ON THE THEORETICAL POSSIBILITY OF THE LIBRATION CLOUD

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Abstract. In the present paper, the problem of whether the interplanetary matter has a tendency to accumulate around the Lagrangian libration points L_4 and L_5, is examined statistically. It is concluded that: (1) If the particles are initially assumed to be distributed uniformly, they keep the uniformity ever after around the libration points. (2) If the particles receive random stochastic perturbations, their distribution tends to become uniform even if initially they have non-uniform distributions. (3) If the particles mutually collide inelastically, they have a tendency to avoid the regions near the libration points. Therefore, the interplanetary matter will not tend to accumulate near the libration points. Even if the observations of the 'libration cloud' so far reported are confirmed, the clouds are likely to be but temporary objects.

1. Introduction

Since Kordylewski reported observations of nebulous matter about the Lagrangian libration points L_4 and L_5, which form two equilateral triangles with the Earth and Moon in the Moon's orbital plane (Kordylewski, 1961a), the arguments whether the clouds really exist or not have been discussed. Kordylewski observed such clouds again in September of the same year (Kordylewski, 1961b), as did Simpson (1967) and Vanýsek (1968) in 1966. But many other observers failed to confirm such clouds, and raised doubts about their existence (Roosen, 1966, 1968; Roosen *et al.*, 1967; Bruman, 1969).

It is well known that a particle near the Lagrangian libration points L_4 and L_5 possessing sufficiently small velocities relative to these points, librates around these points and remains permanently in this region, if the mass-ratio of two finite bodies is below a definite limit. But, theoretically, this reasoning has a weak point. The librating particles must have formerly been wandering outside these regions, and owing to some kind of disturbances – for example, collisions with the other particles – they could have been placed in the state of libration. Hence, these librating particles can also escape from these regions by the operation of the same causes. Is there any tendency to accumulate around the libration points, if particles originally distributed with uniform density in space undergo random perturbations occurring with equal probability in all directions? This is the main subject of the present paper.

2. A Point Set of Uniform Distribution

Before entering into the main subject, the author wishes to set forth some mathematical lemmas. Mathematical rigour will not be required, and almost all statements will be mentioned without proof or with simple explanations.

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(1) Let us suppose an (Euclidian) space, in which coordinate systems $(x_1, x_2, \ldots, x_n)$ are defined so that the distance between two points $\{x_i\}$ and $\{x'_i\}$ is

$$\sqrt{\sum_{i=1}^{n} (x_i - x'_i)^2} \,. \tag{1}$$

In this space, a set of particles is supposed to exist possessing uniform density distribution. This means that, in all parts of the space, the probability that a definite volume element contains definite number of particles, is uniform.

If all particles of the space are displaced so that one of their coordinates – for example, x_i – is replaced by αx_i; the distribution of the particles remains uniform, although the density becomes α^{-1} times.

(2) If all particles of the space are displaced so that any two of their coordinates – for example $(x_i, x_j, i \neq j)$ – are replaced by $(x_i \cos\theta - x_j \sin\theta, x_i \sin\theta + x_j \cos\theta)$, the distribution of the particles remains uniform. In this case, the density of the particle is not altered. This is a rotation by angle θ in the (x_i, x_j)-plane.

(3) If all particles of the space are displaced by the above two manners successively in finite times, the distribution of the particles remains uniform.

(4) If all particles of the space are displaced so that their coordinates are transformed linearly in accordance with

$$x'_i = \sum_{j=1}^{n} \alpha_{ij} x_j \,, \tag{2}$$

where the determinant $|\alpha_{ij}|$ does not vanish, the distribution of the particles remains uniform. The reason is the fact that a linear transformation is always a combination of dilations, contractions and rotations.

