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Asymptotic Location and Recurrence Properties of Maxima of a Sequence of Random Variables Defined on a Markov Chain*

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1. Introduction

Let $\{X_n, n \geqq 1\}$ be a sequence of random variables defined on a finite Markov chain (M.C.) and conditionally independent given the chain. Set

$$M_n = \max\{X_1, \ldots, X_n\}.$$

In [4] we investigated limit laws for $\{M_n\}$. Here we ask where the maximum M_n was achieved. We are interested in the state of the M.C. when the maximum was achieved. Also how often the maximum occurs in a particular state. These questions are concerned with the degree of intimacy between the maximum term and the underlying M.C.

Let $\{J_n, X_n, n \geqq 0\}$ be a two-dimensional stochastic process such that

$$X_0 = -\infty \quad \text{a.s.},$$

$$P[J_0 = k] = p_k, \qquad k = 1, \ldots, m; \qquad \sum_{k=1}^{m} p_k = 1,$$

and

$$\begin{aligned} P\{J_n = j, X_n \leqq x \mid X_0, J_0, X_1, J_1, \ldots, X_{n-1}, J_{n-1} = i\} \\ = P\{J_n = j, X_n \leqq x \mid J_{n-1} = i\} \\ = p_{ij} H_i(x) = Q_{ij}(x) \end{aligned}$$

for $i, j = 1, \ldots, m$. The distributions $H_i(x), i = 1, \ldots, m$ are nondegenerate and honest $(H_i(+\infty) = 1)$. The transition matrix $\mathbf{P} = \{p_{ij}\}$ $i, j = 1, \ldots, m$ is assumed to be stochastic, irreducible and aperiodic. The stationary probabilities associated with $\mathbf{P}$ are $(\pi_1, \ldots, \pi_m)$; $\mathbf{P}^n \to \boldsymbol{\Pi}$ where $\Pi_{ij} = \pi_j$.

Immediate consequences:

(i) The marginal sequence $\{J_n, n \geqq 0\}$ is an irreducible, aperiodic, m-state M.C. with $P\{J_n = j \mid J_{n-1} = i\} = p_{ij}$.

(ii) $P\{X_n \leqq x \mid J_{n-1} = i\} = H_i(x)$.

(iii) $P\{X_1 \leqq x_1, \ldots, X_n \leqq x_n \mid J_0, J_1, \ldots, J_{n-1}\} = \prod_{i=1}^{n} P\{X_i \leqq x_i \mid J_{i-1}\}$.

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The random variables $\{X_n\}$ are defined on the M.C. and conditionally independent given the chain in precisely the sense given by (ii) and (iii).

Remark. There is no loss of generality in allowing the distribution of X_n to depend on J_{n-1} only, rather than J_n and J_{n-1} – Pyke [2, p. 1751]. The case where the distribution of X_n depends on the pair (J_{n-1}, J_n) can be reduced to this case.

Let $M_n = \max\{X_1, \ldots, X_n\}$ and $\mathbf{Q}(x) = \{p_{ij} H_i(x)\}$. The distribution of M_n is obtained from:

(1.1) $$P\{J_n = j, M_n \leqq x \mid J_0 = i\} = Q_{ij}^n(x),$$

where $\mathbf{Q}^n(x) = \{Q_{ij}^n(x)\}$ is the nth power of the Q-matrix. (Here we are not concerned with matrix-convolution powers.) (1.1) was proven in [4].

In [4] we studied the semi-Markov matrix $\mathbf{Q}(x)$. In particular we proved:

(1.2) $\rho(x)$, the Perron-Frobenius eigenvalue of the non-negative matrix $\mathbf{Q}(x)$, is a distribution function.

(1.3) If $\mathbf{Q}(x_0)$ is irreducible then for any $x > x_0$, $\mathbf{Q}(\cdot)$ is (right, left) continuous at x iff $\rho(\cdot)$ is (right, left) continuous at x.

(1.4) Let $\mathbf{r}(x) = (r_1(x), \ldots, r_m(x))$ and $\mathbf{l}(x) = (l_1(x), \ldots, l_m(x))$ be the right and left eigenvectors of $\mathbf{Q}(x)$ corresponding to $\rho(x)$, normalized so that

$$\sum_{i=1}^m l_i(x) = \sum_{i=1}^m l_i(x)\, r_i(x) = 1.$$

Then:

$$\lim_{x\to\infty} \mathbf{r}(x) = (1, \ldots, 1),$$

$$\lim_{x\to\infty} \mathbf{l}(x) = (\pi_1, \ldots, \pi_m).$$

(1.5) If $\mathbf{Q}(x_0)$ is irreducible, then for any $x > x_0$, $\mathbf{Q}(\cdot)$ (right, left) continuous at x implies $\mathbf{r}(\cdot)$ and $\mathbf{l}(\cdot)$ are (right, left) continuous at x.

For matrices $\mathbf{A} = \{a_{ij}\}$ and $\mathbf{B} = \{b_{ij}\}$ with real entries, we write $\mathbf{A} \geqq \mathbf{B}$ ($\mathbf{A} > \mathbf{B}$) if $a_{ij} \geqq b_{ij}$ ($a_{ij} > b_{ij}$) for each i, j. For a matrix $\mathbf{C} = \{c_{ij}\}$ with complex entries, $|\mathbf{C}|$ denotes the matrix $\{|c_{ij}|\}$.

Form the matrix $\mathbf{M}(x) = \{r_i(x)\, l_j(x)\}$ and set $\mathbf{B}(x) = \mathbf{Q}(x) - \rho(x)\,\mathbf{M}(x)$. Then $\mathbf{B}^n(x) = \mathbf{Q}^n(x) - \rho^n(x)\,\mathbf{M}(x)$ [4 – (2.12), (2.14)]. Below, we suppress the power n and write $\mathbf{o}(1) = \mathbf{B}^n(x)$. We strengthen Theorem (2.16) in [4]:

(1.6) **Theorem.** *Let $\mathbf{Q}(x) = \{p_{ij} H_i(x)\}$, $\mathbf{M}(x)$, $\mathbf{B}(x)$ be as above. There exists a real number K such that $\lim_{n\to\infty} \mathbf{B}^n(x) = \lim_{n\to\infty} \mathbf{Q}^n(x) - \rho^n(x)\,\mathbf{M}(x) = \mathbf{0}$ uniformly in $x > K$ at a geometric rate. There exist constants $c > 0$ and $0 < \lambda < 1$ such that for $x > K$,*

$$\mathbf{Q}^n(x) = \rho^n(x)\,\mathbf{M}(x) + \mathbf{o}(1),$$

and

$$|\mathbf{o}(1)| \leqq c\,\lambda^n\,\mathbf{E},$$

for $n = 1, 2, \ldots$, where $\mathbf{E}$ is the $m \times m$ matrix with $E_{ij} = 1$.

Proof. From [4–(2.16)] we have that there exists a real number K and a positive integer N such that for $x>K$,

$$|\mathbf{B}^N(x)| \leqq (\alpha+\varepsilon)\,\mathbf{E} < m^{-1}\,\mathbf{E}.$$

Since $\mathbf{E}^n = m^{n-1}\,\mathbf{E}$ we have

$$\begin{aligned}|\mathbf{B}^{nN}(x)| &\leqq (\alpha+\varepsilon)^n E^n = (\alpha+\varepsilon)^n m^{n-1}\,\mathbf{E}\\ &\leqq \{(\alpha+\varepsilon)\,m\}^{n-1}\,\mathbf{E}.\end{aligned}$$

Therefore:

$$|\mathbf{B}^n(x)| \leqq \left|\mathbf{B}^{[\frac{n}{N}]N}(x)\right| \left|\mathbf{B}(x)^{n-[\frac{n}{N}]N}\right|.$$

We choose K large enough to insure that when $x>K$, $\left|\mathbf{B}(x)^{n-[\frac{n}{N}]N}\right| \leqq \mathbf{E}$. This gives:

$$\begin{aligned}|\mathbf{B}^n(x)| &\leqq \left|\mathbf{B}^{[\frac{n}{N}]N}(x)\right|\,\mathbf{E}\\ &\leqq \left\{((\alpha+\varepsilon)\,m)^{[\frac{n}{N}]-1}\,\mathbf{E}\right\}\mathbf{E}\\ &\leqq m\,[(\alpha+\varepsilon)\,m]^{[\frac{n}{N}]-1}\,\mathbf{E}.\end{aligned}$$

Set $\gamma=(\alpha+\varepsilon)\,m$ so that $0<\gamma<1$. Then

$$m\,\gamma^{[\frac{n}{N}]-1} = (m/\gamma)\,\gamma^{\frac{n}{N}}\,\gamma^{-(\frac{n}{N}-[\frac{n}{N}])} \leqq (m/\gamma^2)\,\gamma^{\frac{n}{N}}.$$

Setting $c=m/\gamma^2$ and $\lambda=\gamma^{\frac{1}{N}}$ gives: $|\mathbf{B}^n(x)| \leqq c\,\lambda^n\,\mathbf{E}$.

