RELATIVE DEFORMATIONS OF SELENODETIC NETS OF COORDINATES

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Abstract. One of the main problems of selenodesy consists of the construction of a net of basic reference points on the surface of the Moon. At present there exist many catalogues containing the coordinates of selected objects on the Moon. These catalogues differ by the presence of both systematic and accidental errors.

The investigations concerning the comparison of catalogues and the elucidation of their systematic differences are of very recent date. Various methods of interpretation of the systematic differences between catalogues have been proposed. Without an attempt to encompass the whole problem in what follows, we shall describe one method for comparative study of catalogues based on the theory of the deformation of continuous media.

1. A Theory of the Deformation of Selenodetic Coordinate Systems

Systematic differences between catalogues are due to systematic errors of observations. The causes of such errors vary and often remain unknown. But the general nature of displacement of one system relatively another can be investigated in sufficient detail by the mathematical techniques of continuum mechanics.

We shall consider the ensemble of basic points on the Moon as a selenodetic system. Because of errors in the coordinates of such points, such systems are always deformed. To study the absolute deformation of the selenodetic net is difficult because of a lack of exact coordinates of the control points. Therefore, we shall consider only the relative displacements of basic nets, interpreting them as a transformation of one selenodetic system into another. Then we treat the linear deformation which is represented by the equations

$$a\xi_{2i} + b\eta_{2i} + c\zeta_{2i} + d = \xi_{1i} - \xi_{2i} = s_i, \tag{1a}$$

$$e\xi_{2i} + f\eta_{2i} + g\zeta_{2i} + h = \eta_{1i} - \eta_{2i} = t_i, \tag{1b}$$

$$m\xi_{2i} + n\eta_{2i} + l\zeta_{2i} + k = \zeta_{1i} - \zeta_{2i} = u_i, \tag{1c}$$

where ξ, η, ζ – are the selenodetic rectangular coordinates of craters relative to main axes of inertia of the Moon. The positive direction of the ξ-axis is to the east (to Mare Crisium); of the η-axis – to the north and of the ζ-axis in the direction of the earth. The indices 1, 2 designate coordinates taken from two different catalogues; i is the number of the same crater in these two catalogues. The $a, b, \ldots l, k$ values are the reduction coefficients of the second catalogue values to the first.

Small displacements of the selenodesic construction which are determined by Equation (1) can be visualized as consisting of three parts:

(1) translational movement of the whole system;

(2) a rotation of the whole system, given by $\Delta\xi$, $\Delta\eta$, $\Delta\zeta$;

(3) a deformation of the system $\overline{\Delta\xi}$, $\overline{\Delta\eta}$, $\overline{\Delta\zeta}$.

The linear shift of the second catalogue relative to the first is given by the displacements d, h, k along the coordinate axes. We shall consider that this kind of shift has been taken into account.

Let us represent by the point K_{1i} $(\xi_{1i}, \eta_{1i}, \zeta_{1i})$ the location of the ith crater with its coordinates in the first catalogue. (Figure 1). The location of the same crater according

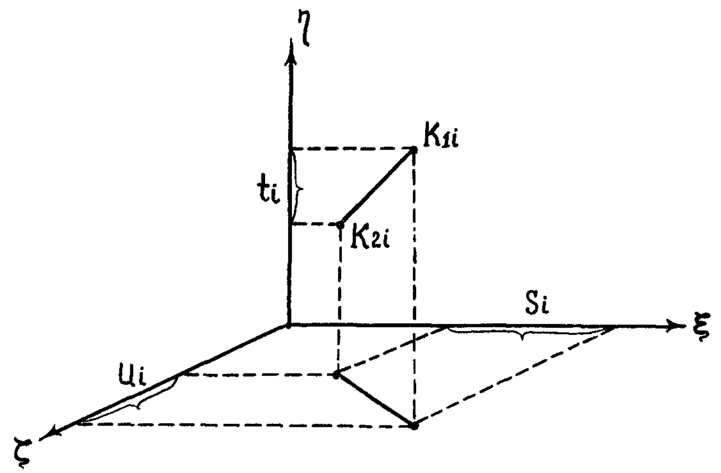

Fig. 1.

to the second catalogue we designate by K_{2i} $(\xi_{2i}, \eta_{2i}, \zeta_{2i})$. The displacements of the points $\xi_{1i}-\xi_{2i}=s_i$, $\eta_{1i}-\eta_{2i}=t_i$, $\zeta_{1i}-\zeta_{2i}=u_i$ are given by the linear deformation matrix

$$\Pi = \begin{pmatrix} a & b & c \\ e & f & g \\ m & n & l \end{pmatrix}. \tag{2}$$

