr{C}_{k-1}); \Lambda) &= E_\mu(Z_k; \Lambda) \\ &= E_\mu(Z_k; \{k \leqq \tau\} \cap \Lambda) \\ &= E_\mu(K(x, X_{\tau-k}); \{k \leqq \tau\} \cap \Lambda) \\ &= P_x(\{k \leqq \tau\} \cap \Lambda). \end{aligned}$$

The last expression comes from Lemma 3.

Now we have

$$\begin{aligned} P_x(\{k \leqq \tau\} \cap \Lambda) \leqq P_x(\{k-1 \leqq \tau\} \cap \Lambda) &= E_\mu(Z_{k-1}; \{k-1 \leqq \tau\} \cap \Lambda) \\ &= E_\mu(Z_{k-1}; \Lambda). \end{aligned}$$

From this inequality the lemma follows.

Let a, b be real numbers, with $a < b$, and let $\gamma_{a,b}(\tau)$ denote the number of up-crossings of the interval (a, b) by Z_k. If follows from a result in [4] that

$$E_\mu(\gamma_{a,b}(\tau)) \leqq \frac{1}{b-a} E_\mu(Z_0) = \frac{1}{b-a} P_x(\tau < \infty).$$

Theorem 4. $\lim_{n \to \infty} K(x, X_n)$ *exists a.s.*

Proof. Let $\tau_1, \tau_2, \ldots$ be a time sequence. Then $\gamma_{a,b}(\tau_k)$ is a non-decreasing sequence and

$$\gamma_{ab} = \lim_{k \to \infty} \gamma_{ab}(\tau_k)$$

is the total number of down-crossings of the interval (a, b) by $K(x, X_n)$. The previous lemma implies

$$E_\mu(\gamma_{ab}) \geqq \frac{1}{b-a} \lim_{k \to \infty} P_x(\tau_k < \infty) \leqq \frac{1}{b-a}$$

hence γ_{ab} is a.s. finite. This proves the theorem.

We turn now to identifying this limit. We begin with a generalization of the corollary to Lemma 3.

Lemma 5. *Let $\tau_1, \tau_2, \ldots$ be a time sequence. Then*

$$K(x, X_{\tau_k}) = \frac{dP_x}{dP_\mu}\bigg|_{\mathscr{B}(X_{\tau_k}, X_{\tau_{k+1}}, \ldots)} \quad P_\mu\text{-}a.s.$$

Proof. Let $\Lambda = \{X_{\tau_{k+1}} \in A_1, \ldots, X_{\tau_{k+m}} \in A_m\}$.

Now Lemma 1 implies that

$$\theta^{-n}[\Lambda \cap \{\tau_k = 0\}] = \Lambda \cap \{\tau_{k+1} \geqq n, \tau_{k+2} \geqq n, \ldots, \tau_{k+m} \geqq n\} \cap \{\tau_k = n\}$$

but since $\tau_k \leqq \tau_{k+j}$ on $\{\tau_k < \infty\}$, this reduces to

$$\theta^{-n}[\Lambda \cap \{\tau_k = 0\}] = \Lambda \cap \{\tau_k = n\}.$$

From this we obtain the expression

$$P_\mu(\Lambda \cap \{\tau_k = n\} | X_n = y) = P_y(\Lambda \cap \{\tau_k = 0\}).$$

Hence, we obtain

$$\begin{aligned} K(x, y)\, P_\mu(\{X_n \in dy\} \cap \{\tau_k = n\} \cap \Lambda) &= K(x, y) \cdot P_\mu(\Lambda \cap \{\tau_k = n\} | X_n = y) \\ &= K(x, y) \cdot P_\mu(X_n \in dy) \cdot \big(P_y(\Lambda \cap \{\tau_k = 0\})\big). \end{aligned}$$

Summing on n, we obtain

$$\begin{aligned} K(x, y)\, P_\mu(\{X_{\tau_k} \in dy\} \cap \Lambda) &= K(x, y)\, g(dy)\, P_y(\Lambda \cap \{\tau_k = 0\}) \\ &= g(x, dy)\, P_y(\Lambda \cap \{\tau_k = 0\}). \end{aligned}$$

The last expression holds over A_x. Now, similarly to above, we see that

$$P_x(X_n \in dy)\, P_y(\Lambda \cap \{\tau_k = 0\}) = P_x(\Lambda \cap \{\tau_k = n\} \cap \{X_{\tau_k} \in dy\}).$$

Summing on n yields

$$g(x, dy)\, P_y(\Lambda \cap \{\tau_k = 0\}) = P_x(\Lambda \cap \{X_{\tau_k} \in dy\}).$$

Hence, for any set $A_0 \subseteq A_x$, let

$$\Lambda = \{X_{\tau_k} \in A_0, X_{\tau_{k+1}} \in A_1, \ldots, X_{\tau_{k+m}} \in A_m\}$$

and it follows that

$$\begin{aligned} E_\mu(K(x, X_{\tau_k}); \Lambda) &= \int_{A_0} K(x, y)\, P_\mu(\{X_{\tau_k} \in dy\} \cap \Lambda) \\ &= \int_{A_0} P_x(\{X_{\tau_k} \in dy\} \cap \Lambda) = P_x(\Lambda) \end{aligned}$$

which proves the lemma.

Theorem 5.

$$\lim K(x, X_n) = \frac{dP_x}{dP_\mu}\bigg|_{\mathscr{G}} \quad P_\mu\text{-}a.s.$$

Proof. The limit exists, so it is sufficient to show that for a time sequence $\tau_1, \tau_2, \ldots,$

$$\lim K(x, X_{\tau_k}) = \left.\frac{dP_x}{dP_\mu}\right|_{\mathscr{G}} \quad P_\mu\text{-a.s.}$$

From the previous lemma, we must show

$$\lim \left.\frac{dP_x}{dP_\mu}\right|_{\mathscr{B}(X_{\tau_k}, X_{\tau_{k+1}}, \ldots)} = \left.\frac{dP_x}{dP_\mu}\right|_{\mathscr{G}} \quad P_\mu\text{-a.s.}$$

But this follows from Theorem 2 and familiar martingale arguments, e.g., those in [7, Complement IV.5.3].

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