Let I_n be the state in which M_n is achieved; i.e. $I_n=j$ iff for some $k=0,1,\ldots,n-1$, $J_k=j$ and $X_{k+1}=M_n$. In order to insure that I_n is well defined, we must preclude the possibility of ties so we assume throughout this paper that $H_1(\cdot),\ldots,H_m(\cdot)$ are continuous.

Corollary (2.19) gives necessary and sufficient conditions for the existence of $\lim_{n\to\infty} P[I_n=j]$ and a procedure for calculating its numerical value. Existence of such limits is not a class property. We refer to these limits as the asymptotic location probabilities. How often is the maximum achieved on state j? State j is maximum-transient or maximum-recurrent according as $P([I_n=j]\text{ i.o.})=0$ or 1. A state must be one or the other and Theorem (3.8) gives criteria for each case.

Conventions. We shall frequently need inverse functions. For a continuous distribution function $F(x)$ we define

$$F^{-1}(y)=\inf\{x\,|\,F(x)=y\}.$$

For $F(\cdot)$ a distribution function set $x_0=\inf\{y\,|\,F(y)=1\}$. If $F(y)<1$ for all y then $x_0=\infty$. If two distribution functions are involved in a discussion we write x_0^F, x_0^G. If no distinction by superscripts is made, it is to be understood that $x_0^F=x_0^G=x_0$.

Two distribution functions $F(\cdot)$ and $G(\cdot)$ are *tail equivalent* iff $x_0^F = x_0^G = x_0$ and $1-F(x) \sim 1-G(x)$ as $x \to x_0-$; i.e. iff

$$\lim_{x \to x_0-} \frac{1-F(x)}{1-G(x)} = 1.$$

We often speak of two distribution functions whose tails have a ratio approaching α where $0 \leqq \alpha < \infty$.

In [5] we proved that $\rho(x)$, $\prod_{i=1}^{m} H_i^{\pi_i}(x)$ and $\sum_{i=1}^{m} \pi_i H_i(x)$ are all tail equivalent.

2. Asymptotic Location of the Maximum

We calculate the distribution of I_n:

$$P\{I_n = j | J_0 = i\} = \sum_{k=0}^{n-1} P\{J_k = j, X_{k+1} = M_n | J_0 = i\}.$$

To utilize the conditional independence of the random variables $\{X_n\}$ we introduce as auxiliary variables $J_1, \ldots, J_{k-1}, J_{k+1}, \ldots, J_{n-1}$ and compensate for this introduction by summing these variables from 1 to m. Hence the desired probability equals:

$$\sum_{k=0}^{n-1} \sum_{j_1, \ldots, j_{k-1}, j_{k+1} \ldots j_{n-1}} P\{X_{k+1} > \max_{\substack{1 \leqq l \leqq n \\ l \neq k+1}} \{X_l\} | J_0 = i, J_\alpha = j_\alpha, 1 \leqq \alpha \leqq n-1, \alpha \neq k, J_k = j\}$$

$$\cdot P\{J_\alpha = j_\alpha, 1 \leqq \alpha \leqq n-1, \alpha \neq k, J_k = j | J_0 = i\}$$

$$= \sum_{k=0}^{n-1} \sum_{j_1, \ldots, j_{k-1}, j_{k+1}, \ldots, j_{n-1}} \int_{-\infty}^{\infty} H_i(x) H_{j_1}(x), \ldots, H_{j_{k-1}}(x) H_{j_{k+1}}(x) \ldots$$

$$\ldots H_{j_{n-1}}(x)\, dH_j(x) \cdot p_{i j_1} p_{j_1 j_2} \cdots p_{j_{k-1} j} p_{j j_{k+1}} \cdots p_{j_{n-2} j_{n-1}}.$$

As usual, $Q_{ij}^0(x) = \delta_{ij}$ and with matrix notation we have:

$$P[I_n = j | J_0 = i] = \sum_{k=0}^{n-1} \int_{-\infty}^{\infty} Q_{ij}^k(x) \sum_{\alpha=1}^{m} p_{j\alpha} \sum_{l=1}^{m} Q_{\alpha l}^{n-k-1}(x)\, dH_j(x). \tag{2.1}$$

We wish to study the limiting behavior of expression (2.1).

First a remark. Consider a sequence of independent random variables $\{y_n, n \geqq 1\}$ such that $P[y_{2n} \leqq x] = F_1(x)$, $P[y_{2n+1} \leqq x] = F_2(x)$, $n \geqq 1$. The probability that the maximum of the first $2n$ random variables comes from $F_2(\cdot)$ is $\int_{-\infty}^{\infty} F_1^n(x)\, dF_2^n(x)$. The study of the limiting behavior of (2.1) reduces to a study of the limiting behavior of this integral. This will be made precise in the Comparison Theorem (2.7).

We begin a study of $\lim_{n \to \infty} \int_{-\infty}^{\infty} F_1^n(x)\, dF_2^n(x)$ by considering the case where $-\infty < x_0^{F_1} < x_0^{F_2} \leqq \infty$. Set $x_1 = x_0^{F_1}$, $x_2 = x_0^{F_2}$. Then:

$$\int_{-\infty}^{\infty} F_1^n(x)\, dF_2^n(x) = \int_{-\infty}^{x_1} F_1^n(x)\, dF_2^n(x) + \int_{x_1}^{x_2} dF_2^n(x).$$

But

$$\int_{-\infty}^{x_1} F_1^n(x)\, dF_2^n(x) \leqq F_2^n(x_1) \to 0 \quad \text{as } n \to \infty$$

and

$$\int_{x_1}^{x_2} dF_2^n(x) = 1 - F_2^n(x_1) \to 1 \quad \text{as } n \to \infty$$

so

$$\int_{-\infty}^{\infty} F_1^n(x)\, dF_2^n(x) \to 1.$$

The interesting cases are when either $F_1(x) < 1$, $F_2(x) < 1$ for all x, or $X_0^{F_1} = x_0^{F_2} < \infty$.

The following lemma is useful and the method of proof will be used repeatedly.

(2.2) **Lemma.** *$F_1(\cdot)$, $F_2(\cdot)$, $H(\cdot)$ are continuous distribution functions and $X_0^H \leqq X_0^{F_1}$. Then*

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} F_1^n(x)\, dF_2^n(x) = \lim_{n\to\infty} \int_{-\infty}^{\infty} F_1^n(x)\, H(x)\, dF_2^n(x)$$

where "lim" *is understood in the sense that the limit of one side exists iff the limit of the other side exists in which case the limits are equal.*

Proof. For any ε, choose M so large that for $x > M$, $|1 - H(x)| < \varepsilon$. Then:

$$\begin{aligned} 0 &\leqq \overline{\lim_{n\to\infty}} \left| \int_{-\infty}^{\infty} F_1^n(x)\, dF_2^n(x) - \int_{-\infty}^{\infty} F_1^n(x)\, H(x)\, dF_2^n(x) \right| \\ &\leqq \overline{\lim_{n\to\infty}} \int_{-\infty}^{\infty} F_1^n(x)\, |1 - H(x)|\, dF_2^n(x) \\ &= \overline{\lim_{n\to\infty}} \int_{M}^{\infty} F_1^n(x)\, |1 - H(x)|\, dF_2^n(x) \\ &\leqq \overline{\lim_{n\to\infty}}\, \varepsilon \int_{M}^{\infty} F_1^n(x)\, dF_2^n(x) \leqq \varepsilon. \end{aligned}$$

Since ε is arbitrary, the proof is complete.

For the following theorem we suppose for simplicity that $F_1(x) < 1$, $F_2(x) < 1$ for all x. Only minor changes are necessary when $x_0^{F_1} = x_0^{F_2} < \infty$.