We decompose the matrix into a sum of a symmetric and an antisymmetric matrix, designated by R and Q, respectively. Thus we have

$$\Pi = Q + R, \tag{3}$$

where

$$Q = \begin{pmatrix} 0 & \dfrac{b-e}{2} & \dfrac{c-m}{2} \\ -\dfrac{b-e}{2} & 0 & \dfrac{g-n}{2} \\ -\dfrac{c-m}{2} & -\dfrac{g-n}{2} & 0 \end{pmatrix} \tag{4}$$

and

$$R = \begin{pmatrix} a & \frac{b+e}{2} & \frac{c+m}{2} \\ \frac{b+e}{2} & f & \frac{g+n}{2} \\ \frac{c+m}{2} & \frac{g+n}{2} & l \end{pmatrix}. \tag{5}$$

First, we consider the displacements characterizing the Q-matrix. We shall imagine the trihedron of axes $0\xi_2, \eta_2, \zeta_2$, which is rigidly connected with the second catalogue points, and which in the initial stage coincides with the axis of the coordinates $0\xi\eta\zeta$ The rotation of this trihedron around at the point O will correspond to the displacement of the selenodetic net ξ_2, η_2, ζ_2 relative to the first system ξ_1, η_1, ζ_1 as a rigid body.

We shall turn the second catalogue system and rotate it through the angles μ, v, π around the coordinate axis ζ, η and ξ, respectively (Figure 2). If the angles μ, π and v

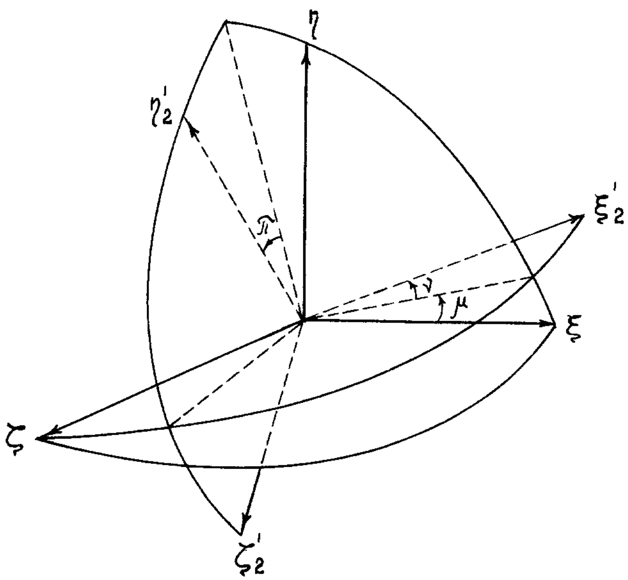

Fig. 2.

are small, the displacements are determined by

$$\begin{aligned} \Delta\xi &= \xi_2' - \xi_2 = 0\xi_2 - \mu\eta_2 + v\zeta_2\,, \\ \Delta\eta &= \eta_2' - \eta_2 = \mu\xi_2 + 0\eta_2 - \pi\zeta_2\,, \\ \Delta\zeta &= \zeta_2' - \zeta_2 = -\, v\xi_2 + \pi\eta_2 + 0\zeta_2\,, \end{aligned} \tag{6}$$

where ξ_2', η_2', ζ_2' are the coordinates of the second catalogue points after the three above-mentioned rotations. It is apparent that the rotation is characterized by the antisymmetric matrix

$$\begin{pmatrix} 0 & -\mu & \nu \\ \mu & 0 & -\pi \\ -\nu & \pi & 0 \end{pmatrix}. \tag{7}$$

If one compares the angles of rotation with the values

$$\mu = -\frac{b-e}{2}, \quad \nu = \frac{c-m}{2}, \quad \pi = -\frac{g-n}{2}, \tag{8}$$

the geometrical meaning of antisymmetric Q-matrix becomes clear. The values $\omega_1 = \pi$, $\omega_2 = \nu$, $\omega_3 = \mu$ are the components of the axial vector ω, around which the system rotates as a whole. The angle of rotation is expressed by

$$\omega = \sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2}. \tag{9}$$

The selenographic coordinates of the pole of the rotation are

$$\lambda_\Pi = \operatorname{arc\,tg} \frac{\omega_1}{\omega_3}, \qquad \beta_\Pi = \arcsin \frac{\omega_2}{\omega}. \tag{10}$$

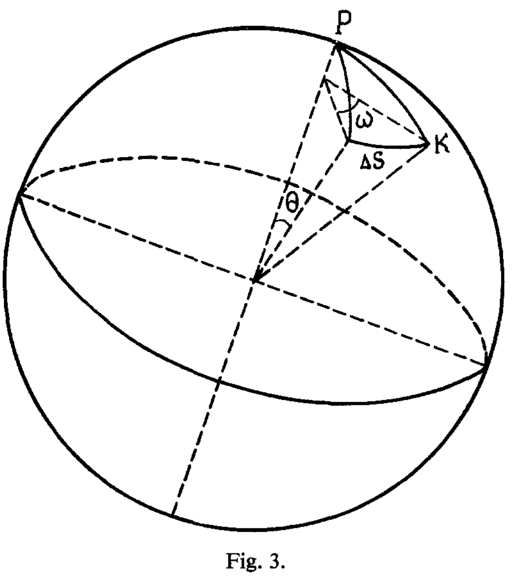

Fig. 3.