(5) Let us suppose that a system of differential equations

$$\dot{x}_i = F_i(x_1, \ldots, x_n, t) \quad (i = 1, \ldots, n) \tag{3}$$

has a general solution which can be expressed as

$$x_i = \sum_{j=1}^{n} a_j f_{ij}(t), \quad (i = 1, \ldots, n), \tag{4}$$

where a_i are integration constants, and determinant $|f_{ij}(t)|$ never vanishes. Then, if the probability distribution of particles, whose coordinates $\{x_i\}$ are ruled by Equation (3), is uniform at initial time, it remains uniform after that time. The reason is as follows. The expression (4) is a linear transformation between $\{x_i\}$ and $\{a_i\}$. If $\{x_i\}$ has uniform distribution at initial time, the $\{a_i\}$ also has, and after that time, the $\{x_i\}$ again has uniform distribution, though density itself may be changed.

(6) Suppose that the particles whose coordinates are given by Equation (3) suffer perturbations of the form

$$\dot{x}_i = F_i(x_1, \ldots, x_n, t) + \lambda_i(t) \,, \tag{5}$$

where $\lambda_i(t)$ are stochastic functions with mean value zero, and have equal probability in all directions. We imagine that each particle has its proper $\lambda_i(t)$ as a molecule of gas has its proper perturbations. Its stochastic character is the same for all particles and does not depend on the positions of the particles. Then if the points have uniform distribution at initial time, this distribution remains uniform also for all other time.

The reason is as follows. The solutions of Equation (5) are easily obtained by regarding the $\{a_i\}$'s as a function of time and inserting them in Equation (5). Thus we get a system of linear differential equations

$$\sum_{i=1}^{n} \dot{a}_i f_{ij}(t) = \lambda_i(t), \tag{6}$$

which is solvable for $\dot{a}_i$ as the determinant $|f_{ij}(t)|$ does not vanish. The vector $\{\dot{a}_i\}$ thus obtained has not necessarily uniform distribution in all directions. But if the distribution of $\{a_i\}$ is initially uniform, it will remain uniform ever after. In this case, the relations (4) also hold although a_i's are functions of time. Hence, if initially $\{x_i\}$ possesses uniform distribution, $\{a_i\}$ at the initial state will also retain it. Consequently, if $\{a_i\}$ possesses it at all times, then $\{x_i\}$ also has uniform distribution for all times.

(7) Let us suppose that the vector $\{\dot{a}_i\}$ has finite probability in all directions and vectors $\{\dot{a}_i\}$ and $\{-\dot{a}\}$ have the same probability of occurrence. This is always possible owing to the relations (6), if $\{\lambda_i(t)\}$ has similar character unless the determinant $|f_{ij}(t)|$ vanishes. Then, if vector $\{a_i\}$ has non-uniform distribution at initial state, the distribution tends to uniform asymptotically after a sufficiently long time interval.

The reason is as follows. The probability that $\{a_i\}$ changes to $\{a_i'\}$ is equal to that of reverse procedure. If state $\{a_i\}$ contains initially more particles than $\{a_i'\}$, the more particles are transferred from $\{a_i\}$ to $\{a_i'\}$. Hence, difference of numbers of particles between two states must diminish exponentially. If $\{a_i\}$ becomes uniform, the distribution of $\{x_i\}$ also becomes uniform.

Considering all lemmas above stated, we arrive at the following conclusions. If a swarm of particles whose motion is governed by Equation (3) suffers random perturbations of the form (5) then, even if the particles initially have a non-uniform distribution, the swarm approaches the state of uniform distribution asymptotically, and never has a tendency to accumulate around any special point of the space.

3. Libration Around L_4 and L_5

Now to return to our main subject. The aim of the present section will be to show that the libration motion around L_4 and L_5 of the restricted problem of three bodies is subject to conditions mentioned in the previous section.

It is well known that the equations of motion near the librational point can be reduced to the following form (cf. Szebehely, 1967)

$$\begin{aligned} \ddot{x} - 2\dot{y} &= ax, \\ \ddot{y} + 2\dot{x} &= by. \end{aligned} \tag{7}$$

These equations apply under the following conditions: two bodies are rotating uniformly around their common gravity centre, interacting with each other by gravitational force, and their distance is assumed to be invariable. The unit of length is their distance, and the unit of time is taken so as to make the revolutional velocity of the two bodies unity. The unit of mass is the sum of both masses. Thus the gravitational constant becomes unity owing to the Kepler's third law. We take the mass of the smaller body as μ, hence the mass of the other body is $1-\mu$. If μ is smaller than a definite value of

$$\mu < \tfrac{1}{2}\{1 - (\tfrac{23}{27})^{1/2}\} = 0.0385\ldots,$$

it is known that the motion of the third body around the L_4 or L_5 is stable. In our case, the ratio of mass of the Moon to that of the Earth is 1:81.3, and so we have $\mu = 0.0122$.