(2.3) **Theorem.** *$F_1(\cdot)$, $F_2(\cdot)$ are distribution functions such that $F_1(x) < 1$, $F_2(x) < 1$ for all x. Then*

(2.4)
$$\lim_{x\to\infty} \frac{1 - F_1(x)}{1 - F_2(x)} = L$$

for $0 \leqq L \leqq \infty$ iff

(2.5)
$$\lim_{n\to\infty} \int_{-\infty}^{\infty} F_1^n(x)\, dF_2^n(x) = (1 + L)^{-1}.$$

Proof. We make a series of substitutions designed to bring the integral into a form where the Karamata Tauberian theorem is applicable. At each stage we keep track of how the substitutions affect $\lim_{x\to\infty}\frac{1-F_1(x)}{1-F_2(x)}$. If

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} F_1^n(x)\,dF_2^n(x)=(1+L)^{-1},$$

then setting $y=F_2(x)$, $G(y)=F_1(F_2^{-1}(y))$, we have

$$\lim_{n\to\infty}\int_0^1 n\,G^n(y)\,y^{n-1}\,dy=(1+L)^{-1}.$$

Also

$$\lim_{x\to\infty}\frac{1-F_1(x)}{1-F_2(x)}=L \quad \text{iff} \quad \lim_{y\to 1}\frac{1-G(y)}{1-y}=L.$$

Set $H(y)=y\,G(y)$ and we get

$$\int_0^1 \frac{H^n(y)}{y}\,dy\sim\frac{1}{n(1+L)} \qquad \text{as } n\to\infty$$

and

$$\lim_{x\to\infty}\frac{1-F_1(x)}{1-F_2(x)}=L \quad \text{iff} \quad \lim_{y\to 1}\frac{y-H(y)}{1-y}=L.$$

Putting $y=e^{-v}$ gives

$$\int_0^\infty H^n(e^{-v})\,dv\sim\frac{1}{n(1+L)}$$

and setting $K(v)=H(e^{-v})$ gives

$$\int_0^\infty K^n(v)\,dv\sim\frac{1}{n(1+L)}.$$

Then $K(0)=1$, $K(\infty)=0$, and

$$\lim_{x\to\infty}\frac{1-F_1(x)}{1-F_2(x)}=L \quad \text{iff} \quad \lim_{v\to 0+}\frac{e^{-v}-K(v)}{1-e^{-v}}=L \quad \text{iff} \quad \lim_{v\to 0+}\frac{e^{-v}-K(v)}{v}=L,$$

iff

$$\frac{1-K(v)}{v}=\frac{1-e^{-v}}{v}+\frac{e^{-v}-K(v)}{v}\to 1+L \qquad \text{as } v\to 0+.$$

If $\log K(v)=-S(v)$, then $\int_0^\infty e^{-nS(v)}\,dv\sim\frac{1}{n(1+L)}$. Substitute $u=S(v)$ and set $\Phi(u)=S^{-1}(u)$ so that

$$\int_0^\infty e^{-nu}\,d\Phi(u)\sim\frac{1}{n(1+L)}.$$

Also

$$\lim_{x\to\infty}\frac{1-F_1(x)}{1-F_2(x)}=L \quad \text{iff} \quad \Phi(u)\sim\frac{u}{1+L} \qquad \text{as } u\to 0.$$

Observe that

$$\int_0^\infty e^{-nu}\,d\Phi(u)\sim\frac{1}{n(1+L)} \quad \text{iff} \quad \lim_{x\to\infty} x\int_0^\infty e^{-xu}\,d\Phi(u)=(1+L)^{-1}.$$

This is shown by the inequalities:

$$[x]\int_0^\infty e^{-([x]+1)u}\,d\Phi(u)\leqq x\int_0^\infty e^{-xu}\,d\Phi(u)\leqq([x]+1)\int_0^\infty e^{-[x]u}\,d\Phi(u)$$

and by multiplying and dividing on the right by $[x]$ and on the left by $[x]+1$.

We have shown that:

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} F_1^n(x)\,dF_2^n(x)=(1+L)^{-1}$$

iff

$$\lim_{x\to\infty} x\int_0^\infty e^{-xu}\,d\Phi(u)=(1+L)^{-1},$$

and also that:

$$\lim_{x\to\infty}\frac{1-F_1(x)}{1-F_2(x)}=L$$

iff

$$\Phi(u)\sim\frac{u}{1+L} \quad \text{as } u\to 0,$$

and the desired result follows by the Karamata Tauberian Theorem [1, p. 422].

(2.6) **Corollary.** $F_1(\cdot)$, $F_2(\cdot)$, $G(\cdot)$ *are continuous distribution functions and* $F_1(\cdot)$ *and* $G(\cdot)$ *are tail equivalent. Then*

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} F_1^n(x)\,dF_2^n(x)=\lim_{n\to\infty}\int_{-\infty}^{\infty} G^n(x)\,dF_2^n(x).$$

(2.7) **Theorem. Comparison Theorem.** *We have:*

$$\begin{aligned}\lim_{n\to\infty} P[I_n=j\,|\,J_0=i]&=\lim_{n\to\infty} P[I_n=j]\\&=\lim_{n\to\infty}\sum_{k=0}^{n-1}\int_{-\infty}^{\infty} Q_{ij}^k(x)\sum_{\alpha=1}^m p_{j\alpha}\sum_{l=1}^m Q_{\alpha l}^{n-k-1}(x)\,dH_j(x)\\&=\lim_{n\to\infty}\int_{-\infty}^{\infty}\Big[\prod_{k\neq j} H_k^{\pi_k}(x)\Big]^n\,dH_j^{\pi_j n}.\end{aligned}$$

As before, the equalities are to be understood in the sense that the limit of any of the quantities exists iff the limit of all the quantities exists and then all the limits are equal.

Proof. If there exists x_0 such that $H_j(x_0)=1$ and $\rho(x_0)<1$ (iff there exists $j'\neq j$ such that $H_{j'}(x_0)<1$) then each of the above limits equals 0 and the theorem holds. Otherwise we proceed in stages as follows:

(1) Suppose that:

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \Big(\prod_{i\neq j} H_i^{\pi_i}(x)\Big)^n \, dH_j^{\pi_j n}(x) = l. \tag{2.8}$$

By Theorem (2.3), this is true iff

$$\lim_{x\to\infty} \frac{1-\prod_{i\neq j} H_i^{\pi_i}(x)}{1-H_j^{\pi_j}(x)} = \frac{1-l}{l}$$

iff

$$\lim_{x\to\infty} \frac{1-\prod_{i=1}^{m} H_i^{\pi_i}(x)}{1-H_j^{\pi_j}(x)} = \frac{1}{l}$$

iff

$$\lim_{x\to\infty} \frac{1-\rho(x)}{1-H_j(x)} = \frac{\pi_j}{l}, \tag{2.9}$$

where we have used the tail equivalence of $\prod_{i=1}^{m} H_i^{\pi_i}(x)$ and $\rho(x)$. Hence (2.8) holds iff (2.9) holds.

(2) The Tauberian argument used in the proof of Theorem (2.3) can be modified to prove that:

$$\lim_{n\to\infty} n \int_{-\infty}^{\infty} \rho^{n-1}(x)\, dH_j(x) = \frac{l}{\pi_j} \tag{2.10}$$

iff

$$\lim_{x\to\infty} \frac{1-H_j(x)}{1-\rho(x)} = \frac{l}{\pi_j}.$$

So (2.8) iff (2.10).

(3) Since we have eliminated the possibility that there exists an x_0 such that $H_j(x_0)=1$, $\rho(x_0)<1$, for any $\varepsilon>0$ there exists M such that $H_j(M)<1$, $\rho(M)<1$ and for $x>M$:

$$|\pi_k - l_k(x)| < \varepsilon,$$

$$|1 - r_k(x)| < \varepsilon,$$

$k=1,\ldots,m$. This and the fact that $n \int_{-\infty}^{\infty} \rho^{n-1}(x)\, dH_j(x)$ is bounded imply by the method of proof of Lemma (2.2) that (2.10) holds iff

$$\lim_{n\to\infty} n \int_{-\infty}^{\infty} \rho^{n-1}(x)\, l_j(x)\, dH_j(x) = l \tag{2.11}$$

iff

$$\lim_{n\to\infty} n \int_{-\infty}^{\infty} \rho^{n-1}(x)\, r_i(x)\, l_j(x)\, dH_j(x) = l$$

iff

$$\lim_{n\to\infty} n \int_{-\infty}^{\infty} \rho^{n-1}(x)\, r_i(x)\, l_j(x) \left(\sum_{\alpha=1}^{m} p_{j\alpha} r_\alpha(x)\right) dH_j(x) = l.$$

Recalling that $M_{ij}(x)=r_i(x)\,l_j(x)$ and $\sum_{k=1}^{m} l_k(x)=1$, we have that the above holds iff

$$\lim_{n\to\infty} n \int_{-\infty}^{\infty} \rho^{n-1}(x)\,M_{ij}(x)\sum_{k=1}^{m}\sum_{\alpha=1}^{m} p_{j\alpha} M_{\alpha k}(x)\,dH_j(x)=l$$

iff

$$\lim_{n\to\infty}\sum_{l=\nu}^{n-\nu}\int_{-\infty}^{\infty} \rho^{n-1}(x)\,M_{ij}(x)\sum_{k=1}^{m}\sum_{\alpha=1}^{m} p_{j\alpha} M_{\alpha k}(x)\,dH_j(x)=l \tag{2.12}$$

since

$$2\nu\int_{-\infty}^{\infty} \rho^{n-1}(x)\,M_{ij}(x)\sum_{k=1}^{m}\sum_{\alpha=1}^{m} p_{j\alpha} M_{\alpha k}(x)\,dH_j(x)\to 0, \qquad n\to\infty.$$

Since $M_{ij}(x)$, $M_{\alpha k}(x)$ are bounded functions, there exists a constant K such that

$$\sum_{l=\nu}^{n-\nu}\int_{-\infty}^{M} \rho^{n-1}(x)\,M_{ij}(x)\sum_{k=1}^{m}\sum_{\alpha=1}^{m} p_{j\alpha} M_{\alpha k}(x)\,dH_j(x)\leqq m\,K\,n\,\rho^{n-1}(M)\,H_j(M)\to 0$$

as $n\to\infty$ for any M such that $\rho(M)<1$. Hence (2.16) holds iff

$$\lim_{n\to\infty}\sum_{l=\nu}^{n-\nu}\int_{M}^{\infty} \rho^{n-1}(x)\,M_{ij}(x)\sum_{k=1}^{m}\sum_{\alpha=1}^{m} p_{j\alpha} M_{\alpha k}(x)\,dH_j(x)=0 \tag{2.13}$$

for any M such that $\rho(M)<1$.