With θ denoting the angle between the axis of rotation and the radius-vector of the point K (Figure 3) the displacement ΔS of this point is

$$\Delta s = r\omega \sin\theta . \tag{11}$$

One can see that the points lying on the equator of a rotation ($\theta = \pi/2$) will exhibit maximum displacements, while points in the polar region will possess minimum displacements.

So far we have considered relative linear shift and rotational displacements of selenodetic systems. Now we shall investigate their pure deformation represented by

$$\begin{aligned}
\overline{\Delta\xi} &= a\xi_2 + \frac{b+e}{2}\eta_2 + \frac{c+m}{2}\zeta_2 ,\\
\overline{\Delta\eta} &= \frac{b+e}{2}\xi_2 + f\eta_2 + \frac{g+n}{2}\zeta_2 ,\\
\overline{\Delta\zeta} &= \frac{m+c}{2}\xi_2 + \frac{g+n}{2}\eta_2 + l\zeta_2 ,
\end{aligned} \tag{12}$$

which is described by the matrix (5).

First of all we shall call some general properties of a linear deformation (Equation 1). In a linear deformation all points lying in the one plane before the deformation will be in another plane after the deformation. If some points of one catalogue are in one plane, in the system of the other catalogue they will also be in another plane.

From the law of conservation of planes the law of conservation of straight lines follows. From last law, the law of conservation of surface order follows, because the order of surface is determined by number of the points of a intersection of this surface and a straight line. Therefore, if all points in the system of the second catalogue are situated on the surface of a sphere, then (after the deformation in the system of the other catalogue) they will lie on an ellipsoidal surface with the same center as the sphere.

The lines perpendicular to each other before the deformation will not generally remain perpendicular thereafter. However, there are always certain three lines which, without change of their direction, will remain perpendicular to each other – both before and after the deformation. Such directions are given by the principal values of the matrix R, and we shall call the principal axes of deformation.

Usually one studies the effect of linear deformations along the principal axes of deformation, because their directions are not affected by the deformation. The coefficients $\lambda_1, \lambda_2, \lambda_3$ characterizing the change in length of a system along the principal axis of deformation are called the principal values of a matrix. They are also called the coefficients of relative elongation.

In order to study the deformation of one selenodetic system relative to another is necessary to determine the directions of the principal axes of deformation and principal values of the R-matrix. The directional cosines of one principal axis at deformation are p, q, r, while λ is the corresponding coefficient of the relative widening along this axis. From a determination of the principal values of a matrix the following formulae obtain:

$$\begin{aligned} ap + \frac{b+e}{2} q + \frac{m+c}{2} r &= \lambda p\,, \\ \frac{b+e}{2} p + f \cdot q + \frac{g+n}{2} r &= \lambda q\,, \\ \frac{m+c}{2} p + \frac{g+n}{2} q + lr &= \lambda r\,. \end{aligned} \tag{13}$$

These linear equations are homogeneous in p, q, r. As all these values cannot vanish simultaneously, the main determinant must be equal zero: i.e.,

$$\begin{vmatrix} a-\lambda & \frac{b+e}{2} & \frac{m+e}{2} \\ \frac{b+e}{2} & f-\lambda & \frac{g+n}{2} \\ \frac{m+c}{2} & \frac{g+n}{2} & l-\lambda \end{vmatrix} = 0\,. \tag{14}$$

The three real roots of Equation (14) are λ_1, λ_2, λ_3, which are the coefficients of the relative lengthening of a system.

The directional cosines p_i, q_i, r_i $(i=1,2,3)$ are determined by any two Equations (13) for corresponding values λ_i and the relation

$$p_i^2 + q_i^2 + r_i^2 = 1\,. \tag{15}$$

Thus one can represent the deformation of the selenodetic system in the form of two operations. With the coordinates taken from the second catalogue we construct in the space $(\xi\eta\zeta)$ the net of basic points ξ_2, η_2, ζ_2. After this we find the positions of these points relative to the principal axis of deformation. Then the coordinates ξ_2, η_2, ζ_2 are transformed in (xyz) according to the formulae

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{pmatrix} \begin{pmatrix} \xi_2 \\ \eta_2 \\ \zeta_2 \end{pmatrix}. \tag{16}$$