In the present paper, we assume that the whole system is confined to a plane. As L_4 and L_5 are situated asymmetrically with respect to the line joining the Earth and Moon, we consider the motion about L_4 only hereafter. The coordinate system (x, y) is taken so as to make its origin coincide with L_4 and the x-axis inclines to the above line at angle α, which is defined by

$$\tan 2\alpha = \sqrt{3}(1 - 2\mu). \tag{8}$$

The constants a and b are expressed as

$$\left.\begin{aligned} a &= \tfrac{3}{2}\{1 - \sqrt{1 - 3\mu(1-\mu)}\}, \\ b &= \tfrac{3}{2}\{1 + \sqrt{1 - 3\mu(1-\mu)}\}. \end{aligned}\right\} \tag{9}$$

Equations (7) can be rewritten as

$$\left.\begin{aligned} \dot{x} &= X, \\ \dot{y} &= Y, \\ \dot{X} - 2Y &= ax, \\ \dot{Y} + 2X &= by. \end{aligned}\right\} \tag{10}$$

Hence, their general solutions must contain four integration constants. They are easily found to be

$$\begin{aligned} x &= A_1 \cos n_1 t + B_1 \sin n_1 t + A_2 \cos n_2 t + B_2 \sin n_2 t, \\ y &= \alpha_1 (A_1 \sin n_1 t - B_1 \cos n_1 t) + \alpha_2 (A_2 \sin n_2 t - B_2 \cos n_2 t), \end{aligned} \tag{11}$$

where A_i and B_i $(i = 1, 2)$ are integration constants and

$$\left.\begin{aligned} n_1 &= \sqrt{\tfrac{1}{2}\{1 + \sqrt{1 - 27\mu(1-\mu)}\}}, \\ n_2 &= \sqrt{\tfrac{1}{2}\{1 - \sqrt{1 - 27\mu(1-\mu)}\}}, \end{aligned}\right\} \tag{12}$$

$$\left.\begin{aligned} -\alpha_1 &= \frac{2n_1}{n_1^2 + b} = \frac{n_1^2 + a}{2n_1}, \\ -\alpha_2 &= \frac{2n_2}{n_2^2 + b} = \frac{n_2^2 + a}{2n_2}. \end{aligned}\right\} \quad (13$$

Inspecting the solutions (10) we can immediately conclude that they consist of linear combinations of the integration constant of the form (4). Hence, owing to the lemma (5) of the previous section, the set of particles having initially uniform distribution keeps the same state thereafter.

When the particles are subject to random perturbations, the Equations (10) are replaced by

$$\left.\begin{aligned} &\dot{x} = X \\ &\dot{y} = Y \\ &\dot{X} - 2Y = ax + \lambda(t), \\ &\dot{Y} + 2X = by + \nu(t). \end{aligned}\right\} \tag{14}$$

The solutions of these equations can be expressed in the same forms as (11), but the quantities A_i, B_i $(i=1,2)$ are not now constants, but functions of time. They are obtained by integrating the differential equations

$$\left.\begin{aligned} \dot{A}_1 &= \frac{\nu(t)\cos n_1 t}{n_1\alpha_1 - n_2\alpha_2} + \frac{\alpha_2\lambda(t)\sin n_1 t}{n_2\alpha_1 - n_1\alpha_2}, \\ \dot{B}_1 &= \frac{\nu(t)\sin n_1 t}{n_1\alpha_1 - n_2\alpha_2} - \frac{\alpha_2\lambda(t)\cos n_1 t}{n_2\alpha_1 - n_1\alpha_2}, \\ \dot{A}_2 &= \frac{\nu(t)\cos n_2 t}{n_2\alpha_2 - n_1\alpha_1} + \frac{\alpha_1\lambda(t)\sin n_2 t}{n_1\alpha_2 - n_2\alpha_1}, \\ \dot{B}_2 &= \frac{\nu(t)\sin n_2 t}{n_2\alpha_2 - n_1\alpha_1} - \frac{\alpha_1\lambda(t)\cos n_2 t}{n_1\alpha_2 - n_2\alpha_1}. \end{aligned}\right\} \tag{15}$$