(4) For any M such that $\rho(M)<1$:

$$\lim_{n\to\infty}\sum_{k=0}^{n-1}\int_{-\infty}^{M} Q_{ij}^{k}(x)\sum_{\alpha=1}^{m} p_{j\alpha}\sum_{l=1}^{m} Q_{\alpha l}^{n-k-1}(x)\,dH_j(x)=0. \tag{2.14}$$

Proof. (2.14) is bounded by:

$$\sum_{k=0}^{n-1}\sum_{l=1}^{m} Q_{ij}^{k}(M)\sum_{\alpha=1}^{m} p_{j\alpha} Q_{\alpha l}^{n-k-1}(M)\,H_j(M)=\sum_{l=1}^{m}\sum_{k=0}^{n-1} Q_{ij}^{k}(M)\,Q_{jl}^{n-k}(M)\leqq\sum_{l=1}^{m} Q_{il}^{n}(M)\to 0$$

as $n\to\infty$ since $\rho(M)<1$.

(5) For any fixed integer ν

$$\lim_{n\to\infty}\sum_{k=0}^{\nu-1}\int_{M}^{\infty} Q_{ij}^{k}(x)\sum_{\alpha=1}^{m} p_{j\alpha}\sum_{l=1}^{m} Q_{\alpha l}^{n-k-1}(x)\,dH_j(x)=0. \tag{2.15}$$

Proof.

$$\begin{aligned}
&\sum_{k=0}^{\nu-1}\int_{M}^{\infty} Q_{ij}^{k}(x)\sum_{\alpha=1}^{m} p_{j\alpha}\sum_{l=1}^{m} Q_{\alpha l}^{n-\nu}(x)\,dH_j(x)\\
&\leqq\sum_{k=0}^{\nu-1}\int_{M}^{\infty} Q_{ij}^{k}(x)\sum_{\alpha=1}^{m} p_{j\alpha}\sum_{l=1}^{m} Q_{\alpha l}^{n-\nu}(x)\,dH_j(x)\\
&\leqq\left(\sum_{k=0}^{\nu-1} p_{ij}^{k}\right)\sum_{\alpha=1}^{m} p_{j\alpha}\int_{M}^{\infty}\sum_{l=1}^{m} Q_{\alpha l}^{n-\nu}(x)\,dH_j(x)\to 0 \qquad \text{as } n\to\infty
\end{aligned}$$

by the Lebesgue Dominated Convergence Theorem.

A similar proof shows that

$$\lim_{n\to\infty} \sum_{k=n-\nu+1}^{n-1} \int_M^\infty Q_{ij}^k(x) \sum_{\alpha=1}^m p_{j\alpha} \sum_{l=1}^m Q_{\alpha l}^{n-k-1}(x)\, dH_j(x) = 0. \tag{2.16}$$

(6) Given any $\varepsilon > 0$, there exists a positive integer ν_0 such that for $\nu > \nu_0$ and n sufficiently large:

$$\begin{aligned}\Bigg| &\sum_{k=\nu}^{n-\nu} \int_M^\infty Q_{ij}^k(x) \sum_{\alpha=1}^m p_{j\alpha} \sum_{l=1}^m Q_{\alpha l}^{n-k-1}(x)\, dH_j(x) \\ &- \sum_{k=\nu}^{n-\nu} \int_M^\infty \rho^{n-1}(x)\, M_{ij}(x) \sum_{l=1}^m \sum_{\alpha=1}^m p_{j\alpha}\, M_{\alpha l}(x)\, dH_j(x)\Bigg| < \varepsilon\end{aligned} \tag{2.17}$$

uniformly in M.

The difference in (2.17) is less than

$$\begin{aligned}&\left|\sum_{k=\nu}^{n-\nu} \int_M^\infty Q_{ij}^k(x) - \rho^k(x)\, M_{ij}(x) \sum_{\alpha=1}^m p_{j\alpha} \sum_{l=1}^m Q_{\alpha l}^{n-k-1}(x)\, dH_j(x)\right| \\ &\quad + \left|\sum_{k=\nu}^{n-\nu} \int_M^\infty \rho^k(x)\, M_{ij}(x) \sum_{\alpha=1}^m \sum_{l=1}^m p_{j\alpha} [Q_{\alpha l}^{n-k-1} - \rho_{(x)}^{n-k-1}\, M_{\alpha l}(x)]\, dH_j(x)\right|.\end{aligned}$$

Using Theorem (1.6) we have that for $n \geqq 2\nu$, the first term is less than

$$c\, m \sum_{k=\nu}^{n-\nu} \lambda^k = c\, m\, \lambda^\nu \frac{1-\lambda^{n-2\nu+1}}{1-\lambda}$$

and the second is less than

$$\|M_{ij}(x)\|\; c\, m \sum_{k=\nu}^{n-\nu} \lambda^{n-k-1} = \|M_{ij}(x)\|\; c\, n\, \lambda^{\nu-1} \frac{1-\lambda^{n-2\nu+1}}{1-\lambda},$$

where $\|M_{ij}(x)\|$ is the supremum of this continuous function with limit at $+\infty$ over any convenient interval $[T, \infty]$, $T < M$.

Thus the difference in (2.17) is less than

$$c\, m \frac{\lambda^\nu + \|M_{ij}(x)\|\, \lambda^{\nu-1}}{1-\lambda}$$

which may be made less than ε by choosing $\nu \geqq \nu_0$.

(7) If (2.13) holds then by (6)

$$\begin{aligned} l - \varepsilon &\leqq \varliminf_{n\to\infty} \sum_{k=\nu}^{n-\nu} \int_M^\infty Q_{ij}^k(x) \sum_{\alpha=1}^m p_{j\alpha} \sum_{l=1}^m Q_{\alpha l}^{n-k-1}(x)\, dH_j(x) \\ &\leqq \varlimsup_{n\to\infty} \leqq l + \varepsilon.\end{aligned}$$

Taking into account (4) and (5) we must have

$$l-\varepsilon \leqq \varliminf_{n\to\infty} \sum_{k=0}^{n-1} \int_{-\infty}^{\infty} Q_{ij}^{k}(x) \sum_{\alpha=1}^{m} p_{j\alpha} \sum_{l=1}^{m} Q_{\alpha l}^{n-k-1}(x)\, dH_j(x)$$
$$\leqq \varlimsup_{n\to\infty} \leqq l+\varepsilon$$

which requires that

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} \int_{-\infty}^{\infty} Q_{ij}^{k}(x) \sum_{\alpha=1}^{m} p_{j\alpha} \sum_{l=1}^{m} Q_{\alpha l}^{n-k-1}(x)\, dH_j(x) = l. \tag{2.18}$$

Similarly (2.18) implies (2.13). Since (2.13) is equivalent to (2.8) we have completed the proof of the Comparison Theorem.

Studying $\lim_{n\to\infty} P[I_n=j]$ is thus equivalent to studying these probabilities in the alternating case as described after (2.1). In fact we can lump all the states $k\neq j$ into a single class, adjust the distribution functions to take into account sojourn times and study the two-state alternating scheme with distribution functions $H_j^{\pi_j}(x)$ and $\prod_{k\neq j} H_k^{\pi_k}(x)$.

The Comparison Theorem (2.6) and Theorem (2.3) combine immediately to give criteria for existence of asymptotic location probabilities.