The deformation of the system is a result of its lengthening along three axes. After this widening the displacements of points will be expressed by the next relative increments of their coordinates

$$\begin{aligned} \Delta x &= \lambda_1 x\,, \\ \Delta y &= \lambda_2 y\,, \\ \Delta z &= \lambda_3 z\,; \end{aligned} \tag{17}$$

or

$$\begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{18}$$

One can also represent the formula (17) as

$$\begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix} = \begin{pmatrix} \lambda_1 p_1 & \lambda_1 q_1 & \lambda_1 r_1 \\ \lambda_2 p_2 & \lambda_2 q_2 & \lambda_2 r_2 \\ \lambda_3 p_3 & \lambda_3 q_3 & \lambda_3 r_3 \end{pmatrix} \begin{pmatrix} \xi_2 \\ \eta_2 \\ \zeta_2 \end{pmatrix}. \tag{19}$$

If these increments are transformed again to the old axes $0\xi\eta\zeta$, we shall have

$$\begin{pmatrix} \overline{\Delta\xi_2} \\ \overline{\Delta\eta_2} \\ \overline{\Delta\zeta_2} \end{pmatrix} = \begin{pmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix} \tag{20}$$

or

$$\begin{pmatrix} \overline{\Delta\xi_2} \\ \overline{\Delta\eta_2} \\ \overline{\Delta\zeta_2} \end{pmatrix} = \begin{pmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{pmatrix} \begin{pmatrix} \lambda_1 p_1 & \lambda_1 q_1 & \lambda_1 r_1 \\ \lambda_2 p_2 & \lambda_2 q_2 & \lambda_2 r_2 \\ \lambda_3 p_3 & \lambda_3 q_3 & \lambda_3 r_3 \end{pmatrix} \begin{pmatrix} \xi_2 \\ \eta_2 \\ \zeta_2 \end{pmatrix}. \tag{21}$$

Because the product of matrices is equal to the expression (4) i.e.,

$$\begin{pmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{pmatrix} \begin{pmatrix} \lambda_1 p_1 & \lambda_1 q_1 & \lambda_1 r_1 \\ \lambda_2 p_2 & \lambda_2 q_2 & \lambda_2 r_2 \\ \lambda_3 p_3 & \lambda_3 q_3 & \lambda_3 r_3 \end{pmatrix} = \begin{pmatrix} a & \frac{b+e}{2} & \frac{c+m}{2} \\ \frac{b+e}{2} & f & \frac{g+n}{2} \\ \frac{c+m}{2} & \frac{g+n}{2} & l \end{pmatrix}, \tag{22}$$

we verify that the increments (21) taken as the result of the above-mentioned deformation are the displacements expressed by Equation (12).

We have considered the kinematics of a transformation of one selenodetic construction into another by Equation (1). In the papers by Arthur (1968), Kisliuk (1970), Gavrilov and Kisliuk (1971) the differences of coordinates of catalogues which are compared are expressed by equations

$$\mu\xi_{2i} + \alpha\eta_{2i} + \beta\zeta_{2i} + e = \xi_{1i} - \xi_{2i}, \tag{23a}$$

$$-\alpha\xi_{2i} + \mu\eta_{2i} + \gamma\zeta_{2i} + f = \eta_{1i} - \eta_{2i}, \tag{23b}$$

$$-\beta\xi_{2i} - \gamma\eta_{2i} + \mu\zeta_{2i} + g = \zeta_{1i} - \zeta_{2i}. \tag{23c}$$

We shall establish the meaning of the Equations (23). In this case, the matrix of relative displacement is of the form

$$\Pi = \begin{pmatrix} \mu & \alpha & \beta \\ -\alpha & \mu & \gamma \\ -\beta & -\gamma & \mu \end{pmatrix}. \tag{24}$$

We decompose it into a sum of a symmetrical and antisymmetrical matrix $\Pi = Q + R$

of the form

$$Q = \begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ -\beta & -\gamma & 0 \end{pmatrix} \tag{25}$$

and

$$R = \begin{pmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{pmatrix}. \tag{26}$$

From the expression for R it is clear that the system of formulae (23) describes a highly particular case of a deformation: namely, the case of an expansion uniform in all directions. If a system expands isotropically, all points which lie on a sphere with radius r_0 will after deformation, also lie on a sphere, but now with the radius $r_i = \mu r_0$. For such a type of expansion any three orthogonal axes may be taken as the expansion axes; in particular, the axes of selenodetic coordinates $0\xi\eta\zeta$.

Later we will consider the deformation of some selenodetic systems and show that the deformation is anisotropic.

2. Analysis of Systematic Differences between Basic Catalogues

An investigation of systematic errors of basic nets has been carried out by one of the authors of the present article (Kisliuk, 1971). For this purpose, the comparisons have been made for eight catalogues, the list of which is given in Table I.

In the Table I the number of common points of corresponding pairs of catalogues is given.