Let us hereafter assume that the vectors (λ, ν), $(\dot{A}_1, \dot{B}_1)$ and $(\dot{A}_2, \dot{B}_2)$ individually constitute three two-dimensional spaces. If a set of these vectors has a density distribution dependent only on its length, and independent of its direction, we call the distribution isotropic. Inspecting the above Equations (15) we see that even if distribution of (λ, ν) is isotropic, that of $(\dot{A}_1, \dot{B}_1)$ and $(\dot{A}_2, \dot{B}_2)$ are not necessarily isotropic. In order to ascertain if the distribution functions of $(\dot{A}_1, \dot{B}_1)$ and $(\dot{A}_2, \dot{B}_2)$ are isotropic, it is sufficient that the absolute values of

$$\frac{n_1\alpha_1 - n_2\alpha_2}{n_1\alpha_2 - n_2\alpha_1}\alpha_2 \quad \text{and} \quad \frac{n_1\alpha_1 - n_2\alpha_2}{n_1\alpha_2 - n_2\alpha_1}\alpha_1$$

are equal to unity, respectively. But if the functions $\lambda(t)$ and $\nu(t)$ are statistically independent of $\sin n_1 t$ and $\sin n_2 t$ the distribution functions of $(\dot{A}_1, \dot{B}_1)$ and $(\dot{A}_2, \dot{B}_2)$ become isotropic in the mean during sufficiently long time intervals.

Hence, from the lemma (7) of the previous section, we can conclude that even if, at the initial state, distribution of the particles is not uniform, it becomes uniform after a sufficiently long time interval. Therefore, even if a 'libration cloud' is produced around the libration points temporarily, it is bound to disperse sooner or later, and therefore there is no tendency to accumulate on the libration points for the particles floating near these points.

By the same reasoning, we see that there should also be no tendency for interplanetary matter to accumulate around the equilibrium points L_1, L_2 and L_3, which lie on the line connecting the Earth and the Moon.

4. Case of Inelastic Collisions

In the above theory, it is assumed that the statistical nature of the perturbations is an utterly random one. But if we assume that the mutual collision of the particles is inelastic, the perturbations will acquire a special character to increase the sum of Jacobi's constants of individual particles.

Integrating Equations (7) we get the Jacobi integral

$$\dot{x}^2 + \dot{y}^2 = ax^2 + by^2 - C, \tag{16}$$

where C is Jacobi's constant (Szebehely, 1967).

If a particle moves freely inside the Earth-Moon system, it has an individual Jacobi constant of its own. If two particles collide mutually inelastically, the sum of each kinetic energy $\frac{1}{2}(\dot{x}^2 + \dot{y}^2)$ must diminish; hence, the sum of each Jacobi constant of each particle increases, and the number of particles which have positive Jacobi constants increases. As these particles cannot reach the Lagrangian libration points L_4 and L_5 beyond the zero-velocity curve, the density of particles near the libration points must be diminished.

This reasoning also applies if the particle is far from the libration points, as the Jacobi integral exists everywhere in the orbital plane.

Owing to these arguments, the author inclines to the opinion that the Lagrangian libration points L_4 and L_5 do not possess a tendency to accumulate interplanetary matter; but, on the contrary they should constitute the 'zones of avoidance' of the matter. It may be possible that interplanetary matter disintegrates around these points and forms a temporary cloud or swarm, but it will disperse into space in the course of time. The objects which Kordylewski and Simpson observed might be such dispersing clouds.

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