(2.19) **Corollary.** *For* $0 \leqq l_i \leqq 1$,

$$\lim_{n\to\infty} P[I_n=i] = l_i$$

iff

$$\lim_{x\to\infty} \frac{1-\prod_{k\neq i} H_k^{\pi_k}(x)}{1-H_i^{\pi_i}(x)} = \frac{1-l_i}{l_i}$$

or equivalently iff:

$$\lim_{x\to\infty} \frac{1-H_i(x)}{1-\rho(x)} = \frac{l_i}{\pi_i}.$$

Remark. In the above, $\rho(x)$ may be replaced by any tail equivalent distribution function such as $\sum_{k=1}^{m} \pi_k H_k(x)$ or $\prod_{k=1}^{m} H_k^{\pi_k}(x)$.

The results obtained in proving the Comparison Theorem (2.7) afford us the following interpretation of Corollary (2.19):

(2.20) **Corollary.** *Given a process* $\{\hat{J}_n, \hat{X}_n, n \geqq 0\}$ *defined as in the introduction with* $P\{\hat{J}_n=j, \hat{X}_n \leqq x \mid \hat{J}_{n-1}=i\} = \hat{Q}_{ij}(x) = \pi_j H_i(x)$. *Then:*

$$\lim_{n\to\infty} P\{I_n=j \mid J_0=i\} = l$$

iff

$$\lim_{n\to\infty} P\{\hat{I}_n=j \mid \hat{J}_0=i\} = l.$$

Remark. The two systems governed by the S.M.M.'s $\mathbf{Q}(x)=\{p_{ij} H_i(x)\}$ and $\hat{\mathbf{Q}}(x)=\{\pi_j H_i(x)\}$ have the same properites as far as the existence and numerical value of $\lim_{n\to\infty} P[I_n=j]$ is concerned. Likewise with respect to the existence of limiting extreme value distributions [4–(3.12)]. The limiting behavior of the sequence $\{M_n\}$ is determined by the quantity of probability contained in the tails of the distributions $H_i(\cdot)$, $i=1,\dots,m$ and also by the relative amounts of time the Markov chain spends in each state after the chain has reached equilibrium.

Proof. If we evaluate (2.1) using the matrix $\hat{\mathbf{Q}}(x)=\{\pi_j H_i(x)\}$, we obtain:

$$\begin{aligned} P\{\hat{I}_n=j|\hat{J}_0=i\}=&\pi_j(n-1)\int_{-\infty}^{\infty} H_i(x)\left(\sum_{k=1}^{m}\pi_k H_k(x)\right)^{n-2} dH_j(x) \\ &+\delta_{ij}\int_{-\infty}^{\infty}\left(\sum_{k=1}^{m}\pi_k H_k(x)\right)^{n-1} dH_j(x). \end{aligned} \tag{2.21}$$

The second term on the right hand side goes to zero and may be neglected. Then

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\Big(\prod_{k\neq j} H_k^{\pi_k}(x)\Big)^n dH_j^{\pi_j n}(x)=l$$

iff

$$\lim_{x\to\infty}\frac{1-H_j(x)}{1-\rho(x)}=\frac{l}{\pi_j} \qquad (2.19)$$

iff

$$\lim_{x\to\infty}\frac{1-H_j(x)}{1-\sum_{k=1}^{m}\pi_k H_k(x)}=\frac{l}{\pi_j}$$

iff

$$\lim_{n\to\infty} n\int_{-\infty}^{\infty}\left(\sum_{k=1}^{m}\pi_k H_k(x)\right)^{n-1} dH_j(x)=\frac{l}{\pi_j}$$

(Theorem (2.7-2)) iff

$$\lim_{n\to\infty} n\pi_j\int_{-\infty}^{\infty} H_i(x)\left(\sum_{k=1}^{m}\pi_k H_k(x)\right)^{n-1} dH_j(x)=l$$

(same proof as Lemma (2.2)).

We have proved:

$$\lim_{n\to\infty} P[I_n=j|J_0=i]=l$$

iff

$$\lim_{n\to\infty} P[\hat{I}_n=j|\hat{J}_0=i]=l.$$

We postpone a discussion of solidarity questions and proceed to investigate recurrence properties of the sequence $\{I_n\}$.

3. Recurrence Properties

Let $(\Omega,\mathscr{F},P)$ be the underlying probability space. Then:

(3.1) **Definition.** State j is *maximum-recurrent* (*max-rec*) iff $P\{[I_n=j]\text{ i.o.}\}=1$; i.e., for any integer N and for almost all $\omega\in\Omega$, there exists some $n(\omega)>N$ such that $I_{n(\omega)}(\omega)=j$.

(3.2) **Definition.** State j is *maximum-transient* (*max-trans*) if $P\{[I_n=j]\text{ i.o.}\}=0$; i.e. iff for almost all ω there exists a positive integer $N(\omega)$ such that for all $n>N(\omega)$ $I_n(\omega)\neq j$.

(3.3) **Definition.** (Cf. [3].) For a sequence of random variables $\{X_n, n \geqq 1\}$, X_j is a *record value* of the sequence if it is strictly greater than all preceding values, i.e., if $X_j > \max(X_1, \ldots, X_{j-1})$. By convention X_1 is a record value.

For $n \geqq 1$ define the events A_n^j by

$$A_n^j = [X_n \text{ is a record}, J_{n-1} = j].$$

A_n^j is the event that a record occurs at time n in state j.

We have that

(3.4) $$j \text{ is max-trans iff } P\{A_n^j \text{ i.o.}\} = 0,$$

(3.5) $$j \text{ is max-rec iff } P\{A_n^j \text{ i.o.}\} = 1.$$

To calculate PA_n^j, let $\{p_l\}$, $l = 1, \ldots, m$ be some initial distribution. Then

$$\begin{aligned} PA_n^j &= P[X_n > \max\{X_1, \ldots, X_{n-1}\}, J_{n-1} = j] \\ &= \sum_{l=1}^{m} p_l \sum_{j_1=1}^{m} \cdots \sum_{j_{n-2}=1}^{m} \\ &\quad \cdot P[X_n > \max\{X_1, \ldots, X_{n-1}\} \mid J_0 = l, J_1 = j_1, \ldots, J_{n-2} = j_{n-2}, J_{n-1} = j] \\ &\quad \cdot P[J_{n-1} = j, J_{n-2} = j_{n-2}, \ldots, J_1 = j_1 \mid J_0 = l] \\ &= \sum_{l=1}^{m} p_l \sum_{j_1=1}^{m} \cdots \sum_{j_{n-2}=1}^{m} p_{l j_1} p_{j_1 j_2} \cdots p_{j_{n-2} j} \int_{-\infty}^{\infty} H_l(x) H_{j_1}(x) \ldots H_{j_{n-2}}(x)\, dH_j(x). \end{aligned}$$

Introducing matrix notation gives

(3.6) $$PA_n^j = \sum_{l=1}^{m} p_l \int_{-\infty}^{\infty} Q_{lj}^{n-1}(x)\, dH_j(x).$$

For an i.i.d. sequence $\{X_n, n \geqq 1\}$, the events $A_n = [X_n \text{ is a record}]$ are independent—Renyi [3]. Although our events A_n^j are not independent they exhibit some properties of an independent sequence, namely they satisfy a zero-one law. We will show that the only values for $P\{A_n^j \text{ i.o.}\}$ are 0 or 1. Hence a state must be either max-trans or max-rec. Before a formal statement of these results, we prove a lemma:

(3.7) **Lemma.**

$$\sum_{n=1}^{\infty} PA_n^j < \infty \quad \textit{iff} \quad \int_{-\infty}^{\infty} \frac{dH_j(x)}{1-\rho(x)} < \infty,$$

$$\sum_{n=1}^{\infty} PA_n^j = \infty \quad \textit{iff} \quad \int_{-\infty}^{\infty} \frac{dH_j(x)}{1-\rho(x)} = \infty.$$

Proof. From (3.6) we have that

$$\sum_{n=1}^{\infty} PA_n^j = \sum_{l=1}^{m} p_l \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} Q_{lj}^n(x)\, dH_j(x).$$

For any M such that $\rho(M) < 1$:

$$\sum_{l=1}^{m} p_l \int_{-\infty}^{M} \sum_{n=0}^{\infty} Q_{lj}^n(x)\, dH_j(x) \leqq \sum_{l=1}^{m} p_l \sum_{n=0}^{\infty} Q_{lj}^n(M)\, H_j(M).$$

Choose M such that $M>K$ and $\rho(M)>\lambda$ where K and λ are given in Theorem (1.6). Then there exists a positive constant k_M so large that $Q_{ij}^n(M)\leqq k_M\,\rho^n(M)$ for $1\leqq i,j\leqq m$, $n\geqq 1$.

So the above is dominated by $\sum_{l=1}^{m} p_l\,H_j(M)\dfrac{k_M}{1-\rho(M)}<\infty$. Therefore $\sum_{n=1}^{\infty} PA_n^j$ converges or diverges according as

$$\sum_{l=1}^{m} p_l \int_M^\infty \sum_{n=0}^{\infty} Q_{jl}^n(x)\,dH_j(x)$$

converges or diverges.