TABLE I

		Schr I	Bald	AMS	ACIC	Schr II	Kiev	Mills	Arthur
Schrutka I	(1958)	150	70	60	41	61	143	1	17
Baldwin	(1963)		696	86	94	75	166	77	31
AMS	(1964)			256	28	67	107	3	15
ACIC	(1965)				196	24	78	61	39
Schrutka II	(1966)					137	91	0	12
Kiev	(1967)						500	18	29
Mills	(1968)							942	2
Arthur	(1968)								48

For the catalogues ACIC and 'Mills' we have given the selenographic coordinates λ, β and relative heights of points h. Therefore, the rectangular coordinates for them have been computed by

$$\begin{aligned} \xi &= (1 + h) \cos\beta \sin\lambda, \\ \eta &= (1 + h) \sin\beta, \\ \zeta &= (1 + h) \cos\beta \cos\lambda. \end{aligned} \tag{27}$$

In the catalogue by Baldwin (1963) the coordinates ξ, η, h are published. We computed, therefore, the values ζ by

$$\zeta = \sqrt{(1 + h)^2 - \xi^2 - \eta^2}\,. \tag{28}$$

The differences in coordinates of the compared catalogues are expressed in two different ways. In the paper by Kisliuk (1970) they were given by Equation (23), while in a later paper of Kisliuk (1971) they were given by Equation (1). The coefficients of the formula (1) for each pair of catalogues were found by methods of least-squares.

Table II contains the coefficients $a, b, \ldots\, l, k$. They were the initial data for a description of relative deformation of the above-mentioned catalogues. The values of the coefficients are given in the units $10^{-5}\,R_{☾}$.

TABLE II

Pair	a	b	c	d
	l	f	g	h
	m	n	h	k
Schr I-Bald.	17 ± 8	0 ± 10	60 ± 22	− 55 ± 18
	1 ± 6	− 6 ± 7	62 ± 16	− 59 ± 13
	− 67 ± 53	45 ± 65	442 ± 140	− 348 ± 120
-AMS	− 53 ± 7	23 ± 10	− 50 ± 21	40 ± 17
	− 54 ± 6	− 84 ± 8	− 18 ± 18	12 ± 14
	− 140 ± 39	− 189 ± 51	− 175 ± 113	49 ± 90
-ACIC	7 ± 6	− 16 ± 7	0 ± 15	2 ± 12
	− 12 ± 5	− 19 ± 6	− 10 ± 13	10 ± 11
	− 63 ± 23	− 24 ± 25	82 ± 57	− 73 ± 47
-Schr.II	− 23 ± 8	− 6 ± 9	12 ± 31	− 9 ± 17
	− 9 ± 5	− 13 ± 6	19 ± 3	− 8 ± 11
	− 67 ± 42	− 16 ± 49	297 ± 108	− 137 ± 87
-Kiev	1 ± 5	19 ± 7	16 ± 14	− 28 ± 11
	− 6 ± 4	1 ± 5	− 12 ± 11	− 16 ± 9
	115 ± 26	− 2 ± 32	15 ± 68	− 6 ± 55
-Arthur	− 11 ± 3	4 ± 3	5 ± 23	5 ± 19
	− 26 ± 3	− 18 ± 3	− 44 ± 20	35 ± 17
	− 203 ± 15	5 ± 16	− 214 ± 123	164 ± 102
Bald.-AMS	− 66 ± 6	24 ± 6	− 82 ± 14	70 ± 11
	− 55 ± 6	− 70 ± 6	− 46 ± 13	44 ± 10
	− 99 ± 42	− 281 ± 45	− 730 ± 99	483 ± 79
-ACIC	− 4 ± 5	− 14 ± 6	− 70 ± 15	61 ± 12
	− 15 ± 4	− 12 ± 4	− 62 ± 10	65 ± 8
	159 ± 29	− 61 ± 34	− 369 ± 81	231 ± 66
-Schr.II	− 51 ± 8	− 12 ± 8	− 66 ± 23	58 ± 18
	− 16 ± 6	− 6 ± 6	− 39 ± 17	54 ± 13
	− 12 ± 54	− 55 ± 55	− 309 ± 117	319 ± 72
-Kiev	− 6 ± 6	− 18 ± 6	− 47 ± 14	53 ± 11
	2 ± 6	− 2 ± 6	− 21 ± 14	44 ± 11
	− 26 ± 36	− 18 ± 36	− 192 ± 80	135 ± 65
-Mills	27 ± 16	36 ± 14	− 13 ± 43	8 ± 37
	26 ± 9	− 9 ± 7	100 ± 24	− 95 ± 20
	98 ± 59	− 9 ± 50	− 1220 ± 159	992 ± 137
-Arthur	− 18 ± 8	− 8 ± 9	− 134 ± 22	124 ± 17
	− 6 ± 9	− 14 ± 9	− 89 ± 22	72 ± 17
	34 ± 64	8 ± 73	− 672 ± 174	540 ± 128

Table II (Continued)