Now for $x>M$, $\mathbf{Q}^n(x)=\rho^n(x)\,\mathbf{M}(x)+\mathbf{o}(1)$ where $|\mathbf{o}(1)|\leqq c\,\lambda^n E$, $o<\lambda<1$ (Theorem (1.6)). Hence:

$$\begin{aligned}&\sum_{l=1}^{m} p_l \int_M^\infty \sum_{n=0}^{\infty} Q_{lj}^n(x)\,dH_j(x)\\ &\quad=\sum_{l=1}^{m} p_l \int_M^\infty \sum_{n=0}^{\infty} \rho^n(x)\,M_{lj}(x)\,dH_j(x)+\sum_{l=1}^{m} p_l \int_M^\infty \sum_{n=0}^{\infty} o(1)\,dH_j(x).\end{aligned}$$

The last term is dominated by

$$\sum_{n=0}^{\infty} c\,\lambda^n\big(1-H_j(M)\big)=\frac{c\big(1-H_j(M)\big)}{1-\lambda}<\infty.$$

So $\sum_{n=1}^{\infty} PA_n^j$ converges or diverges according as

$$\sum_{l=1}^{m} p_l \int_M^\infty \sum_{n=0}^{\infty} \rho^n(x)\,M_{lj}(x)\,dH_j(x).$$

But we have:

$$\begin{aligned}\min_{1\leqq l\leqq m}\ \inf_{T\leqq x\leqq\infty}|M_{lj}(x)|\int_M^\infty \frac{dH_j(x)}{1-\rho(x)}&\leqq\sum_{l=1}^{m} p_l \int_M^\infty \sum_{n=0}^{\infty} \rho^n(x)\,M_{lj}(x)\,dH_j(x)\\ &\leqq \max_{1\leqq l\leqq m}\ \sup_{T\leqq x\leqq\infty}|M_{lj}(x)|\int_M^\infty \frac{dH_j(x)}{1-\rho(x)}\end{aligned}$$

where T is chosen less than M but large enough so that $\min_{1\leqq l\leqq m}\ \inf_{T\leqq x\leqq\infty}|M_{lj}(x)|>0$. Hence

$$\sum_{n=1}^{\infty} PA_n^j \quad\text{and}\quad \int_M^\infty \frac{dH_j(x)}{1-\rho(x)}$$

converge or diverge together and since

$$\int_{-\infty}^{M} \frac{dH_j(x)}{1-\rho(x)}\leqq\frac{H_j(M)}{1-\rho(M)}<\infty,$$

this suffices to show the desired result.

Let $V_1^j=\inf\{n>1\,|\,X_n$ is a record, $J_{n-1}=j\}$, i.e. V_1^j is the index of the first non-trivial record in state j.

(3.8) **Theorem. Recurrence Criteria.** *State j is max-trans iff*

(i) $$P\{A_n^j \text{ i.o.}\}=0 \quad \textit{iff}$$

(ii) $$\sum_{n=1}^{\infty} PA_n^j<\infty \quad \textit{iff}$$

(iii) $$\int_{-\infty}^{\infty} \frac{dH_j(x)}{1-\rho(x)}<\infty \quad \textit{iff}$$

(iv) $$P\{V_1^j=\infty \mid X_1=y, J_0=j\}>0 \quad \textit{for some } y.$$

State j is max-rec iff

(v) $$P\{A_n^j \text{ i.o.}\}=1 \quad \textit{iff}$$

(vi) $$\sum_{n=1}^{\infty} PA_n^j=\infty \quad \textit{iff}$$

(vii) $$\int_{-\infty}^{\infty} \frac{dH_j(x)}{1-\rho(x)}=\infty \quad \textit{iff}$$

(viii) $$P\{V_1^j=\infty \mid X_1=y, J_0=j\}=0 \quad \textit{for all } y.$$

Remark. $\sum_{n=1}^{\infty} PA_n^j=$ Expected number of records in state j.

Proof. The equivalence of (ii) and (iii), and (vi) and (vii) follows from Lemma (3.7). That (ii) implies (i) is the statement of the Borel-Cantelli Lemma.

We have that:

$$\begin{aligned} P\{\varliminf_{n\to\infty}(A_n^j)^c\} &= P[\text{The number of records in state } j \text{ is finite}] \\ &= \sum_{n=1}^{\infty} P[\text{The last record in state } j \text{ is at index } n] \\ &= \sum_{n=1}^{\infty} P[X_n \text{ is a record in state } j; \text{ there are no records in state } j \text{ among } X_{n+1}, X_{n+2}, \ldots] \\ &= \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} P\{\text{There are no records in state } j \text{ among } X_{n+1}, X_{n+2}, \ldots \mid X_n \text{ is a record in } j, X_n=y\}\cdot dP[X_n \text{ is a record in } j, X_n\leqq y]. \end{aligned}$$

Now

$$\begin{aligned} P[X_n \text{ is a record in } j, X_n\leqq y] &= P[y\geqq X_n>\max\{X_1,\ldots,X_{n-1}\}, J_{n-1}=j] \\ &= \sum_{l=1}^{m} p_l \int_{-\infty}^{y} Q_{lj}^{n-1}(x)\, dH_j(x). \end{aligned}$$

Therefore setting

$$\begin{aligned} &P\{\text{There are no records in state } j \text{ among } X_{n+1}, X_{n+2}, \ldots \mid X_n \text{ is a record in } j, X_n=y\} \\ &= P\{V_1^j=\infty \mid J_0=j, X_1=y\} \end{aligned}$$

gives:

$$(3.9)\qquad P\{\varliminf_{n\to\infty}(A_n^j)^c\} = \sum_{l=1}^{m} p_l \int_{-\infty}^{\infty} P\{V_1^j=\infty \mid J_0=j, X_1=y\} \sum_{n=0}^{\infty} Q_{lj}^n(y)\, dH_j(y).$$

If j is max-trans, then $P\{A_n^j \text{ i.o.}\}=0$ and $P\{\varliminf_{n\to\infty}(A_n^j)^c\}=1$ so that (3.9) requires that we have for some y, $P\{V_1^j=\infty \mid X_1=y, J_0=j\}>0$ and (i) implies (iv). Assuming (iv) we note that $P\{V_1^j=\infty \mid X_1=y, J_0=j\}$ is non-decreasing in y and hence $\lim_{y\to\infty} P\{V_1^j=\infty \mid X_1=y, J_0=j\}$ exists and is strictly positive. Therefore

$$\sum_{l=1}^{m} p_l \int_{-\infty}^{\infty} P\{V_1^j=\infty \mid J_0=j, X_1=y\} \sum_{n=0}^{\infty} Q_{lj}^n(y)\, dH_j(y)<\infty$$

implies that

$$\sum_{l=1}^{m} p_l \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} Q_{lj}^n(y)\, dH_j(y)<\infty$$

and this happens when and only when

$$\int_{-\infty}^{\infty} \frac{dH_j(x)}{1-\rho(x)}<\infty \qquad \text{by Lemma (3.7).}$$

Thus by virtue of (3.9), (iv) implies (iii) and we have shown the equivalence of (i)–(iv).

If state j is max-rec, $P\{A_n^j \text{ i.o.}\}=1$ and $P[\varliminf_{n\to\infty}(A_n^j)^c]=0$ so that from (3.9) we have that:

$$0=\int_{-\infty}^{\infty} P\{V_1^j=\infty \mid J_0=j, X_1=y\} \sum_{l=1}^{m} p_l \sum_{n=0}^{\infty} Q_{lj}^n(y)\, dH_j(y).$$

Let $y_0=\inf\{y \mid \min_{1\leq k\leq m} H_k(y)>0\}$. Then for $y>y_0$, $\sum_{l=1}^{m} p_l \sum_{n=0}^{\infty} Q_{lj}^n(y)>0$ so that if $P\{V_1^j=\infty \mid J_0=j, X_1=y_0\}>0$, then $H_j(y_0)=1$. But by the definition of y_0 and the continuity of the H's, there must be a subscript l_0 such that $H_{l_0}(y_0)=0$ so that j could not possibly be max-rec. Therefore $P\{V_1^j=\infty \mid J_0=j, X_1=y_0\}=0$. Suppose there exists $y_1>y_0$ such that for $y>y_1$ $P\{V_1^j=\infty \mid J_0=j, X_1=y\}>0$. Then $H_j(y_1)=1$ and if there were a subscript k, $1\leq k\neq j\leq m$, such that $H_k(y_1)<1$ then j could not be max-rec. Therefore for all k, $H_k(y_1)=1$. In particular if there exists an index α, $1\leq\alpha\leq m$ such that $H_\alpha(y)<1$ for all y, then $P[V_1^j=\infty \mid J_0=j, X_1=y]=0$ for all y. Otherwise we observe that $P\{V_1^j=\infty \mid J_0=j, X_1=y]\}=0$ for all y such that $\min_{1\leq k\leq m} H_k(y)<1$. For other values of y, the conditional probability is not well defined and we can arbitrarily assign it the value zero. Therefore (v) implies (viii). Conversely if $P\{V_i^j=\infty \mid J_0=j, X_1=y\}=0$ for all y, then $P\{A_n^j \text{ i.o.}\}=1$ by (3.9) so that (viii) implies (v).

Suppose (viii) holds. Then by (3.9) $P\{\varliminf_{n\to\infty}(A_n^j)^c\}=0$ and j is not max-trans. Thus (vii) holds. Suppose (vii) holds. We now show that the assumption that $P\{V_1^j=\infty \mid X_1=y, J_0=j\}>0$ for some y leads to a contradiction. For if this were true, then

$$\lim_{y\to\infty} P\{V_1^j=\infty | X_1=y, J_0=j\}>0$$

and by the chain of reasoning following (3.9), this implies the RHS of (3.9) is infinite which contradicts the equality in (3.9). Thus (vii) holds iff (viii) holds. This completes the proof of Theorem (3.8).

4. Connections between Recurrence Properties and Asymptotic Location Probabilities; Solidarity Results

We now investigate the connections between recurrence properties and asymptotic location probabilities and give solidarity results. For the construction of counterexamples recall the following:

(4.1) If $p_{ij}=\pi_j$, $1\leqq i, j\leqq m$, then $\rho(x)=\sum_{i=1}^{m}\pi_i H_i(x)$. It is often convenient to take $p_{ij}=m^{-1}$ so that $\rho(x)=m^{-1}\sum_{i=1}^{m} H_i(x)$.

(4.2) For two arbitrary distributions the ratio of the tails need not have a limit as $x\to x_0-$. As an example, let $F(x)$ be any continuous, strictly increasing distribution function. Pick x^* such that $F(x^*)<1$ and set x_n to be the (unique) solution of the equation $1-F(x)=2^{-n}(1-F(x^*))$. Define $G(x)$ as follows: $G(x)=F(x)$ for $x\leqq x^*$, $1-G(x_{2n-1})=1-G(x_{2n})=1-F(x_{2n})$. For other values of x, define $G(\cdot)$ by linear interpolation.

Then $\lim_{n\to\infty}\frac{1-F(x_{2n})}{1-G(x_{2n})}=1$ and $\lim_{n\to\infty}\frac{1-F(x_{2n+1})}{1-G(x_{2n+1})}=2$ which shows $\frac{1-F(x)}{1-G(x)}$ does not have a limit as $x\to\infty$.

We give our results as a sequence of propositions.

(4.3) **Proposition.** *If* $\lim_{n\to\infty} P[I_n=j]=l_j>0$ *then* j *is max-rec.*

Proof. Observe that $\int_{-\infty}^{\infty}\frac{dH_j(x)}{1-H_j(x)}=\infty$. Now $\lim_{n\to\infty} P[I_n=j]=l_j>0$ iff $\lim_{x\to\infty}\frac{1-H_j^{\pi_j}(x)}{1-\rho(x)}=l_j$ (Corollary (2.19)) iff $\lim_{x\to\infty}\frac{1-H_j(x)}{1-\rho(x)}=\frac{l_j}{\pi_j}>0.$

Therefore the integrals $\int_{-\infty}^{\infty}\frac{dH_j(x)}{1-\rho(x)}$ and $\int_{-\infty}^{\infty}\frac{dH_j(x)}{1-H_j(x)}$ converge or diverge together and j is max-rec. (Theorem (3.8).)

(4.4) **Proposition.** *If* $\lim_{n\to\infty} P[I_n=j]=0$, *then* j *can be either max-rec or max-trans. An asymptotic location probability of zero gives no information about the recurrence properties of the state.*

Proof. Take a 2×2 stochastic matrix with entries $p_{ij}=\frac{1}{2}$, $i,j=1,2$. Take any two distribution functions $H_1(x)$, $H_2(x)$ such that there exists x_0 with $H_1(x_0)=1$ but $H_2(x)<1$ for all x. Then $\rho(x)=\frac{1}{2}H_1(x)+\frac{1}{2}H_2(x)$ and $\frac{1-H_1(x)}{1-\rho(x)}\to 0$ as $x\to\infty$. Also $\int_{-\infty}^{\infty}\frac{dH_1(x)}{1-\rho(x)}\leqq\int_{-\infty}^{x_0}\frac{dH_1(x)}{1-\rho(x)}\leqq(1-\rho(x_0))^{-1}<\infty$.

Therefore: $\lim_{n\to\infty} P[I_n=1]=0$ and 1 is max-trans.

We now give an example where the asymptotic location probability is zero but the state is max-rec.

Consider again the stochastic matrix $p_{ij}=\frac{1}{2}$, $i,j=1,2$. It suffices to find two distribution functions $H_1(\cdot)$ and $H_2(\cdot)$ such that

$$\lim_{x\to\infty}\frac{1-H_1(x)}{1-H_2(x)}=0 \quad\text{and}\quad \int_{-\infty}^{\infty}\frac{dH_1(x)}{1-H_2(x)}=\infty.$$

This is sufficient since $\lim_{x\to\infty}\frac{1-H_1(x)}{1-H_2(x)}=0$ implies

$$\frac{1-H_1(x)}{1-\rho(x)}=\frac{1-H_1(x)}{\frac{1}{2}(1-H_1(x))+\frac{1}{2}(1-H_2(x))}\to 0 \quad \text{as } x\to\infty.$$

Also

$$\frac{1-H_2(x)}{1-\rho(x)}=\frac{1-H_2(x)}{\frac{1}{2}(1-H_1(x))+\frac{1}{2}(1-H_2(x))}=\frac{1}{\frac{1}{2}\left(\frac{1-H_1(x)}{1-H_2(x)}\right)+\frac{1}{2}}\to 2 \quad \text{as } x\to\infty$$

so that $\int_{-\infty}^{\infty}\frac{dH_1(x)}{1-H_2(x)}$ and $\int_{-\infty}^{\infty}\frac{dH_1(x)}{1-\rho(x)}$ will converge or diverge together.

It is sufficient to find a continuous function $f(z)$ on $(0,1]$ with the following properties:

i) $f(1)=1$.

ii) $\lim_{z\to 0+} z f(z)=0$.

iii) $f(\cdot)$ is decreasing on $(0,1]$.

iv) $\lim_{z\to 0} f(z)=\infty$.

v) $\int_0^1 f(z)\,dz=\infty$.

Given such a function $f(\cdot)$, select any continuous distribution function $H_1(\cdot)$ such that $H_1(x)<1$ for all x and set $\frac{1}{1-H_2(x)}=f(1-H_1(x))$. Then $H_2(x)=1-\frac{1}{f(1-H_1(x))}$. We have that $H_2(-\infty)=0$, $H_2(\infty)=1$, and $H_2(\cdot)$ is non-decreasing so that $H_2(\cdot)$ is a distribution function. Furthermore

$$\lim_{x\to\infty}\frac{1-H_1(x)}{1-H_2(x)}=\lim_{x\to\infty}(1-H_1(x))\,f(1-H_1(x))=0$$

and

$$\int_{-\infty}^{\infty}\frac{dH_1(x)}{1-H_2(x)}=\int_{-\infty}^{\infty} f(1-H_1(x))\,dH_1(x)=\int_0^1 f(z)\,dz=\infty.$$

To construct the required function we define f as follows:

$$f(1)=1,$$

$$f(x)=1, \qquad \tfrac{1}{2}\leqq x\leqq 1,$$

$$f\left(\frac{1}{n!}\right)=(n-2)! \qquad n\geqq 2.$$

For other values of x in $(0, 1]$ define $f(\cdot)$ by linear interpolation. $f(\cdot)$ has the required properties. For any $x \in (0, 1]$ there exists n such that $x \in \left[\frac{1}{(n+1)!}, \frac{1}{n!}\right]$ so that

$$x f(x) \leqq \frac{f\left(\frac{1}{(n+1)!}\right)}{n!} = \frac{1}{n} \to 0 \quad \text{as } x \to 0.$$

Also $\lim_{x\to 0} f(x) = \infty$ and $\int_0^1 f(x)\,dx = \infty$. This follows by a direct summation of the areas of the rectangles and triangles under the curve:

$$\int_0^1 f(x)\,dx = \sum_n \left\{\left(\frac{1}{n!} - \frac{1}{(n+1)!}\right)(n-2)! + \frac{1}{2}\left(\frac{1}{n!} - \frac{1}{(n+1)!}\right)((n-1)! - (n-2)!)\right\}$$

$$= \sum_n \frac{1}{(n+1)(n-1)} + \frac{1}{2} \sum_n \frac{n-2}{(n+1)(n-1)} = \infty.$$

(4.5) **Proposition.** *If state j is max-trans, then the asymptotic location probability exists and* $\lim_{n\to 0} P[I_n = j] = 0$.