Pair	*a* *l* *m*	*b* *f* *n*	*c* *g* *h*	*d* *h* *k*
AMS-ACIC	60 ± 6	− 17 ± 7	34 ± 17	− 25 ± 14
	30 ± 8	37 ± 8	− 4 ± 20	11 ± 16
	146 ± 59	293 69	− 50 ± 169	26 ± 139
-Schr.II	11 ± 10	− 48 ± 8	5 ± 22	5 ± 18
	41 ± 6	68 ± 5	− 3 ± 14	19 ± 11
	77 ± 57	197 ± 47	197 ± 129	92 ± 103
-Kiev	49 ± 7	− 70 ± 8	4 ± 19	18 ± 15
	58 ± 6	72 ± 7	6 ± 7	19 ± 13
	36 ± 44	219 ± 47	− 163 ± 116	− 29 ± 91
-Arthur	43 ± 7	− 33 ± 5	− 25 ± 13	35 ± 9
	42 ± 12	44 ± 9	2 ± 21	4 ± 15
	119 ± 108	143 ± 79	− 227 ± 192	279 ± 135
ACIC-Schr.II	− 48 ± 10	− 10 ± 10	4 ± 28	3 ± 22
	− 8 ± 6	17 ± 6	7 ± 17	2 ± 13
	− 184 ± 49	− 57 ± 48	113 ± 141	10 ± 110
-Kiev	− 2 ± 8	7 ± 8	− 22 ± 20	11 ± 17
	− 12 ± 6	− 10 ± 7	− 14 ± 17	− 5 ± 14
	− 56 ± 36	− 52 ± 39	− 221 ± 93	195 ± 77
-Mills	− 33 ± 8	− 5 ± 8	− 34 ± 22	26 ± 19
	− 3 ± 4	20 ± 4	4 ± 11	− 8 ± 10
	− 24 ± 18	80 ± 616	− 111 ± 48	96 ± 41
-Arthur	− 12 ± 4	7 ± 5	− 28 ± 10	34 ± 8
	− 4 ± 4	− 1 ± 5	− 28 ± 12	16 ± 9
	− 131 ± 22	79 ± 28	− 150 ± 62	133 ± 47
Schr.II-Kiev	− 33 ± 8	18 ± 9	− 14 ± 24	8 ± 18
	− 21 ± 6	0 ± 7	− 13 ± 16	− 11 ± 13
	− 18 ± 39	− 14 ± 42	95 ± 113	21 ± 89

We now consider the relative progressive shifts of these catalogues. From the table we see that in the majority of the cases the relative shifts of the systems in the direction of the Earth (the coefficients k) are ten times larger than the shifts in the normal plane. They vary for different catalogues between 0.2 km (ACIC-Schr.I) and 17.2 km (Bald.-Mills). The large displacements in the direction of the Earth are explained by uncertain determinations of the ζ-values.

We now proceed to analyse mutual displacements of different catalogues caused by their rigid-body rotation. Using Equations (8) to (10) and the values from Table II, we computed the angles of rotation π, ν, μ and the coordinates of the pole of the rotation (β, λ) of one catalogues relative to the others. The results are given in Table III. It is clear from the table that, in the majority of cases, the displacements of the selenodetic nets in the normal planes, which are given by the rotation around the 0ζ-axis through the angle μ, are smaller than the displacements caused by the change in the orientation of a system related to the direction to the Earth (the angles ν and π). Relative to the system 'Schrutka I' (with the exception of AMS) the vectors of rotation of catalogues ω are directed towards the polar region of the Moon.

TABLE III

Pair	π	ν	μ	β	λ
Schr.I-Bald	− 17″.5	130″.9	1″.0	82°21′	−86°38′
-AMS	−176.4	92.8	− 79.4	25 38	65 46
-ACIC	− 14.4	65.0	4.1	77 01	−74 04
-Schr.II	− 36.0	81.5	− 3.1	66 03	85 06
-Kiev	10.3	−102.1	− 25.8	−74 48	−21 48
-Arthur	50.5	214.5	− 30.9	74 34	−58 32
Bald.-Schr.I	17.5	−130.9	− 1.0	−82 21	−86 38
-AMS	−242.4	17.5	− 81.5	3 55	−71 25
-ACIC	1.0	−236.2	− 1.0	−90 00	−45 00
-Schr.II	− 16.5	− 80.4	− 4.1	−78 06	−75 59
-Kiev	3.1	− 21.6	20.6	−46 06	8 32
-Arthur	100.0	173.3	2.1	60 00	88 49
Mills	−112.4	−114.5	− 10.3	−45 24	84 46
AMS-Schr.I	176.4	− 92.8	79.4	−25 38	65 46
-Bald.	242.5	− 17.5	81.5	− 3 55	71 25
-ACIC	306.3	−115.5	48.5	−20 25	81 01
-Schr.II	206.3	− 74.2	91.8	−18 12	66 01
-Kiev	232.0	− 33.0	132.0	− 7 03	60 22
-Arthur	149.5	−148.5	77.3	−41 25	62 39
ACIC-Schr.I	14.4	− 65.0	− 4.1	−77 00	−74 04
-Bald.	− 1.0	236.2	1.0	90 00	−45 00
-AMS	−306.3	115.5	− 48.5	20 26	81 00
-Schr.II	− 66.0	193.9	4.1	71 10	−86 26
-Kiev	− 39.2	− 80.4	− 19.6	−61 26	63 26
-Arthur	110.4	106.2	− 11.3	43 45	84 08
-Mills	− 78.4	− 10.3	2.1	− 7 30	−88 30
Schr.II-Schr.I	36.1	− 81.5	3.1	−66 03	85 06
-Bald.	16.5	80.4	4.1	78 06	75 59
-AMS	−206.3	74.2	− 91.8	18 12	66 01
-Kiev	− 1.0	4.1	− 40.2	5 51	1 28

Consequently, the systematic differences relative to the catalogue 'Schrutka I', caused by rigid rotation, will be largest for the points in the equatorial region of the Moon. The angles of rotation for the different catalogues are in the interval 1′.7–3′.7.

For Baldwin's system the situation becomes more complicated. For the majority of the catalogues (Schr.I, ACIC, Schr.II, Arthur) the ω-vector is directed towards the polar region; for two other catalogues (Kiev, Mills) – in the direction of intermediate latitudes. For the AMS-system the vector is directed towards the equatorial region. The values of the rotations are from 0′.5 to 4′.3. The AMS-catalogue is particularly interesting: relative to this catalogue all investigated systems have the ω-vectors, directed in the south-east region of Mare Foecunditatis. The maximum systematic differences are for points situated in the region around the great circle going through Mare Humboldtianum in the north-east, and the southern part of Mare Nubium in the south-west. The rotational angles for all catalogues are rather large: namely, between 0′.4–5′. The linear displacements on the surface of the Moon are about 2.5 km. When comparing the catalogues with the ACIC-system we observe the large scatter in the ω-vector direction. The values of the rotations are between 1′.1 and 5′.5.

In order to study the relative deformations of selenodetic constructions, we computed the directional cosines of the principal axes of deformation and the coefficient of relative lengthening by Equations (13) to (15). In the first three lines of Table IV we give, for each pair of catalogues, the matrices of directional cosines between the $\xi\eta\zeta$-axis and the axis of deformation (designated as x, y, z). In the fourth row we have given the coefficients of the relative expansion along the x-y-z axes.

The results show that, in the majority of cases, the maximum deformation takes place along the OZ-axis.

We now consider a sphere with the center at O and a radius $R_{☾}$ in an undeformed

TABLE IV

	p_1 p_2 p_3 $\lambda_1 \times 10^5$	q_1 q_2 q_3 $\lambda_2 \times 10^5$	r_1 r_2 r_3 $\lambda_3 \times 10^5$
Schr.I-Bald	0.0468	− 0.9921	0.1168
	0.9995	0.0316	0.0043
	− 0.0079	0.1169	0.9931
	− 12.33	17.00	448.33
-AMS	0.6544	− 0.7460	0.1235
	− 0.6582	− 0.4816	0.5787
	0.3722	0.4600	0.8061
	− 53.26	19.18	− 277.92
-ACIC	0.4973	0.8280	0.2592
	0.8055	− 0.5516	0.2165
	− 0.3223	− 0.1011	0.9412
	− 32.73	0.12	94.61
-Schr.II	0.9033	0.4225	0.0743
	0.4205	− 0.9063	0.0421
	− 0.0852	0.0068	0.9963
	− 28.77	− 9.59	299.36
-Kiev	− 0.7349	0.1556	0.6611
	− 0.1237	− 0.9877	0.0952
	0.6678	− 0.0119	0.7442
	− 59.37	2.49	73.88
-Arthur	− 0.0221	− 0.9938	0.1093
	− 0.9263	0.0620	0.3717
	0.3761	0.0928	0.9219
	− 16.11	29.46	− 256.36
Bald.-AMS	− 0.9662	0.2483	0.0696
	− 0.2233	− 0.9424	0.2491
	0.1273	0.2252	0.9660
	− 55.51	− 30.45	− 780.05
-ACIC	− 0.6841	− 0.7282	0.0414
	0.7213	− 0.6688	0.1874
	− 0.1089	0.1581	0.9814
	− 22.13	20.97	− 383.85
-Schr.II	− 0.9765	− 0.1699	0.1326
	0.2116	− 0.9739	0.0818
	0.1069	0.1522	0.9826
	− 49.77	2.99	− 319.22