Proof. If state j is max-trans then $P([I_n = j] \text{ i.o.}) = 0$ or equivalently $\lim_{n\to\infty} P(\bigcup_{k\geqq n} [I_k = j]) = 0$. Therefore

$$P[I_n = j] \leqq P(\bigcup_{k\geqq n} [I_k = i]) \to 0 \quad \text{as } n \to \infty.$$

(4.6) **Proposition.** *Suppose j is max-rec. This gives no information about the existence of* $\lim_{n\to\infty} P[I_n = j]$.

Proof. We construct an example where j is max-rec yet $\lim_{n\to\infty} P[I_n = j]$ does not exist. In (4.2) we showed how to construct two distribution functions $H_1(\cdot)$, $H_2(\cdot)$, such that $\lim_{x\to\infty} \frac{1 - H_1(x)}{1 - H_2(x)}$ does not exist. If necessary the method of construction can be slightly modified to insure $1 - H_1(x) \geqq 1 - H_2(x)$ for all large x. Let the stochastic matrix $\mathbf{P}$ be defined by $p_{ij} = \frac{1}{2}$, $i, j = 1, 2$ so that $\rho(x) = \frac{1}{2} H_1(x) + \frac{1}{2} H_2(x)$. Then $\lim_{n\to\infty} P[I_n = 1]$ does not exist since

$$\lim_{x\to\infty} \frac{1 - H_1(x)}{1 - \rho(x)} = \lim_{x\to\infty} \left(\frac{1}{2} + \frac{1}{2}\left(\frac{1 - H_2(x)}{1 - H_1(x)}\right)\right)^{-1}$$

does not exist. But 1 is max-rec since for any M such that $\rho(M) < 1$:

$$\int_M^\infty \frac{dH_1(x)}{1 - \rho(x)} = \int_M^\infty \frac{dH_1(x)}{\frac{1}{2}(1 - H_1(x)) + \frac{1}{2}(1 - H_2(x))} \geqq \int_M^\infty \frac{dH_1(x)}{1 - H_1(x)} = \infty.$$

Proposition (4.3) showed how to construct an example where j was max-rec and $\lim_{n\to\infty} P[I_n = j] = l_j > 0$ and Proposition (4.4) showed how to construct an example where j was max-rec and $\lim_{n\to\infty} P[I_n = j] = 0$.

(4.7) **Proposition.** *Maximum-transience is not a class property. In fact, it is impossible for all states to be max-trans.*

Proof. Suppose all states are max-trans. Setting $A_n^k = [X_n \text{ is a record}, J_{n-1} = k]$ gives $\sum_{k=1}^{m} PA_n^k = P[X_n \text{ is a record}]$. Therefore

$$\sum_{n=1}^{\infty} P[X_n \text{ is a record}] = \sum_{n=1}^{\infty} \sum_{k=1}^{m} PA_n^k = \sum_{k=1}^{m} \left(\sum_{n=1}^{\infty} PA_n^k \right) < \infty$$

by Theorem (3.8, ii). Hence by the Borel-Cantelli Lemma: $P\{[X_n \text{ is a record}] \text{ i.o.}\} = 0$. It is impossible for there to be only a finite number of records a.s. as the following dissection argument shows. Pick an arbitrary state j and let τ_0 be the time of the first visit to state j and let τ_n, $n \geqq 1$ be the waiting times between visits to j. $\{\tau_n, n \geqq 1\}$ is an i.i.d. sequence. Set $S_n = \sum_{k=0}^{n} \tau_k$ and $\chi_0 = \max\{X_1, \dots, X_{\tau_0+1}\}$, $\chi_1 = \max\{X_{S_0+2}, \dots, X_{S_1+1}\}, \dots, \chi_n = \max\{X_{S_{n-1}+2}, \dots, X_{S_n+1}\}$. The sequence $\{\chi_n, n \geqq 1\}$ is i.i.d. and χ_n is a record value of the sequence $\{\chi_n, n \geqq 0\}$ iff at least one of the random variables $X_{S_{n-1}+2}, \dots, X_{S_n+1}$ is a record value of the sequence $\{X_n, n \geqq 1\}$. But the events $\{[\chi_k \text{ is a record value of the sequence } (\chi_n, n \geqq 1)]\}$ for $k \geqq 1$ are independent and have probabilities k^{-1} [3]. Hence $\sum_{k=1}^{\infty} P[\chi_k$ is a record value of the sequence $(\chi_n, n \geqq 1)] = \infty$ and by the Borel Zero-One Law:

$$P\{[\chi_k \text{ is a record value of the sequence } (\chi_n, n \geqq 1)] \text{ i.o.}\} = 1.$$

With probability 1, the sequence $\{\chi_n, n \geqq 1\}$ has infinitely many records and this is true for the sequence $\{\chi_n, n \geqq 0\}$ since χ_0 is exceeded a.s. This completes the proof.

(4.8) **Proposition.** *Maximum-recurrence is not a class property. State j max-rec does not necessitate all states being max-rec.*

Proof. Pick two distribution functions $H_1(\cdot)$, $H_2(\cdot)$ such that $H_1(x_0) = 1$ for $x_0 < \infty$ and $H_2(x) < 1$ for all x. Let $p_{ij} = \frac{1}{2}$, $i, j = 1, 2$. Then $\rho(x) = \frac{1}{2} H_1(x) + \frac{1}{2} H_2(x)$ and

$$\int_{-\infty}^{\infty} \frac{dH_1(x)}{1 - \rho(x)} \leqq \frac{1}{1 - \rho(x_0)} < \infty.$$

Therefore state 1 is max-trans and by Proposition (4.7), state 2 is max-rec.

(4.9) **Proposition.** *Existence of asymptotic location probabilities is not a class property. The existence of* $\lim_{n \to \infty} P[I_n = j]$ *does not imply* $\lim_{n \to \infty} P[I_n = k]$ *exists for* $k \neq j$. *However if all the asymptotic location probabilities exist, then they form a probability distribution:*

$$\sum_{j=1}^{m} \lim_{n \to \infty} P[I_n = j] = 1.$$

Proof. The last statement is proved by integrating by parts:

$$\int_{-\infty}^{\infty} \Big(\prod_{k \neq j} H_k^{\pi_k}(x)\Big)^n dH_j^{\pi_j n}(x) = 1 - \sum_{\alpha \neq j} \int_{-\infty}^{\infty} \Big(\prod_{k \neq \alpha} H_k^{\pi_k}(x)\Big)^n dH_\alpha^{\pi_\alpha n}(x).$$

Hence:
$$\sum_{\alpha=1}^{m} \int_{-\infty}^{\infty} \Big(\prod_{k\neq\alpha} H_k^{\pi_k}(x)\Big)^n \, dH_\alpha^{\pi_\alpha n}(x) = 1.$$

It is easy to show that $\lim_{n\to\infty} P[I_n=3]=0$ does not imply that other states need have asymptotic location probabilities: Take $H_3(\cdot)$ for which there exists $x_0<\infty$ and $H_3(x_0)=1$. As in (4.2) construct two distribution functions $H_1(\cdot)$, $H_2(\cdot)$ such that $H_1(x)<1$, $H_2(x)<1$ for all x and $\lim_{x\to\infty} \frac{1-H_1(x)}{1-H_2(x)}$ does not exist. Set $p_{ij}=\frac{1}{3}$, $1\leqq i,j\leqq 3$ and we have that $\lim_{n\to\infty} P[I_n=3]=0$ but neither $\lim_{n\to\infty} P[I_n=1]$ nor $\lim_{n\to\infty} P[I_n=2]$ exist.

One can also construct an example where the asymptotic location probability exists for one state and is positive, but does not exist for any other state. Again take two distribution functions $H_1(\cdot)$, $H_2(\cdot)$ such that $\lim_{x\to\infty} \frac{1-H_1(x)}{1-H_2(x)}$ does not exist. Set $1-H_3(x)=\frac{1}{2}(1-H_1(x)+1-H_2(x))$, $p_{ij}=\frac{1}{3}$, $1\leqq i,j\leqq 3$. Then $1-\rho(x)=1-H_3(x)$ so

$$\lim_{x\to\infty} \frac{1-H_3(x)}{1-\rho(x)}=1 \quad \text{but} \quad \lim_{x\to\infty} \frac{1-H_1(x)}{1-\rho(x)} \quad \text{and} \quad \lim_{x\to\infty} \frac{1-H_2(x)}{1-\rho(x)}$$

do not exist. This suffices for the desired conclusion.

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