Table IV (Continued)

	p_1 p_2 p_3 $\lambda_1 \times 10^5$	q_1 q_2 q_3 $\lambda_2 \times 10^5$	r_1 r_2 r_3 $\lambda_3 \times 10^5$
Bald.-Kiev	− 0.6531	− 0.7307	0.1986
	− 0.7478	0.6594	0.0768
	0.1875	0.1035	0.9768
	− 3.85	4.80	− 200.94
-Mills	− 0.5094	0.8604	0.0146
	0.8599	0.5083	0.0470
	− 0.0329	− 0.0370	0.9988
	26.58	47.65	− 1223.07
-Arthur	− 0.8101	− 0.5781	0.0976
	0.5702	− 0.8214	0.0089
	0.0756	0.0617	0.9952
	− 16.97	− 8.71	− 678.32
AMS-ACIC	− 0.2876	− 0.5281	0.7990
	− 0.8224	0.5637	0.0766
	0.4910	0.6348	0.5966
	− 177.97	47.16	177.81
-Schr.II	− 0.7733	− 0.5033	0.3857
	0.6182	− 0.7340	0.2811
	0.1415	0.4553	0.8790
	− 11.73	33.81	253.93
-Kiev	− 0.0652	0.3569	0.9319
	− 0.9791	− 0.2030	0.0086
	0.1910	− 0.9122	0.3624
	− 205.16	47.58	115.57
-Arthur	− 0.8279	0.5607	0.0139
	0.5554	0.7842	0.2767
	− 0.1499	− 0.2335	0.9607
	39.14	72.77	− 251.91
ACIC-Schr.II	0.8928	0.1772	0.4141
	− 0.2171	0.9748	0.0508
	− 0.3945	− 0.1354	0.9089
	− 91.73	17.93	155.81
-Kiev	− 0.5539	− 0.8283	0.0841
	0.8291	− 0.5405	0.1429
	− 0.0729	0.1489	0.9861
	− 8.32	2.56	− 227.24
-Mills	− 0.9457	− 0.2553	0.2012
	− 0.1866	0.9334	0.3064
	0.2661	− 0.2521	0.9304
	− 27.91	34.59	− 130.68
-Arthur	− 0.3566	− 0.9338	0.0306
	0.8228	− 0.4021	− 0.4017
	0.4071	− 0.1258	0.9047
	− 1.26	27.55	− 189.29
Schr.II-Kiev	0.9871	0.0921	0.1308
	− 0.1079	0.9869	0.1202
	− 0.1181	− 0.1329	0.9841
	− 35.26	− 1.48	98.74

medium that is constructed from the points $(\xi_2\eta_2\zeta_2)$ of the second catalogue. Referred to the principal axes of deformation *Oxyz* the equation of the sphere is

$$x^2 + y^2 + z^2 = R_{\text{☾}}^2 . \tag{29}$$

According to Equation (17) the coordinates of the points will after the deformation have the values

$$\begin{aligned} x_1 &= x + \lambda_1 x, \\ y_1 &= y + \lambda_2 y, \\ z_1 &= z + \lambda_3 z; \end{aligned} \tag{30}$$

from which

$$\frac{x_1^2}{(1+\lambda_1)^2} + \frac{y_1^2}{(1+\lambda_2)^2} + \frac{z_1^2}{(1+\lambda_3)^2} = R_{\text{☾}}^2 . \tag{31}$$

We see that the deformed surface is an ellipsoid with semi-axes

$$R_{\text{☾}}(1+\lambda_1), \qquad R_{\text{☾}}(1+\lambda_2), \qquad R_{\text{☾}}(1+\lambda_3). \tag{32}$$

For the majority of the catalogues the prolongation along this axis varies between 2–4 km. An exception is the Baldwin system, for which the deformation relatively to all catalogues is very large and reaches -21 km (Baldwin-Mills). It also is in a direction opposite to the linear displacement $(+17)$ given above.

Therefore, the characteristic feature of the systematic catalogue errors by pure deformation is the fact that they cause maximum distortion of selenodetic systems in the direction of the Earth.

The catalogue analysis shows that the hypothesis of relative isotropic expansion of the selenodetic constructions is not real. This is clear from the fact that the ζ-coordinates of lunar objects are determined with errors ten times as large as the ξ- and η-coordinates. Because of this the expansion coefficients cannot be equal (that is, $\lambda_1 = \lambda_2 = \lambda_3 = \mu$). As the calculations show, the λ_3-value is larger than the coefficients λ_1, λ_2 in almost all cases. Moreover, the axes of relative deformation do not coincide with the axis $0\xi\eta\zeta$ and, therefore, one must not take the last-mentioned axis as the principal axis of an expansion.

From the above-mentioned facts it is clear that the basic lunar coordinate systems are deformed anisotropically, and for a transformation of one catalogue into another it is necessary to use the reduction formulae given by the general expression (1).

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